

# Manifestly Gauge Invariant Exact Renormalization Group

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Renormalization and Universality in Mathematical Physics,  
Fields Institute, Oct 2005

# Adapting Exact RG to gauge invariant systems

Obvious Benefits: Exact RG & Gauge invariance

Wilson's Exact RG is a powerful framework for computing in quantum field theory.

- RG invariance / continuum limit built in.
- All physical quantities can be computed.
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- Gauge invariance underlies particle physics and much more besides.

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  - recover with broken Ward identities
- incorporate a gauge invariant momentum cutoff.
  - extra regularisation structure

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Surprise benefits:

Use freedom to design the ERG:

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- no wavefunction renormalization for connection
- simple tightly constrained expressions

# $SU(N)$ Yang-Mills

without fixing the gauge . . .

$$D_\mu = \partial_\mu - iA_\mu, \quad F_{\mu\nu} = i[D_\mu, D_\nu], \quad A_\mu \equiv A_\mu^a \tau^a$$

$$S[A](g) = \frac{1}{2g^2} \text{tr} \int F_{\mu\nu}^2 + \text{higher dim}^n \text{ ops} + \text{vacuum energy}$$

- Renormalization condition
- Preserve exactly  $\delta A_\mu = [D_\mu, \omega]$  invariance
- $\Rightarrow$  no wavefunction renormalization
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  - with solution - see Olly Rosten
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# Generalised Exact RGs

Infinitely many Kadanoff blockings

Define in standard fashion:

$$e^{-S[\varphi]} = \int \mathcal{D}\varphi_0 \delta\left[\varphi - b[\varphi_0]\right] e^{-S_{bare}[\varphi_0]}$$

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e.g.  $b_x[\varphi_0] = \int_y K(x-y)\varphi_0(y)$ , for some kernel  $K(z)$  which is steeply decaying once  $|z| > 1$ .

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⇒ equality of microscopic and blocked partition functions:

$$\mathcal{Z} = \int \mathcal{D}\varphi e^{-S[\varphi]} = \int \mathcal{D}\varphi_0 e^{-S_{bare}[\varphi_0]}.$$

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⇒ infinitely many ERGs parametrised by  $\Psi$

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By construction  $\mathcal{Z} = \int \mathcal{D}\varphi e^{-S[\varphi]}$  is invariant

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Polchinski's ERG for a single scalar field

$$\Psi_x = \frac{1}{2} \int_y \dot{\Delta}_{xy} \frac{\delta \Sigma_1}{\delta \varphi(y)}$$

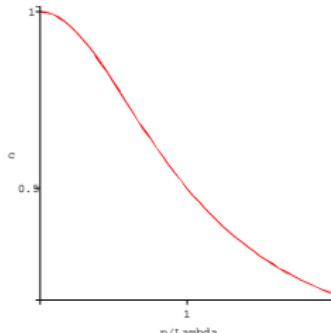
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ERG kernel.  $\dot{\Delta} \equiv -\Lambda \partial_\Lambda \Delta$ .

$\Delta = \frac{c(p^2/\Lambda^2)}{p^2}$  is effective propagator.  $c$  is the effective cutoff.



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$\Sigma_1 = S - 2\hat{S}$ . seed action:  $\hat{S} = \frac{1}{2} \partial_\mu \varphi \cdot c^{-1} \cdot \partial_\mu \varphi$

N.B. for any kernel

$$f \cdot W \cdot g = \int_{x,y} f(x) W_{xy} g(y) = \int_x f(x) W(-\partial^2/\Lambda^2) g(x)$$

$$W_{xy} = W(-\partial^2/\Lambda^2) \delta(x-y) = \int_p W(p^2/\Lambda^2) e^{ip \cdot (x-y)}.$$

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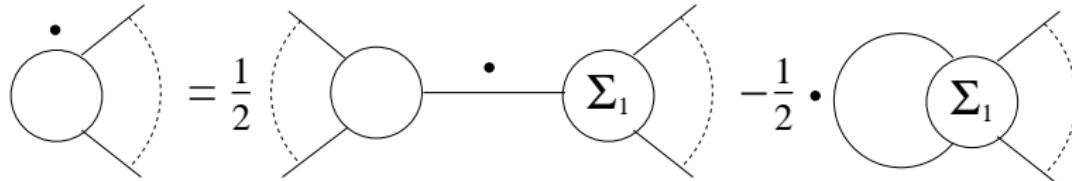
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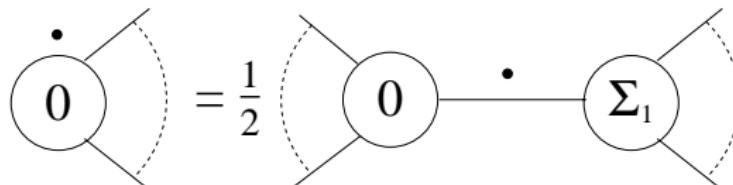
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Classical level

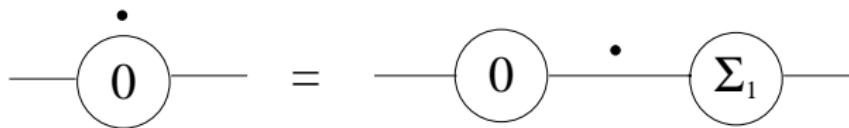
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$$S_0^{\varphi\varphi}(p) = \hat{S}^{\varphi\varphi}(p) \implies \dot{S}_0^{\varphi\varphi} = -S_0^{\varphi\varphi} \dot{\Delta} S_0^{\varphi\varphi} \implies \Delta = (S_0^{\varphi\varphi})^{-1}.$$

