Precise determination of critical exponents by QFT methods

J. Zinn-Justin

Dapnia, CEA/Saclay,

www-dapnia.cea.fr

and

Institut de Mathématiques de Jussieu-Chevaleret, Paris 7

Email: zinn@dsm-mail.saclay.cea.fr

We review here the methods, based on renormalized ϕ^4 QFT and RG, which have led to precise determination of critical exponents of the N-vector model, and more recently of the equation of state of the 3D Ising model.

The starting point is the perturbative expansion for RG functions or the effective potential to the order presently available.

Perturbation theory is known to be divergent and its divergence has been related to instanton contributions.

This has allowed characterizing the large order behaviour of perturbation series, an information that can be used to efficiently ``sum'' them.

Practical summation methods based on Borel transformation and conformal mapping have been developed.

They have led to the most accurate results available probing field theory in a non-perturbative regime.

Compared to exponents, the determination of the the scaling equation of state involves a few additional (non-trivial) steps.

A general reference on the topic is
J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, (Oxford Univ. Press 1989, fourth ed. 2002.)
The first precise estimates appeared in
J.C. Le Guillou and J. Zinn-Justin, *Phys. Rev. Lett*. 39 (1977) 95; *Phys. Rev*. B21 (1980) 3976
using six-loop series for RG functions reported in
B.G. Nickel, D.I.Meiron, G.B. Baker, Univ. of Guelph Report 1977.
Additional terms were used in
R. Guida and J. Zinn-Justin, *J. Phys*. A31 (1998) 8103.
For the equation of state see
R. Guida and J. Zinn-Justin, *Nucl. Phys*. B489 [FS] (1997) 626, [hep-th/9610223];
Many relevant articles about large order behaviour are reprinted in
Large Order Behaviour of Perturbation Theory, *Current Physics* vol. 7, J.C. Le Guillou and J. Zinn-Justin eds., (North-Holland, Amsterdam 1990).

The first steps

- Wilson's renormalization group.
- The $\varepsilon = 4 - d$ expansion: K.G. Wilson and M.E. Fisher, *Phys. Rev. Lett.* 28 (1972) 240.
- Callan-Symanzik equations of massive renormalized perturbation theory: E. Brézin, J.-C. Le Guillou and J. Zinn-Justin, *Phys. Rev.* D8, 434-440 (1973).
- The use of massive perturbation theory at fixed dimension: G. Parisi, Cargèse Lectures 1973, published in *J. Stat. Phys.* 23 (1980) 49.
- Large order behaviour of perturbative expansions: E. Brézin, J.-C. Le Guillou and J. Zinn-Justin, *Phys. Rev.* D15, 1544-1557 (1977).

The N vector model

Number of interesting phase transitions are described by the N vector model, a $O(N)$ symmetric model with an N-component field $\phi(x)$ ϕ_i, $i = 1, \ldots, N$. The partition function then reads

$$\mathcal{Z} = \int [\mathrm{d}\phi(x)] \exp\left[-\mathcal{H}(\phi)\right],$$

where the hamiltonian (or euclidean action) is given by

$$\mathcal{H}(\phi) = \int \left\{ \frac{1}{2} \left[\partial_\mu \phi(x)\right]^2 + \frac{1}{2} r \phi^2(x) + \frac{u}{4!} \left[\phi^2(x)\right]^2 \right\} \mathrm{d}^d x.$$

The first values of N correspond to the transitions:
$N = 1$: liquid–vapour, binary mixtures, Ising systems
$N = 2$: Helium superfluidity
$N = 3$: ferromagnetic systems
and the limit $N = 0$ to statistical properties of long polymers.

