

# ENTANGLEMENT

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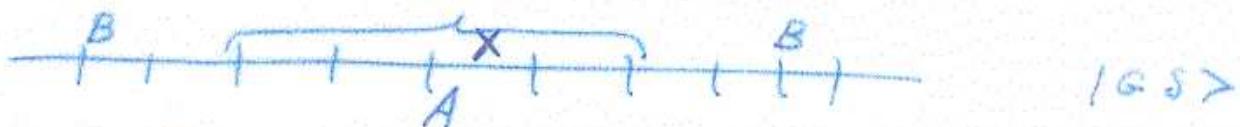
$$|A, B\rangle \neq |A\rangle \otimes |B\rangle \quad \text{pure states}$$

$$|A, B\rangle = \sum_{j=1}^n |A\rangle_j \otimes |B\rangle_j; \quad \text{Bennett, Bernstein, Popescu, Shumacher, 1996}$$

measure entropy of a subsystem,

$$\rho_A = \text{tr}_B (|A, B\rangle \langle A, B|)$$

$$S_A = -\text{tr}_A (\rho_A \ln \rho_A)$$



$$H_{xx} = - \sum_{j=-\infty}^{\infty} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + h \sigma_j^z \right)$$

Subsystem A is X spins

|GS> Lieb, Schultz, Mattis, Barouch, McCoy 1961-1971

$$S_A = S(X)$$

Determinant representation

$$\sigma^a = \{I, \sigma^x, \sigma^y, \sigma^z\}$$

$$a = 0, x, y, z$$

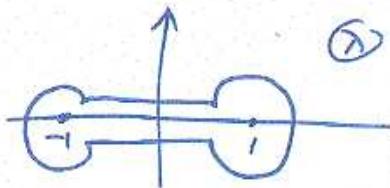
$$\rho_A = 2^{-L} \sum_{\{q_j\}} \text{tr}_B (\rho_{AB} \prod_{j=1}^L \sigma_j^{q_j}) \prod_{j=1}^L \sigma_j^{q_j} =$$

$$= 2^{-L} \sum_{\{q_j\}} \langle GS | \prod_{j=1}^L \sigma_j^{q_j} | GS \rangle \left( \prod_{j=1}^L \sigma_j^{q_j} \right)$$

$\sigma^z$ -Pauli matrices

$L=x$

$$\hat{\gamma}_A = \frac{1}{2\pi i} \oint e(x+i\varepsilon, \lambda) d\ln D_L(\lambda) = S(x)$$



$$e(x, r) = -\frac{x+r}{2} \text{Erf}\left(\frac{x+r}{2}\right) - \frac{x-r}{2} \text{Erf}\left(\frac{x-r}{2}\right)$$

$D_L$  - Toeplitz Determinant

$$D_L(\gamma) = \det(\gamma \hat{I} - G)$$

$$g_{\ell, k} = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i\theta(\ell-k)} g(\theta)$$

$$g(\theta) = \begin{cases} -1 & -K_F < \theta < K_F \\ 1 & K_F < \theta < 2\pi - K_F \end{cases}$$

$$K_F = \arccos(1h1/\gamma)$$

$$\hat{t}_{xy} = - \sum_j \left( b_j^x b_{j+1}^x + b_j^y b_{j+1}^y + h \partial_j^z \right)$$

Fisher-Hartwig formula

$$S_A = S(x) = \int e(\gamma) d \ln D(\gamma)$$

$$-e(\gamma) = \frac{1+\lambda}{2} \ln \frac{1+\lambda}{2} + \frac{1-\lambda}{2} \ln \frac{1-\lambda}{2}$$

$$D(\gamma) = \det G$$

$$\hat{g}_{k\ell} = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) e^{-i\theta(k-\ell)} d\theta; \quad k, \ell = 1 \dots x$$

$$g(\theta) = \begin{cases} \lambda - 1 & , \quad -K_F < \theta < K_F \\ \lambda + 1 & \quad K_F < \theta < 2\pi - K_F \end{cases}$$

$$K_F = \arccos(1h/12)$$

Asymptotics      Fisher-Hartwig formula

$$\mathcal{Z} = x \sqrt{1 - (h/h_c)^2} ; \quad h_c = 2 \text{ ferro}$$

$$S(x) = \frac{2\mathcal{Z}}{\pi} \ln \left( \frac{\pi}{2\mathcal{Z}} \right) ; \quad \text{if } \mathcal{Z} \rightarrow 0$$

$$i(x) = \frac{1}{3} \ln(\mathcal{Z} \cdot 2) + \Gamma + O(\frac{1}{\mathcal{Z}}), \quad \text{if } \mathcal{Z} \rightarrow \infty$$

$$= - \int_0^\infty dt \left\{ \frac{e^{-t}}{3t} + \frac{1}{t \sinh^2(t/2)} - \frac{\cosh(t/2)}{2 \sinh^3(t/2)} \right\}$$