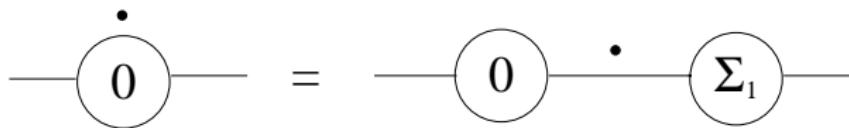
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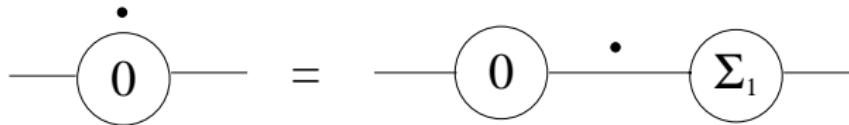
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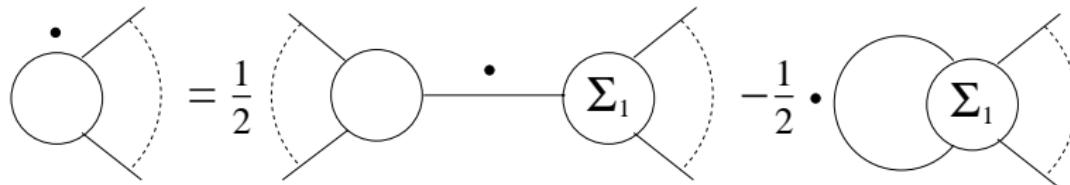
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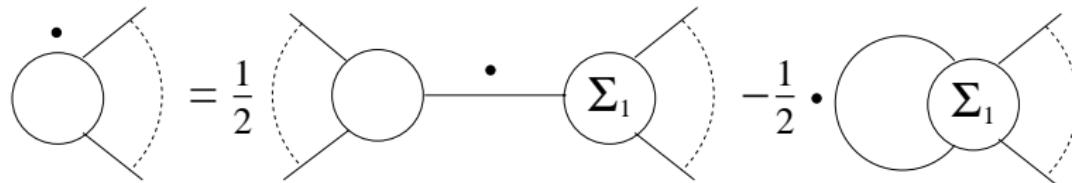
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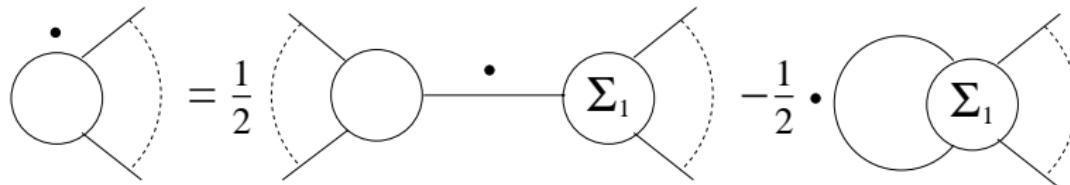
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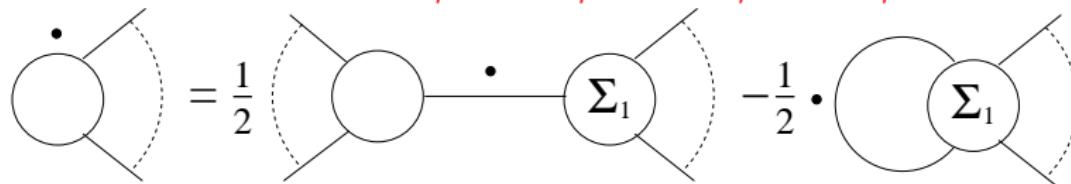
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# Manifestly gauge invariant ERG for $SU(N)$ YM

Generalising from scalar case ...

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E.g.

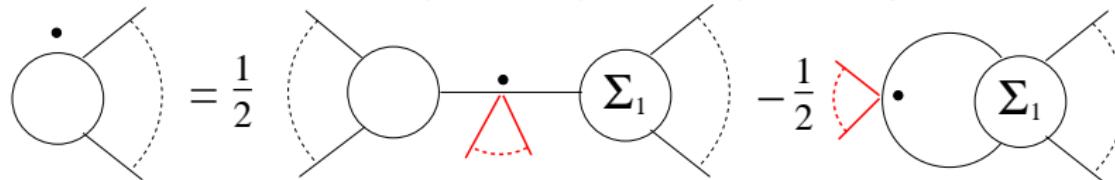
$$\dot{\Delta}(-\partial^2/\Lambda^2) \mapsto \dot{\Delta}(-D^2/\Lambda^2)$$

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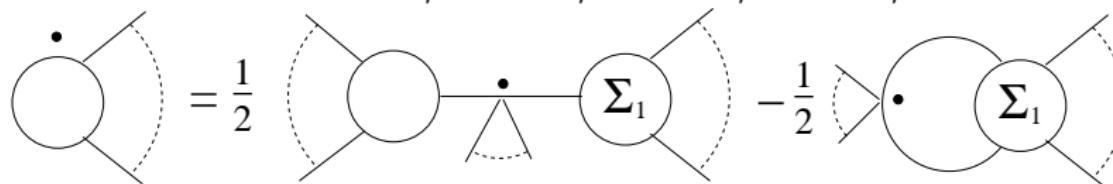


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$$S = \frac{1}{g^2} S_0 + S_1 + g^2 S_2 + \dots$$

$$\beta = \beta_1 g^3 + \beta_2 g^5 + \dots$$

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Gauge invariance  $\Rightarrow S_{0\mu\nu}^{AA} = 2\Box_{\mu\nu}(p)/c(p^2/\Lambda^2)$ .

$$\Box_{\mu\nu}(p) = \delta_{\mu\nu} p^2 - p_\mu p_\nu$$

$$S_{0\mu\nu}^{AA} = \hat{S}_{\mu\nu}^{AA} \Rightarrow \dot{S}_{0\mu\nu}^{AA} = -S_{0\mu\alpha}^{AA} \dot{\Delta} S_{0\alpha\nu}^{AA} \Rightarrow \Delta = c/2p^2$$

$$\Delta S_{0\mu\nu}^{AA} = \delta_{\mu\nu} - p_\mu p_\nu / p^2 \quad (\text{generates gauge transformations})$$

# Manifestly gauge invariant ERG for $SU(N)$ YM

Generalising from scalar case ...

$$\Psi = \frac{1}{2} \{\dot{\Delta}\} \frac{\delta \Sigma_g}{\delta A_\mu}. \quad \Sigma_g = g^2 S - 2\hat{S}. \quad \hat{S} = \dots$$

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$$S = \frac{1}{g^2} S_0 + S_1 + g^2 S_2 + \dots$$

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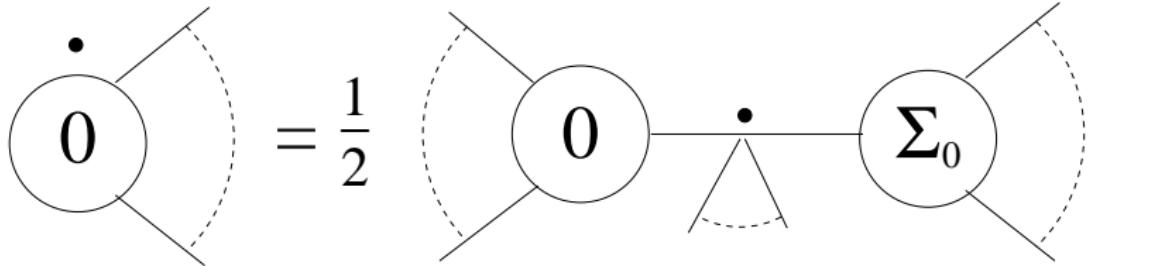
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Classical level

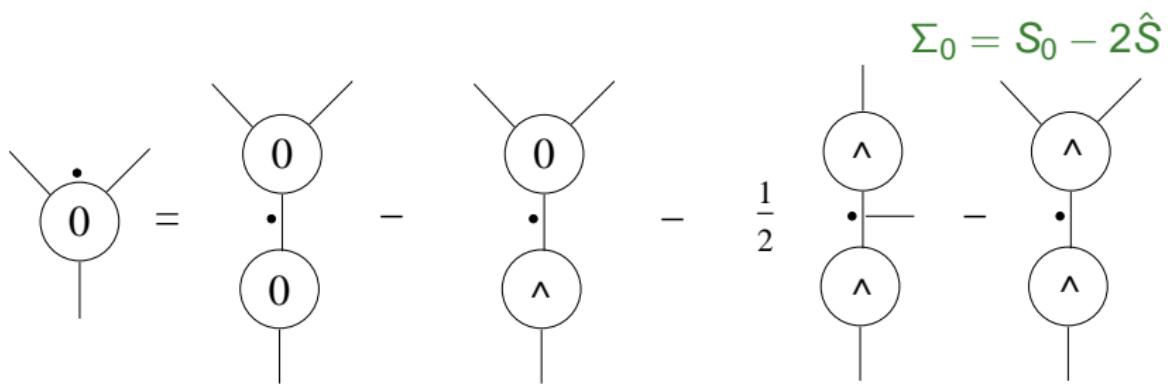
$$\dot{S}_0 = \frac{1}{2} \frac{\delta S_0}{\delta A_\mu} \{\dot{\Delta}\} \frac{\delta \Sigma_0}{\delta A_\mu}$$