The massless scheme and the ε-expansion

This scheme is based on expanding around the critical (massless) theory. Therefore, calculations cannot be performed at fixed dimension $d < 4$ because perturbation theory is IR divergent. The $\varepsilon = 4 - d$-expansion is an essential part of the scheme. The hamiltonian is then parametrized as

$$\mathcal{H}(\phi) = \int \left\{ \frac{1}{2} \left[\partial_\mu \phi(x)\right]^2 + \frac{1}{2} t \phi^2(x) + \frac{g \mu^{4-d}}{4!} \left[\phi^2(x)\right]^2 \right\} \mathrm{d}^d x \,,$$

where dimensional regularization is assumed. Here, μ is the renormalized scale, g a dimensionless coupling constant and $t = r - r_c \propto T - T_c$. Renormalized vertex (1PI) functions satisfy RG equations of the form

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \frac{1}{2} n \eta(g) + \eta_2(g) t \frac{\partial}{\partial t} \right] \Gamma^{(n)} = 0 \,,$$

The RG functions β, η, η_2 can be derived from the calculation of the UV divergent part of two-point and four-point vertex functions when the dimension $d \to 4$ in the massless theory.

The fixed point equation reduces to

$$\beta(g^*) = 0\,.$$

To determine universal properties in the critical domain (near the transition temperature), one must first determine the zeros g^* of the β-function, and then calculate all other physical quantities for $g = g^*$. For $\varepsilon = 4 - d$ small,

$$\beta(g) = -\varepsilon g + (N+8)g^2/48\pi^2 + O(g^3)\,.$$

Therefore, near dimension 4 one finds an IR stable fixed point

$$g^* = 48\pi^2\varepsilon/(N+8) + O(\varepsilon^2)\,.$$

Then, all universal quantities can be calculated in the form of an ε expansion.

In particular, the values at $g = g^*$ of the two functions $\eta(g)$ and $\eta_2(g)$ are related to the two independent critical exponents η and ν:

$$\langle\phi(x)\phi(0)\rangle_{T=T_c} \underset{|x|\to\infty}{\propto} 1/x^{d-2+\eta}\,,$$

$$\xi(T) \underset{T\to T_c}{\propto} |T - T_c|^{-\nu}\,,$$

where ξ is the correlation length.

Callan–Symanzik equations

Another scheme involves working in the massive field theory (the critical domain) where the mass $m = 1/\xi$. Then, renormalized correlation functions are defined by the renormalization conditions

$$\tilde{\Gamma}^{(2)}(p; m, g) = m^2 + p^2 + O(p^4)$$
$$\tilde{\Gamma}^{(4)}(0, 0, 0, 0) = gm^{4-d}.$$

Vertex functions then satisfy the CS equations

$$\left[m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} - \frac{n}{2}\eta(g) \right] \Gamma^{(n)}(p_i; m, g) = m^2(2 - \eta)\Gamma^{(n)}_{\phi^2}(p_i; m, g),$$

where $\Gamma^{(n)}_{\phi^2}$ correspond to correlation functions with one insertion of the operator $\int d^d x\, \phi^2(x)$.

In dimension 4, and within the ε expansion scheme, the r.h.s. is negligible and the scaling relations can be proved (with more work). However, within this massive scheme, the perturbation expansion exists in any dimension.

Practical calculations: the ε expansion

One first calculates all renormalized quantities as an expansion in a double series expansion in powers of g and ε, in particular the β-function. One solves the equation $\beta(g) = 0$ in the form of a ε expansion. One inserts the fixed point value of g in other physical quantities. It then remains to extract, from a small number of terms of the expansion, information about the relevant physical dimensions, for example $\varepsilon = 1$.

Table 1 shows the successive partial sums for two exponents up to order ε^5. It is immediately apparent that the series do not converge.

Table 1

Sum of the successive terms of the ε-expansion of γ and η for $\varepsilon = 1$ and $N = 1$.

k	0	1	2	3	4	5
γ	1.000	1.1667	1.2438	1.1948	1.3384	0.8918
η	$0.0\ldots$	$0.0\ldots$	0.0185	0.0372	0.0289	0.0545

The fixed dimension scheme

Following Parisi'suggestion, one can also evaluate the β-function directly in dimension 3 but, then, one has no longer a "small" expansion parameter. However, it has been noticed by Nickel that Feynman diagrams in dimension 3 can be more easily evaluated than near dimension 4. At present, Nickel has managed to calculate all diagrams up to seven loops (in the terminology of Feynman diagrams) contributing to η, η_2, but the diagrams contributing to the β-function, which are more difficult, only up to six loops.

For example, to six loop order, for $N = 1$, Nickel has obtained

$$\beta(\tilde{g}) = -\tilde{g} + \tilde{g}^2 - \frac{308}{729}\tilde{g}^3 + 0.3510695978\tilde{g}^4$$
$$- 0.3765268283\tilde{g}^5 + 0.49554751\tilde{g}^6 - 0.749689\tilde{g}^7 + O\left(\tilde{g}^8\right),$$

where $\tilde{g} = 3g/(16\pi))$.

One must first determine numerically the zero of the β-function, which is a number of order 1. This clearly requires some summation of the series.

Large order behaviour and instanton calculus

Let function $F(g)$ be a real function, analytic in the plane with a cut on the whole real negative semi-axis. Moreover, we assume that $F(g)$ has an asymptotic expansion for $g \to 0_+$. Under simple technical conditions, $F(g)$ has the Cauchy representation

$$F(g) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{\operatorname{Im} F(g')}{g' - g} \mathrm{d}g'.$$

Expanding $F(g)$ in a power series for $g \to 0_+$,

$$F(g) = \sum_k F_k g^k,$$

one finds

$$F_k = \frac{1}{\pi} \int_{-\infty}^{0} \operatorname{Im} F(g) g^{-k-1} \mathrm{d}g.$$

The behaviour of the integral for $k \to \infty$, is governed by the behaviour of $\operatorname{Im} F(g)$ for $g \to 0_-$.

In a $g\phi^4$ field theory, correlation functions are analytic in a cut-plane. For $g \to 0_-$, the field integral is dominated by saddle points solutions of the euclidean classical field equations. Only those that have a finite action contribute to the integral. They are called instantons.

For $g \to 0_-$, one finds again the trivial solution $\phi(x) = 0$, but it contributes only to the real part. The imaginary part is dominated by non-trivial solutions of

$$(-\Delta_x + m^2)\phi(x) + gm^{4-d}\phi^3(x)/6 = 0\,.$$

The leading contribution comes from solutions of the form

$$\phi(x) = \frac{1}{\sqrt{-g}}m^{(d-2)/2}f(mr), \quad r = |x|.$$

The function f for $d < 4$ satisfies the non-linear differential equation

$$-\ddot{f} + (d-1)\dot{f}/r + f - f^3/6 = 0\,.$$

The contribution of the saddle point to the field integral then has the form $\exp(A/g)$, $A > 0$, up to less singular pre-factors. Then,

$$F_k \sim \int_{-\infty}^{0} \mathrm{e}^{A/g}g^{-k-1}\mathrm{d}g \sim k!(-A)^{-k}\,.$$

Thus, the perturbative expansion is divergent. A more precise calculation leads to, for example, for the β-function in three dimensions:

$$\beta_k \underset{k \to \infty}{\propto} (-a)^k k^b k!$$

with $a = 0.147774232\ldots$.

To deal with this problem, when the coupling constant g is not small, it is necessary to introduce summation techniques. In three dimensions, the perturbative expansion is proved to be Borel summable. It is thus natural to introduce the Borel–Laplace transformation (here, Borel–Leroy):

$$B_\sigma(g) = \sum_k \frac{\beta_k}{\Gamma(k + \sigma + 1)} g^k.$$

Then, formally in the sense of power series

$$\beta(g) = \int_0^{+\infty} t^\sigma e^{-t} B_\sigma(gt) \mathrm{d}t.$$

The function $B(g)$ is analytic in a circle of radius $1/a$. The series is said Borel summable if, in addition, $B(g)$ is analytic in a neighbourhood of the real positive semi-axis and the integral converges.