# Manifestly gauge invariant ERG for $SU(N)$ YM

Classical level

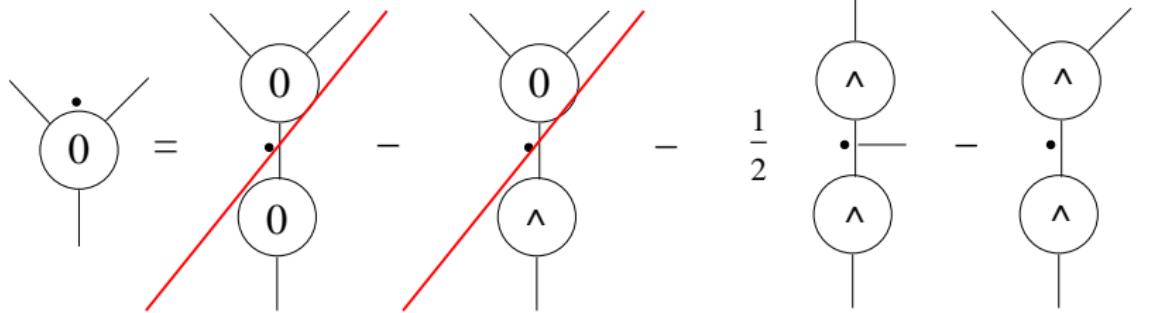
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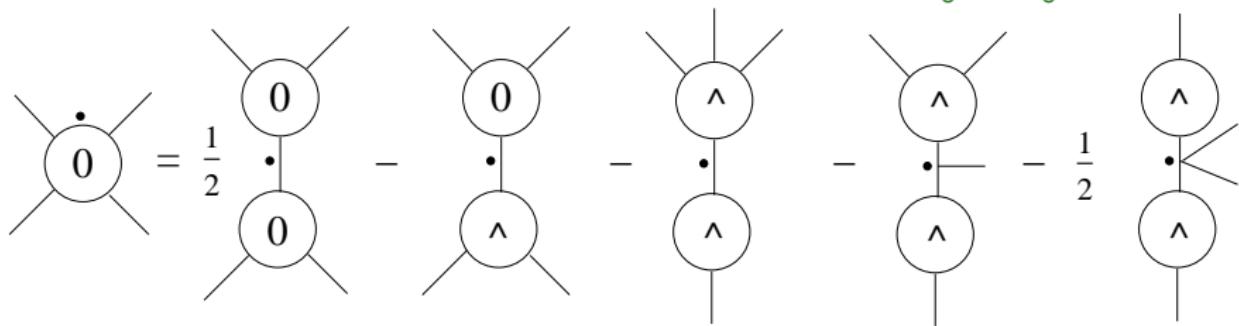


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$$\Sigma_0 = S_0 - 2\hat{S}$$



# $SU(N)$ Yang-Mills

without fixing the gauge . . .

$$D_\mu = \partial_\mu - iA_\mu, \quad F_{\mu\nu} = i[D_\mu, D_\nu], \quad A_\mu \equiv A_\mu^a \tau^a$$

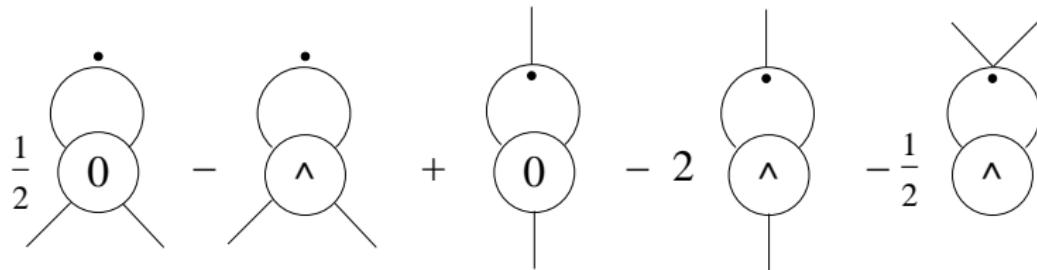
$$S[A](g) = \frac{1}{2g^2} \text{tr} \int F_{\mu\nu}^2 + \text{higher dim}^n \text{ ops} + \text{vacuum energy}$$

- Renormalization condition
- Preserve exactly  $\delta A_\mu = [D_\mu, \omega]$  invariance
- $\Rightarrow$  no wavefunction renormalization
- so there really is only  $g$  that runs!

# Manifestly gauge invariant ERG for $SU(N)$ YM

One loop

$$-4\beta_1 \square_{\mu\nu}(p) + O(p^4) = \frac{1}{2} \frac{\delta}{\delta A_\alpha} \{\dot{\Delta}\} \left. \frac{\delta \Sigma_0}{\delta A_\alpha} \right|_{\mu\nu}^{AA}$$

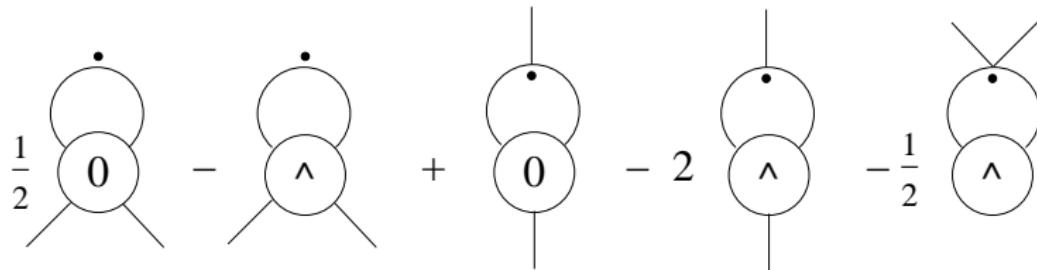


plus extra diags if  $\hat{S}$  has one-loop terms

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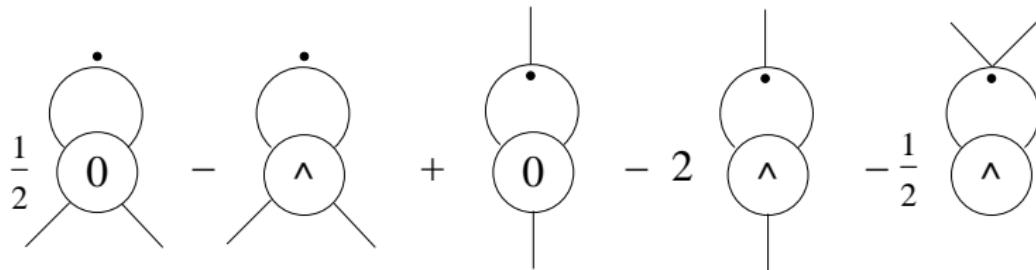


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- But this  $\sim$  covariant higher derivatives which fails at one loop!
- $\Rightarrow$  Need gauge invariant real cutoff  $\wedge$  which can naturally be incorporated in effective action framework.

# $SU(N|N)$ regularisation

- Embed  $SU(N)$  YM in  $SU(N|N)$

$$\mathcal{A}_\mu = \begin{pmatrix} A_\mu^1 & B_\mu \\ \bar{B}_\mu & A_\mu^2 \end{pmatrix} + \mathcal{A}_\mu^0 \mathbb{1}$$

Bosonic subgroup  $SU_1(N) \otimes SU_2(N) \otimes U(1)$

# $SU(N|N)$ regularisation

- Embed  $SU(N)$  YM in  $SU(N|N)$

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centre of superalgebra

e.g.

$$\left\{ \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \right\} = 2\mathbb{1}$$

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- Embed  $SU(N)$  YM in  $SU(N|N)/U(1)$

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- Nothing depends on centre  $\Leftrightarrow$  no- $\mathcal{A}^0$  symm:  $\delta \mathcal{A}_\mu^0 = \lambda_\mu(x)$

$$\nabla_\mu = \partial_\mu - i \mathcal{A}_\mu$$

$$S \sim \frac{1}{2g^2} \text{str} \int \mathcal{F}_{\mu\nu}^2 + \dots$$

$$\text{str} \begin{pmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{pmatrix} = \text{tr}_1 X^{11} - \text{tr}_2 X^{22}$$

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- Quadratic Casimir vanishes.  $\text{tr} \mathbb{1} = N \rightarrow \text{str} \mathbb{1} = 0$ .  
One loop finite & no quantum corr<sup>ns</sup> at all in large  $N$  limit.

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- $B_\mu$  fermionic violates spin-statistics.
- Wrong sign kinetic term for  $A^2 \Rightarrow$  -ve norm states.

$$\text{str} \begin{pmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{pmatrix} = \text{tr}_1 X^{11} - \text{tr}_2 X^{22}$$

$$SU(N|N) \rightarrow SU_1(N) \otimes SU_2(N)$$

$$\mathcal{C} = \begin{pmatrix} C_1 & D \\ \bar{D} & C_2 \end{pmatrix}$$

Choose potential so  $\langle \mathcal{C} \rangle = \Lambda \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ .

Spontaneously breaks all and only fermionic directions.  
Gives masses to  $B$ s by eating  $D$ s.

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$$\frac{1}{2} [\nabla_\mu, \mathcal{C}]^2 \sim -2\Lambda^2 \begin{pmatrix} 0 & -B_\mu \\ \bar{B}_\mu & 0 \end{pmatrix}^2$$

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Lowest dim<sup>n</sup> int<sup>n</sup> between the two low energy sectors:

$$\frac{1}{\Lambda^4} \text{tr}_1 \left( F_{\mu\nu}^1 \right)^2 \text{tr}_2 \left( F_{\mu\nu}^2 \right)^2$$

Appelquist-Carazzone  $\implies$  decouples.

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▶ + covariantized cutoff f<sup>n</sup>s  $\implies$  all finite

# Manifestly $SU(N|N)$ gauge invariant ERG

Putting the two parts together ...

$$\dot{S} = \frac{1}{2} \frac{\delta S}{\delta A_\mu} \{ \dot{\Delta}^A \} \frac{\delta \Sigma_g}{\delta A_\mu} + \frac{1}{2} \frac{\delta S}{\delta C} \{ \dot{\Delta}^C \} \frac{\delta \Sigma_g}{\delta C} + \text{quantum}$$

Requiring invariance under

$$\delta C = -i[C, \Omega],$$

$$\delta A_\mu = [\nabla_\mu, \Omega] + \lambda_\mu \mathbb{1},$$

$$\delta \left( \frac{\delta}{\delta A_\mu} \right) = -i \left[ \frac{\delta}{\delta A_\mu}, \Omega \right] + \tilde{\lambda}_\mu \mathbb{1}$$

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$$+ \frac{1}{2} [C, \frac{\delta S}{\delta A_\mu}] \{\dot{\Delta}_m^A\} [C, \frac{\delta \Sigma_g}{\delta A_\mu}] + \frac{1}{2} [C, \frac{\delta S}{\delta C}] \{\dot{\Delta}_m^A\} [C, \frac{\delta \Sigma_g}{\delta C}] + \text{quantum}$$

if we are to keep  $\hat{S}^{(2)} = S_0^{(2)}$  in spontaneously broken phase

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$$+ \frac{1}{2} [\mathcal{C}, \frac{\delta S}{\delta \mathcal{A}_\mu}] \{ \dot{\Delta}_m^{\mathcal{A}} \} [\mathcal{C}, \frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu}] + \frac{1}{2} [\mathcal{C}, \frac{\delta S}{\delta \mathcal{C}}] \{ \dot{\Delta}_m^{\mathcal{A}} \} [\mathcal{C}, \frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu}]$$

$$+ \frac{1}{2} \frac{\delta S}{\delta \mathcal{A}_\mu} \{ \dot{\Delta}_\sigma^{\mathcal{A}} \} \left( \{ \mathcal{C}, \frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu} \} \text{str} \mathcal{C} - 2 \mathcal{C} \text{str} \left\{ \mathcal{C} \frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu} \right\} \right) + \text{quantum}$$

if we are to keep  $\hat{S}^{(2)} = S_0^{(2)}$  when  $g_2(\Lambda)$  runs

# Manifestly $SU(N|N)$ gauge invariant ERG

## Kernels

- Specify classical two-point vertices consistent with broken supergauge invariance & renormal<sup>n</sup> cond<sup>ns</sup>.
- Kernels follow from insisting  $\hat{S}^{(2)} = S_0^{(2)}$

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$$\dot{S}_0{}^{A_1 A_1}_{\mu \nu} = - S_0{}^{A_1 A_1}_{\mu \alpha} \dot{\Delta}^1 S_0{}^{A_1 A_1}_{\alpha \nu} \quad \dot{S}_0{}^{A_2 A_2}_{\mu \nu} = - S_0{}^{A_2 A_2}_{\mu \alpha} \dot{\Delta}^2 S_0{}^{A_2 A_2}_{\alpha \nu}$$

$$\dot{\Delta}^1 = \Delta^A + 8N\Delta_\sigma^A \quad \dot{\Delta}^2 = \Delta^A - 8N\Delta_\sigma^A$$

Similarly all rest are linear combinations ...