The series defines the function in a circle. It is thus necessary to perform an analytic continuation. In practice, with a small number of terms, the continuation requires a larger domain of analyticity. Le Guillou and Z.-J. (1977–1980) have assumed maximal analyticity, i.e., analyticity in a cut-plane. The continuation has then be obtained by a conformal mapping of the cut-plane onto a circle.

Table 2

Series summed by the method based on Borel transformation and mapping for the zero $\tilde{g}^*$ of the $\beta(g)$ function and the exponents γ and ν in the ϕ_3^4 field theory.

k	2	3	4	5	6	7
$\tilde{g}^*$	1.8774	1.5135	1.4149	1.4107	1.4103	1.4105
ν	0.6338	0.6328	0.62966	0.6302	0.6302	0.6302
γ	1.2257	1.2370	1.2386	1.2398	1.2398	1.2398

Improved summation techniques and the additional seven-loop contributions have lead to new estimates of critical exponents (Guida, Z-J. 1998).

Critical exponents from the $O(N)$ symmetric $(\phi^2)_3^2$ field theory

N	0	1	2	3
$\tilde{g}^*$	1.413 ± 0.006	1.411 ± 0.004	1.403 ± 0.003	1.390 ± 0.004
g^*	26.63 ± 0.11	23.64 ± 0.07	21.16 ± 0.05	19.06 ± 0.05
γ	1.1596 ± 0.0020	1.2396 ± 0.0013	1.3169 ± 0.0020	1.3895 ± 0.0050
ν	0.5882 ± 0.0011	0.6304 ± 0.0013	0.6703 ± 0.0015	0.7073 ± 0.0035
η	0.0284 ± 0.0025	0.0335 ± 0.0025	0.0354 ± 0.0025	0.0355 ± 0.0025
β	0.3024 ± 0.0008	0.3258 ± 0.0014	0.3470 ± 0.0016	0.3662 ± 0.0025
α	0.235 ± 0.003	0.109 ± 0.004	-0.011 ± 0.004	-0.122 ± 0.010
ω	0.812 ± 0.016	0.799 ± 0.011	0.789 ± 0.011	0.782 ± 0.0013
$\omega\nu$	0.478 ± 0.010	0.504 ± 0.008	0.529 ± 0.009	0.553 ± 0.012

Reference: R. Guida and J. Zinn-Justin, *J. Phys. A* 31 (1998) 8103, cond-mat/9803240, an improvement over the results published in
J.C. Le Guillou and J. Zinn-Justin, *Phys. Rev. Lett.* 39 (1977) 95; *Phys. Rev.* B21 (1980) 3976.

For a comparison

Critical exponents from the $O(N)$ symmetric $(\phi^2)_3^2$ field theory

N	0	1	2	3
γ	1.1596 ± 0.0020	1.2396 ± 0.0013	1.3169 ± 0.0020	1.3895 ± 0.0050
ν	0.5882 ± 0.0011	0.6304 ± 0.0013	0.6703 ± 0.0015	0.7073 ± 0.0035
α	0.235 ± 0.003	0.109 ± 0.004	-0.011 ± 0.004	-0.122 ± 0.010
β	0.3024 ± 0.0008	0.3258 ± 0.0014	0.3470 ± 0.0016	0.3662 ± 0.0025
$\omega\nu$	0.478 ± 0.010	0.504 ± 0.008	0.529 ± 0.009	0.553 ± 0.012

Critical exponents from $O(N)$ symmetric lattice models

N	0	1	2	3
γ	1.1575 ± 0.0006	1.2385 ± 0.0025	1.322 ± 0.005	1.400 ± 0.006
ν	0.5877 ± 0.0006	0.631 ± 0.002	0.674 ± 0.003	0.710 ± 0.006
α	0.237 ± 0.002	0.103 ± 0.005	-0.022 ± 0.009	-0.133 ± 0.018
β	0.3028 ± 0.0012	0.329 ± 0.009	0.350 ± 0.007	0.365 ± 0.012
$\omega\nu$	0.56 ± 0.03	0.53 ± 0.04	0.60 ± 0.08	0.54 ± 0.10