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$$\dot{S}_0^{CC} = -S_0^{CC} \dot{\Delta}^C S_0^{CC}$$

$$\begin{pmatrix} \dot{S}_0{}^{\bar{B}\bar{B}}_{\mu\nu} & -\dot{S}_0{}^{\bar{B}\bar{D}}_{\mu} \\ \dot{S}_0{}^{\bar{B}\bar{D}}_{\nu} & -\dot{S}_0{}^{\bar{D}\bar{D}}_{\mu} \end{pmatrix} = - \begin{pmatrix} S_0{}^{\bar{B}\bar{B}}_{\mu\alpha} & -S_0{}^{\bar{B}\bar{D}}_{\mu} \\ S_0{}^{\bar{B}\bar{D}}_{\alpha} & -S_0{}^{\bar{D}\bar{D}}_{\mu} \end{pmatrix} \begin{pmatrix} \dot{\Delta}^B \delta_{\alpha\beta} & 0 \\ 0 & -\dot{\Delta}^D \end{pmatrix} \begin{pmatrix} S_0{}^{\bar{B}\bar{B}}_{\beta\nu} & -S_0{}^{\bar{B}\bar{D}}_{\beta} \\ S_0{}^{\bar{B}\bar{D}}_{\nu} & -S_0{}^{\bar{D}\bar{D}}_{\beta} \end{pmatrix}$$

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$$\dot{S}_0^{CC} = -S_0^{CC} \dot{\Delta}^C S_0^{CC}$$

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# Manifestly $SU(N|N)$ gauge invariant ERG

## Effective propagator relations

$$S_0{}^{\mu\nu}{}^{A_1 A_1} \Delta^1 = \delta_{\mu\nu} - p_\mu p_\nu / p^2$$

$$S_0{}^{\mu\nu}{}^{A_2 A_2} \Delta^2 = \delta_{\mu\nu} - p_\mu p_\nu / p^2$$

generates gauge transformations

# Manifestly $SU(N|N)$ gauge invariant ERG

## Effective propagator relations

$$S_0{}^{\mu\nu}{}^{A_1 A_1} \Delta^1 = \delta_{\mu\nu} - p_\mu p_\nu / p^2$$

$$S_0{}^{\mu\nu}{}^{A_2 A_2} \Delta^2 = \delta_{\mu\nu} - p_\mu p_\nu / p^2$$

$$S_0^{CC} \Delta^C = 1$$

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generates broken supergauge transformations

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Effective propagator relations

$$S_0{}^{\bar{A}_1 A_1}_{\mu\nu} \Delta^1 = \delta_{\mu\nu} - \mathbf{p}_\mu p_\nu / p^2$$

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$$= 1 - \blacktriangleright$$

will simplify using Ward ids

# 'Naïve' Ward Identities

*SU(N) gauge theory*

$$S = \dots + \int S_{\mu_1 \mu_2 \dots \mu_n}(p_1, p_2, \dots, p_n) \operatorname{tr} A_{\mu_1}(p_1) A_{\mu_2}(p_2) \dots A_{\mu_n}(p_n) + \dots$$

$$p_1^{\mu_1} S_{\mu_1 \mu_2 \dots \mu_n}(p_1, p_2, \dots, p_n) =$$

$$S_{\mu_2 \dots \mu_n}(p_1 + p_2, p_3, \dots, p_n) - S_{\mu_2 \dots \mu_n}(p_2, p_3, \dots, p_n + p_1)$$

# 'Naïve' Ward Identities

$SU(N)$  gauge theory

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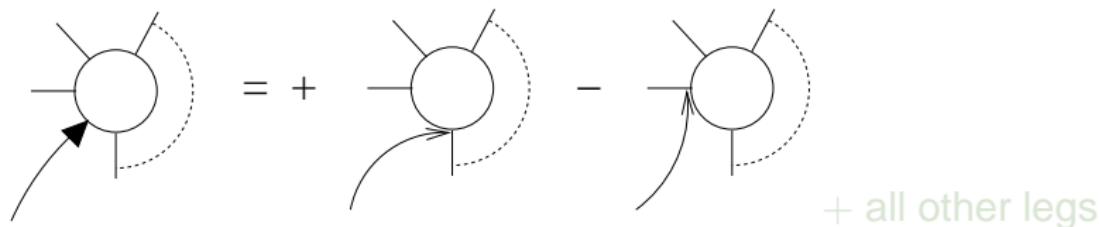
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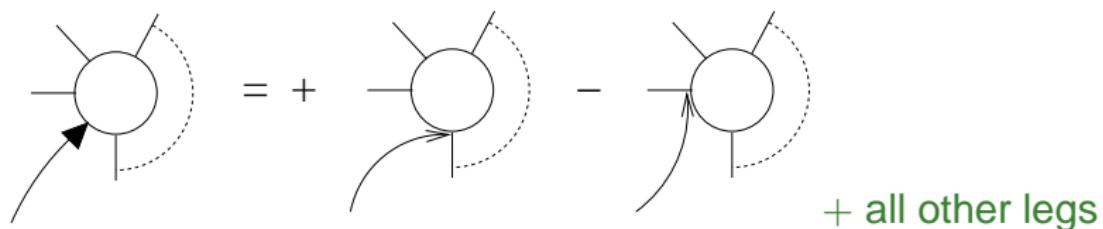
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# Diagrammatic method

## Flow equation

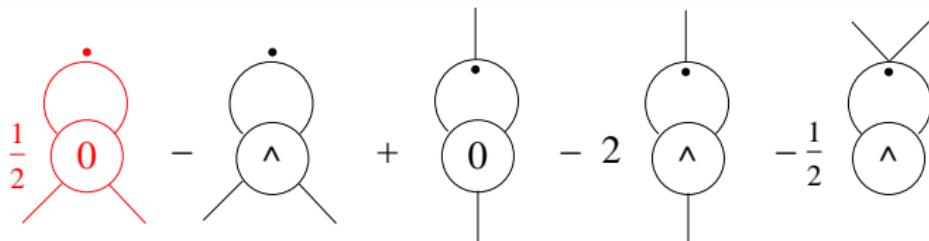
$$\text{Diagrammatic term} = \frac{1}{2} \left( \text{Diagram 1} + \text{Diagram 2} \right)$$

The diagram shows a flow equation. On the left is a circular vertex with a dot inside and a dashed loop around it. An equals sign follows. To the right of the equals sign is a fraction  $\frac{1}{2}$ . After the fraction is a plus sign, followed by a sum of two diagrams. The first diagram in the sum is a circle with a dot inside and a dashed loop around it. The second diagram is a circle with a dot inside and a dashed loop around it, with a red circle overlaid on it. A red line connects the two circles. Below the first circle is a triangle.

- Sum over all intermediate flavours

# Diagrammatic method

One loop

$$\frac{1}{2} \text{ (red circle with 0)} - \text{ (white circle with \wedge)} + \text{ (white circle with 0)} - 2 \text{ (white circle with \wedge)} - \frac{1}{2} \text{ (white circle with \wedge)}$$


# Diagrammatic method

One loop

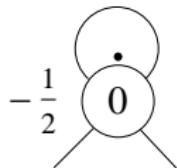
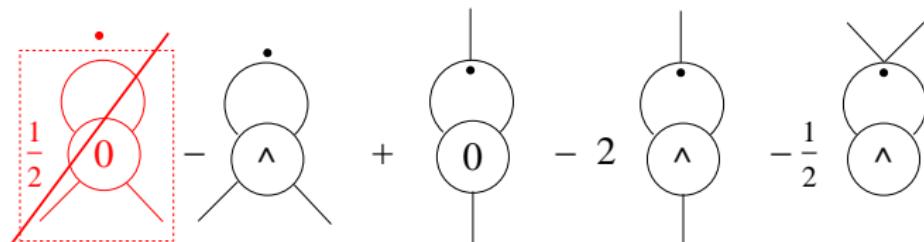
$$\frac{1}{2} \begin{array}{c} \bullet \\ \boxed{\text{0}} \end{array} - \begin{array}{c} \bullet \\ \wedge \end{array} + \begin{array}{c} \bullet \\ 0 \end{array} - 2 \begin{array}{c} \bullet \\ \wedge \end{array} - \frac{1}{2} \begin{array}{c} \bullet \\ \wedge \end{array}$$

$$-\frac{1}{2} \begin{array}{c} \bullet \\ 0 \end{array}$$

integrate by  
parts

# Diagrammatic method

One loop



dim<sup>n</sup>less  
UV & IR  
finite!

# Diagrammatic method

One loop

$$-\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \textcircled{\wedge} \end{array} + \begin{array}{c} \bullet \\ | \\ \textcircled{0} \end{array} - 2 \begin{array}{c} \bullet \\ | \\ \textcircled{\wedge} \end{array} - \frac{1}{2} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \textcircled{\wedge} \end{array}$$

$$-\frac{1}{2} \begin{array}{c} \bullet \\ | \\ \textcircled{0} \end{array}$$

expand  
using flow  
eqn

$$\begin{array}{c} \bullet \\ | \\ \textcircled{0} \end{array} = \frac{1}{2} \begin{array}{c} \bullet \\ | \\ \textcircled{0} \end{array} - \begin{array}{c} \bullet \\ | \\ \textcircled{0} \end{array} - \begin{array}{c} \bullet \\ | \\ \textcircled{\wedge} \end{array} - \begin{array}{c} \bullet \\ | \\ \textcircled{\wedge} \end{array} - \frac{1}{2} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \textcircled{\wedge} \end{array}$$

# Diagrammatic method

One loop

$$-\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \textcircled{\wedge} \end{array} + \begin{array}{c} \bullet \\ \mid \quad \mid \\ \textcircled{0} \end{array} - 2 \begin{array}{c} \bullet \\ \mid \quad \mid \\ \textcircled{\wedge} \end{array} - \frac{1}{2} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \textcircled{\wedge} \end{array}$$

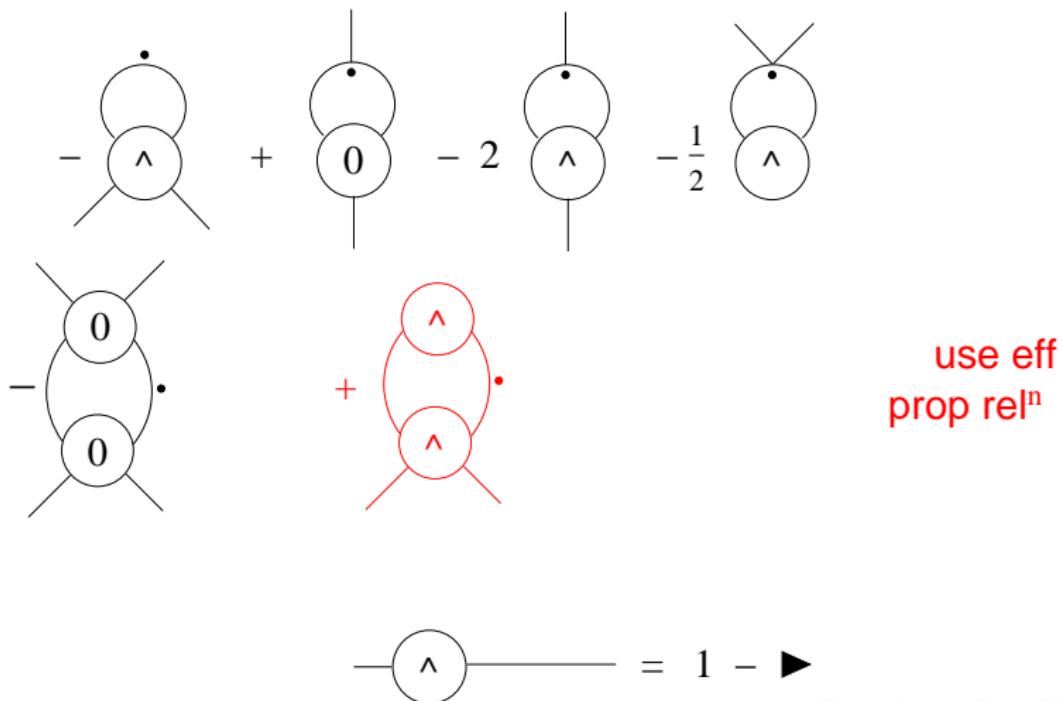
$$-\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \textcircled{0} \\ \mid \quad \mid \\ \textcircled{0} \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \textcircled{\wedge} \\ \mid \quad \mid \\ \textcircled{\wedge} \end{array}$$

$$\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \textcircled{0} \\ \mid \quad \mid \\ \textcircled{0} \end{array} = \frac{1}{2} \begin{array}{c} \bullet \\ \mid \quad \mid \\ \textcircled{0} \\ \mid \quad \mid \\ \textcircled{0} \end{array} - \begin{array}{c} \bullet \\ \mid \quad \mid \\ \textcircled{0} \end{array} - \begin{array}{c} \bullet \\ \mid \quad \mid \\ \textcircled{\wedge} \end{array} - \begin{array}{c} \bullet \\ \mid \quad \mid \\ \textcircled{\wedge} \end{array} - \begin{array}{c} \bullet \\ \mid \quad \mid \\ \textcircled{\wedge} \end{array} - \frac{1}{2} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \textcircled{\wedge} \end{array}$$