A systematic comparison between these field theory and renormalization-group based calculations and available experimental results, as well as lattice calculations, shows excellent agreement but progress in the latter, should encourage us to further improve field theory results. For example, in the case of superfluid Helium transition, low gravity experiments have given

$$\nu = 0.6705 \pm 0.0006\,, \quad \nu = 0.6708 \pm 0.0004$$

$$\alpha = -0.01285 \pm 0.00038\,,$$

a precision that is now a challenge to field theory, which yields:

$$\nu = 0.6703 \pm 0.0015\,, \quad \alpha = -0.011 \pm 0.004\,.$$

A noticeable improvement can be expected from a seven-loop calculation of the β- function, since the value of g^* enters in the calculation of all other universal quantities.

Moreover, it would be useful that another group verifies Nickel's calculations to check the precision of the evaluated diagrams.

Table 6
Critical exponents in the $(\phi^2)_3^2$ field theory from the ε-expansion.

N	0	1	2	3
γ (free) γ (bc)	1.1575 ± 0.0060 1.1571 ± 0.0030	1.2355 ± 0.0050 1.2380 ± 0.0050	1.3110 ± 0.0070 1.317	1.3820 ± 0.0090 1.392
ν (free) ν (bc)	0.5875 ± 0.0025 0.5878 ± 0.0011	0.6290 ± 0.0025 0.6305 ± 0.0025	0.6680 ± 0.0035 0.671	0.7045 ± 0.0055 0.708
η (free) η (bc)	0.0300 ± 0.0050 0.0315 ± 0.0035	0.0360 ± 0.0050 0.0365 ± 0.0050	0.0380 ± 0.0050 0.0370	0.0375 ± 0.0045 0.0355
β (free) β (bc)	0.3025 ± 0.0025 0.3032 ± 0.0014	0.3257 ± 0.0025 0.3265 ± 0.0015	0.3465 ± 0.0035	0.3655 ± 0.0035
ω	0.828 ± 0.023	0.814 ± 0.018	0.802 ± 0.018	0.794 ± 0.018
θ	0.486 ± 0.016	0.512 ± 0.013	0.536 ± 0.015	0.559 ± 0.017

Finally, using the series provided by Nickel, combined with a few new technical tricks, it has been possible to obtain a precise representation of the equation of state for models in the $N = 1$ Ising class (Guida and Z.-J.). In particular, from the equation of state, a number of universal combinations of amplitudes of the singularities at T_c can be derived (see table 10). For example, the magnetic susceptibility, diverges at T_c with susceptibility exponents γ, and

$$\chi_+ \sim C_+(T - T_c)^{-\gamma},$$
$$\chi_- \sim C_-(T_c - T)^{-\gamma}.$$

The ratio C_+/C_- is universal. In the same way, the singular part of the specific heat behaves like

$$\mathcal{C}_+ \sim A_+(T - T_c)^{-\alpha},$$
$$\mathcal{C}_- \sim A_-(T_c - T)^{-\alpha},$$

and the ratio A_+/A_- is also universal.

Reference: R. Guida and J. Zinn-Justin, *Nucl. Phys.* B489 [FS] (1997) 626.

Amplitude ratios: models and binary critical fluids.