examples

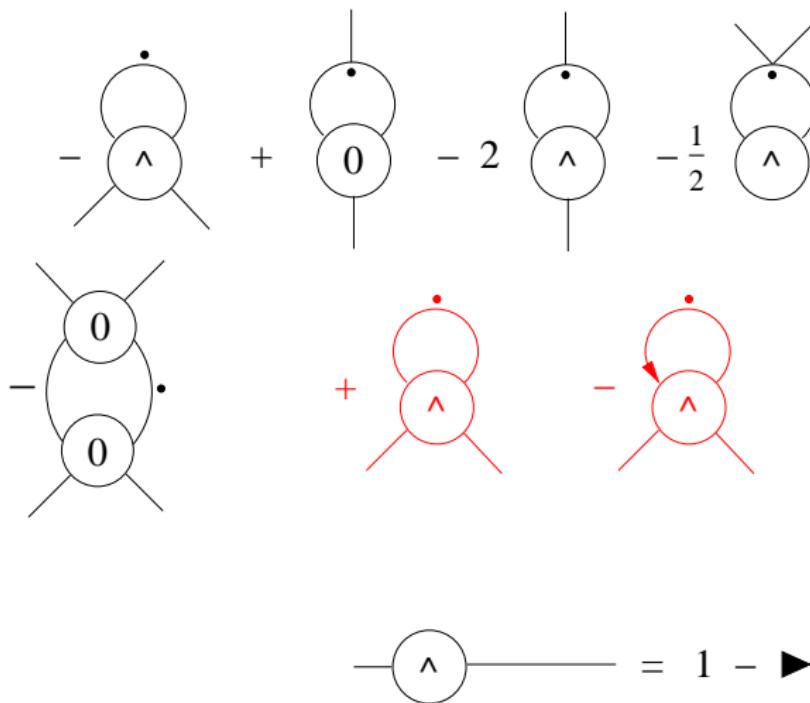
# Diagrammatic method

One loop



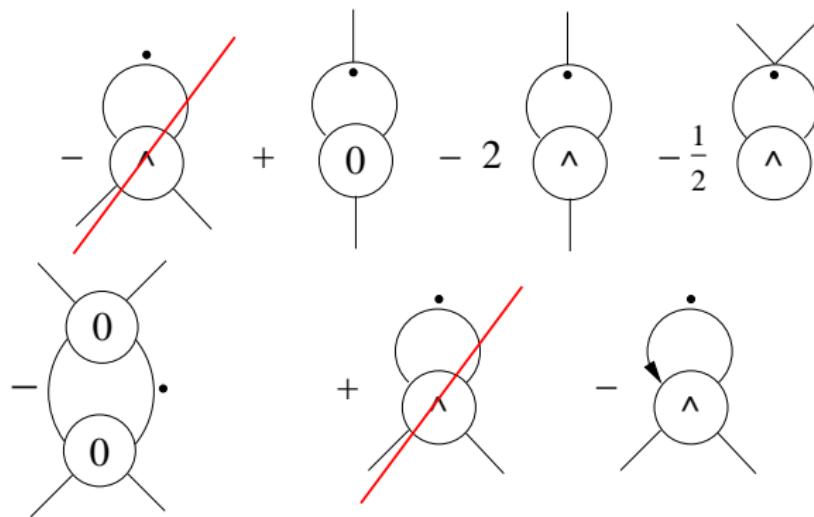
# Diagrammatic method

One loop



# Diagrammatic method

One loop



# Diagrammatic method

One loop

$$+ \begin{array}{c} \bullet \\ \circ \\ 0 \end{array} - 2 \begin{array}{c} \bullet \\ \circ \\ \wedge \end{array} - \frac{1}{2} \begin{array}{c} \bullet \\ \circ \\ \wedge \end{array}$$

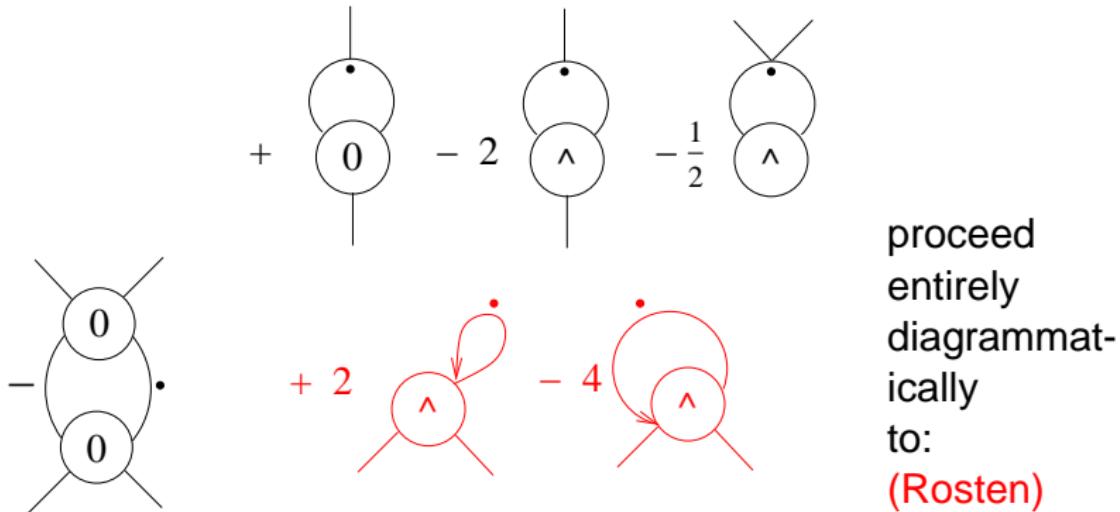
$$- \begin{array}{c} \bullet \\ \circ \\ 0 \\ \circ \\ 0 \end{array}$$

$$- \begin{array}{c} \bullet \\ \circ \\ \wedge \end{array}$$

Use  
Ward  
identities

# Diagrammatic method

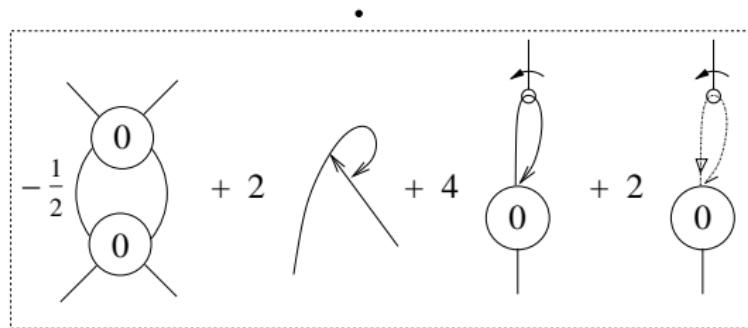
One loop



# Diagrammatic method

## Total derivative terms

$$-4\beta_1 \square_{\mu\nu}(p) + O(p^4) = \frac{1}{2} \frac{\delta}{\delta A_\alpha} \{\dot{\Delta}\} \left. \frac{\delta \Sigma_0}{\delta A_\alpha} \right|_{\mu\nu}^{AA}$$

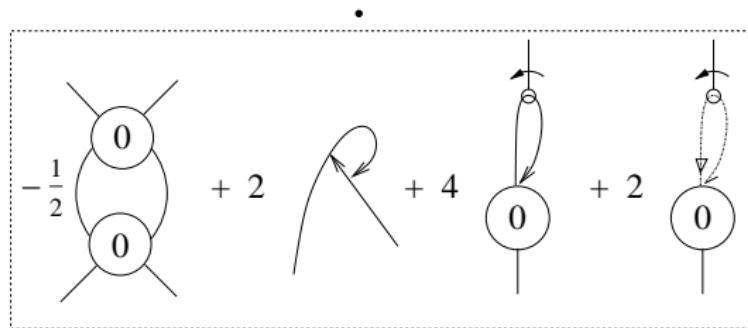


- depends only on far IR where elements determined by renorm<sup>ln</sup> cond<sup>n</sup>

# Diagrammatic method

Total derivative terms

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$$= \frac{44N}{3(4\pi)^2} \square_{\mu\nu}(p) + O(p^4)$$

# Diagrammatic method

Two loops

Two loop  $\beta$  function confirmed similarly.

Subtractions needed to isolate universal parts (Rosten).

# Incorporating massless quarks

- Cannot use fundamental repn  $SU(N|N)$  because  $\mathcal{A}^0$  couples
- Use block off-diagonal elements  $\Psi = \begin{pmatrix} \varphi^1 & \psi \\ \bar{\psi} & \varphi^2 \end{pmatrix}$
- $SU_1(3) \otimes SU_2(3) \equiv$  gauged family symmetry
- send  $g_2 \rightarrow 0$  at end of calculation
- Works for  $2 \times 3$  quarks.