	ε-expansion	Fixed dim. $d = 3$	Lattice models	Experiment
A^+/A^-	0.527 ± 0.037	0.537 ± 0.019	$\begin{cases} 0.523 \pm 0.009 \\ 0.560 \pm 0.010 \end{cases}$	0.56 ± 0.02
C^+/C^-	4.73 ± 0.16	4.79 ± 0.10	$\begin{cases} 4.75 \pm 0.03 \\ 4.95 \pm 0.15 \end{cases}$	4.3 ± 0.3
f_1^+/f_1^-	1.91	2.04 ± 0.04	1.96 ± 0.01	1.9 ± 0.2
R_ξ^+	0.28	0.270 ± 0.001	0.266 ± 0.001	0.25–0.32
R_c	0.0569 ± 0.0035	0.0574 ± 0.0020	0.0581 ± 0.0010	0.050 ± 0.015
$R_\xi^+ R_c^{-1/3}$	0.73	0.700 ± 0.014	0.650	0.60–0.80
R_χ	1.648 ± 0.036	1.669 ± 0.018	1.75	1.75 ± 0.30
Q_2	1.13		1.21 ± 0.04	1.1 ± 0.3
Q_3	0.96		0.896 ± 0.005	

However, to give an idea of the problem one faces, at seven-loop about 3500 diagrams have to be evaluated, which are integrals of rather singular functions over 21 variables.

A set of technical tricks, some already used Nickel, and a complete automatization of the calculation (Guida–Ribeca), which in particular allows finding many sub-integrations that can be performed analytically, reduces somewhat the difficulty. It remains to optimize the numerical integration methods, to secure the required precision.
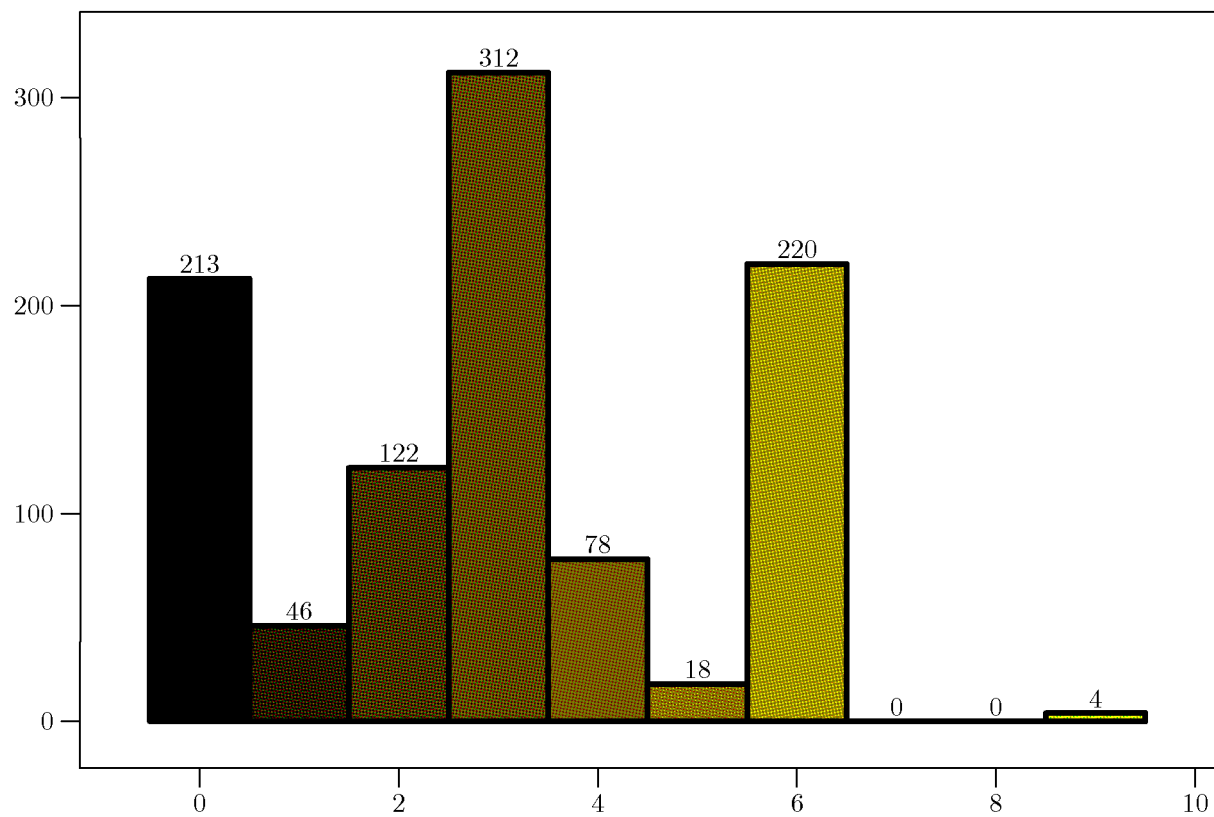
WORK STILL IN PROGRESS.

Effective vertices up to the triangle
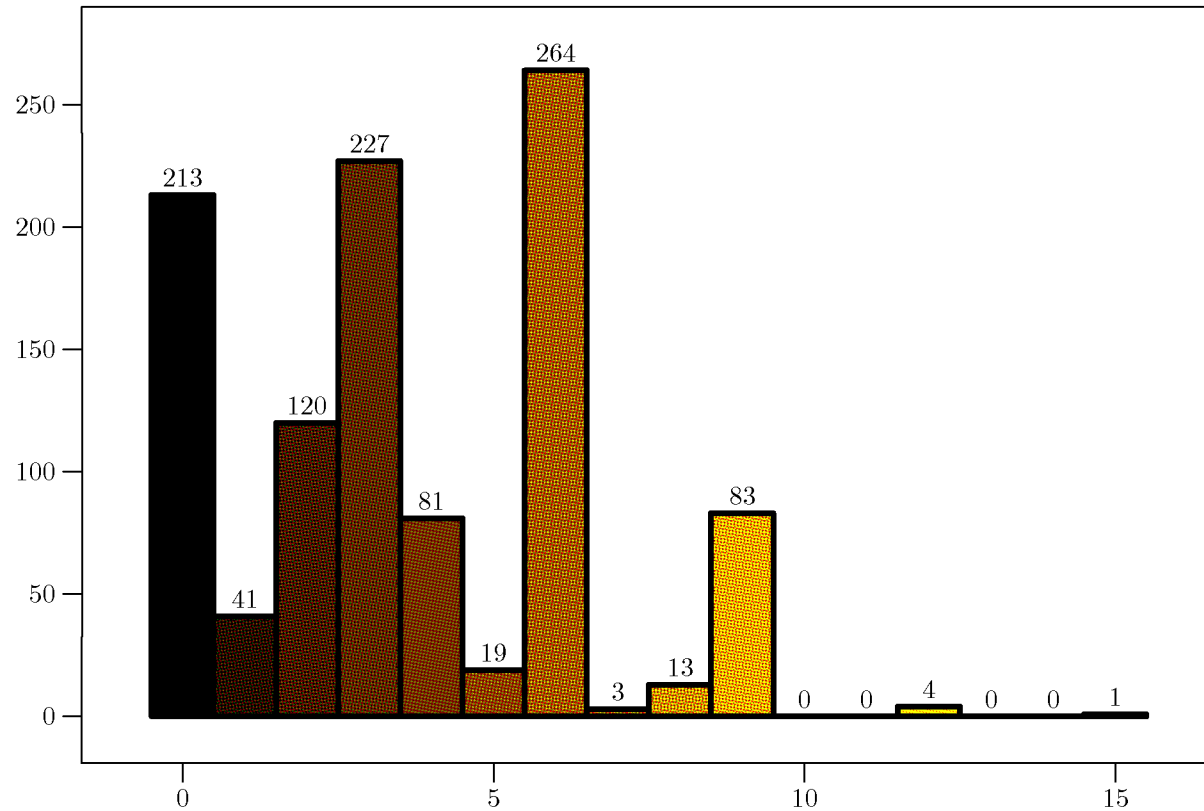
Cumulative number of diagrams vs. number of residual integrations for loops 5 and 6.

Effective vertices up to the pentagon

Cumulative number of diagrams vs. number of residual integrations for loops 5 and 6.