# Renormalised strong coupling expansion

- Assume  $g(\Lambda) \rightarrow \infty$  as  $\Lambda \rightarrow 0$
- $\implies S_{\mu\nu}^{AA}(p) \sim O(p^4)$ : signal of confinement
- Assume  $S(g)$  analytic in  $1/g^2$  in strong coupling regime:

$$S(g) = \tilde{S}_0 + \frac{1}{g^2} \tilde{S}_1 + \frac{1}{g^4} \tilde{S}_2 + \dots$$

- Dimensional transmutation to expansion in  $p^2/\Lambda_{QCD}^2$

# $SU(N)$ Yang-Mills

without fixing the gauge . . .

$$D_\mu = \partial_\mu - iA_\mu, \quad F_{\mu\nu} = i[D_\mu, D_\nu], \quad A_\mu \equiv A_\mu^a \tau^a$$

$$S[A](g) = \frac{1}{2g^2} \text{tr} \int F_{\mu\nu}^2 + \text{higher dim}^n \text{ ops} + \text{vacuum energy}$$

- Renormalization condition
- Preserve exactly  $\delta A_\mu = [D_\mu, \omega]$  invariance
- $\Rightarrow$  no wavefunction renormalization
- so there really is only  $g$  that runs!

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# Summary

- Manifestly gauge invariant ERG
- Compute without gauge fixing
- Incorporate real gauge invariant cutoff  $\Lambda$
- Compute without specifying details of regularisation
- One-loop & two-loop  $\beta$  function coeffs confirmed
- Quarks, strong coupling expansion
- Non-perturbative approximations, expectation value of Wilson loops etc.

# Proof that $A_\mu$ is not renormalized.

$\delta A_\mu = \partial_\mu \omega - i[A_\mu, \omega]$  is an exact symmetry

$$A_\mu = Z^{1/2} A_\mu^R$$

$\implies$

$$\delta A_\mu^R = Z^{-1/2} \partial_\mu \omega - i[A_\mu^R, \omega]$$

$\implies$

$$Z = 1 \quad \& \quad A_\mu^R = A_\mu$$



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⇒

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# Statement of finiteness

If:

$$\begin{aligned} S = \text{str} \int \frac{1}{2g^2} \mathcal{F}_{\mu\nu} c^{-1} \left( -\frac{\nabla^2}{\Lambda^2} \right) \mathcal{F}_{\mu\nu} + \frac{1}{2} \nabla_\mu \mathcal{C} \tilde{c}^{-1} \left( -\frac{\nabla^2}{\Lambda^2} \right) \nabla_\mu \mathcal{C} \\ + \frac{\lambda}{4} (c^2 - \Lambda^2)^2, \end{aligned}$$

where  $c^{-1}$  is a polynomial of rank  $r$  and  $\tilde{c}^{-1}$  is a polynomial of rank  $\tilde{r}$ , and with  $\mathcal{C} \rightarrow \mathcal{C} + \Lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , then all amplitudes are finite to all orders in perturbation theory providing  $r > \tilde{r} + 1$  &  $\tilde{r} > 1$ . □

# Sketch of proof of finiteness

- Gauge fix to 't Hooft gauge introducing  $\hat{c}^{-1}$  (rank  $\hat{r}$ ) for ghosts and gauge fixing term
- $\hat{r} \geq r > \tilde{r} - 1$  &  $\tilde{r} > -1$  ensure cross-over to symmetric regime at high mom
- Power counting:  $\hat{r} \geq r$ ,  $r - \hat{r} > 1$  &  $\hat{r} > 1 \implies$  all diags superficially finite except "remainder contrib<sup>s</sup>": symmetric phase parts of one-loop graphs with no external  $\mathcal{C}$  or ghost lines and up to 4 external  $\mathcal{A}$ s
- 2-pt  $\mathcal{A}$  and 3- $\mathcal{A}$  pt remainder diags vanish by superalgebra viz. contribs  $\sim \text{str}\mathcal{A} \text{ str}\mathcal{A}$  or  $\text{str}\mathcal{A} \mathcal{A} \text{ str}\mathbb{1}$ ,  $\text{str}\mathcal{A} \mathcal{A} \text{ str}\mathcal{A}$  or  $\text{str}\mathcal{A}^3 \text{ str}\mathbb{1}$ .
- 4- $\mathcal{A}$  pt: finite by Lee-Zinn-Justin identities. N.B. requires a 'pre-regl<sup>n</sup>' ( $\dim^n \neq 4$ ) to define conditionally convergent integrals.

# How $g_2$ runs

Classically:

$$S_0 = \frac{1}{2g^2} \text{str} \int \begin{pmatrix} (F_{\mu\nu}^1)^2 & 0 \\ 0 & (F_{\mu\nu}^2)^2 \end{pmatrix} + \dots$$

One-loop correction:

$$-\frac{\beta_1}{g^2} \ln \frac{\Lambda}{\mu} \text{str} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} (F_{\mu\nu}^1)^2 & 0 \\ 0 & (F_{\mu\nu}^2)^2 \end{pmatrix}$$

$$\implies g_2 \neq g$$

Arises from  $\sim \text{str}\mathcal{C} \text{str}\mathcal{C} \mathcal{F}_{\mu\nu}^2 - (\text{str}\mathcal{C} \mathcal{F}_{\mu\nu})^2 |_{\mathcal{C} \rightarrow \mathcal{C} + \Lambda \sigma_3}$

Invariant under no- $\mathcal{A}^0$   $\delta\mathcal{A}_\mu = \lambda_\mu \mathbb{1}$ :  $\delta\mathcal{F}_{\mu\nu} = (\partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu) \mathbb{1}$  □

# Classical two-point vertices

$$S_{0\mu\nu}^{11}(p) = \frac{\alpha + 1 + c_p(\alpha - 1)}{\alpha c_p} \square_{\mu\nu}(p),$$

$$S_{0\mu\nu}^{22}(p) = \frac{\alpha + 1 + c_p(1 - \alpha)}{\alpha c_p} \square_{\mu\nu}(p),$$

$$S_{0\mu\nu}^{B\bar{B}}(p) = \frac{\alpha + 1}{\alpha c_p} \square_{\mu\nu}(p) + \frac{4\Lambda^2}{\tilde{c}_p} \delta_{\mu\nu},$$

$$S_0^{D\bar{B}}{}_\mu(p) = \frac{2\Lambda^2 p_\mu}{\tilde{c}_p},$$

$$S_0^{D\bar{D}}(p) = \frac{\Lambda^2 p^2}{\tilde{c}_p},$$

$$S_0^{C^i C^i}(p) = \frac{\Lambda^2 p^2}{\tilde{c}_p} + 2\lambda\Lambda^4. \quad \square$$



# other defns

$$\alpha = g_2^2/g_1^2$$

$$f_p = \frac{(1+\alpha)\tilde{c}_p}{(1+\alpha)\tilde{c}_p p^2/\Lambda^2 + 4\alpha c_p},$$

$$g_p = \frac{2\alpha\tilde{c}_p}{(1+\alpha)\tilde{c}_p p^2/\Lambda^2 + 4\alpha c_p},$$



# effective propagators (integrated kernels)

$$\Delta^1(p) = \frac{1}{p^2} \frac{\alpha c_p}{\alpha + 1 + c_p(\alpha - 1)},$$

$$\Delta^2(p) = \frac{1}{p^2} \frac{\alpha c_p}{\alpha + 1 + c_p(1 - \alpha)},$$

$$\Delta^B(p) = \frac{1}{2\Lambda^2} \tilde{c}_p g_p,$$

$$\Delta^D(p) = \frac{1}{\Lambda^4} \tilde{c}_p f_p,$$

$$\Delta^C(p) = \frac{1}{\Lambda^4} \frac{\tilde{c}_p}{p^2/\Lambda^2 + 2\lambda \tilde{c}_p}.$$