We also calculated Rényi & Tsallis entropies

$$\left( \frac{1}{1-d} \right) \ln \mathcal{Z} \epsilon p^d \rightarrow \frac{1+d-1}{6} \ln \mathcal{Z}, \quad \mathcal{Z} \rightarrow \infty, \quad 0 < d \leq 1$$

In the information theory not only von Neumann entropy is used. Sometimes different generalizations of Neumann entropy are useful. The paper [3] consider Rényi entropy of the same subsystem:

$$S_{\text{Rényi}} = S_\alpha(x) = \frac{\ln \text{tr}_x \rho^\alpha}{1 - \alpha}.$$

When deformation parameter  $\alpha$  approaches 1 then Rényi entropy turns into von Neumann entropy. Here we also using the density matrix (4). Rényi entropy of the subsystem also scales logarithmically with the size:

$$S_\alpha(x) = (1 + \alpha^{-1}) \frac{c}{6} \ln x, \quad \text{as } x \rightarrow \infty.$$

Another generalized entropy is Tsallis entropy:

$$S_{\text{Tsallis}} := \frac{\text{Tr} \rho^\alpha - 1}{1 - \alpha}$$

Tsallis entropy and Rényi entropy are algebraically related:

$$S_{\text{Tsallis}} = \frac{e^{(1-\alpha)S_\alpha} - 1}{1 - \alpha}$$

Conformal field theory [1] is useful for the description of low temperature behavior of gap-less models in one space and one time dimensions. Important characteristic of conformal field theory is a central charge. It can be defined by considering an energy-momentum tensor  $T_{\mu,\nu}(z)$ . Here  $z$  is complex space-time variable  $z = x + ivt$  and  $v$  is a Fermi velocity. The component of energy-momentum tensor with  $\mu = \nu = z$  is denoted by  $T_{z,z} = T$ . Correlation function of this operator has a singularity:

$$\langle T(z)T(0) \rangle = \frac{c}{z^4}$$

The coefficient  $c$  is the central charge.

In this paper we are mainly interested in specific entropy  $s$  [entropy per unit length]. Let us start with specific heat  $C = T ds/dT$ . Low temperature behavior was obtained in [9, 8]:

$$C = \frac{\pi T c}{3v}, \quad \text{as } T \rightarrow 0. \quad (1)$$

Here  $c$  is a central charge of corresponding Virasoro algebra and  $v$  is Fermi velocity. We are more interested in  $s$ . We can integrate the equation and fix the integration constant from the third law of thermodynamics ( $s = 0$  at  $T = 0$ ). So for specific entropy we have the same low temperature behavior:

$$s = \frac{\pi T c}{3v}, \quad \text{as } T \rightarrow 0. \quad (2)$$

For quantum spin chains this formula agrees with [6]. To formulate the problem precisely let us consider Bose gas with delta interaction. The Hamiltonian of the model is:

$$H = \int dx [\partial\psi_x^\dagger \partial\psi_x + g\psi_x^\dagger \psi_x^\dagger \psi_x \psi_x]. \quad (3)$$

Here  $\psi$  is a canonical Bose field and  $g > 0$  is a coupling constant. The model was solved in [10]. Physics of the model is described in detail in book [4]. First let us consider the model at zero temperature and in the infinite volume. The ground state is unique  $|gs\rangle$ . We consider positive density case. We are interested in the entropy  $S(x)$  of the part of the gas present on the space

interval  $(0, x)$ . Formally we can define it by means of the density matrix

$$\rho = \text{tr}_{\infty} (|gs\rangle\langle gs|) \quad (4)$$

Here we traced out the 'external' degrees of freedom, they describe the gas on the rest of the ground state: on the unification of the intervals  $(-\infty, 0)$  and  $(x, \infty)$ . The density matrix  $\rho$  describes gas on the interval  $(0, x)$ . Now we can calculate von Neumann entropy  $S(x)$  of the part of the gas on the interval  $(0, x)$ :

$$S(x) = -\text{tr}_x \rho \ln \rho \quad (5)$$

Here we are taking the trace with respect to the degrees of freedom representing the part of the gas on the interval  $(0, x)$ . In the major test books it is shown that the laws of thermodynamics can be derived from statistical mechanics, see for example [17, 16]. Second law of thermodynamics states that the entropy is extensive parameter: the entropy of a subsystem  $S(x)$  is proportional to the system size  $x$ :

$$S(x) = sx \quad \text{at } T > 0 \quad (6)$$

Thermodynamics is applicable to the subsystem of macroscopical size, meaning large  $x$ . Here specific entropy  $s$  depends on the

temperature. For small temperatures the dependence simplifies, see (2):

$$S(x) = \frac{\pi T \epsilon}{3v} x, \quad x > \frac{1}{T}. \quad (7)$$

Let us try to find out how  $S(x)$  depends on  $x$  for zero temperature. It is some functions of the size:

$$S(x) = f(x), \quad \text{at } T = 0 \quad (8)$$

Now let us apply the ideas of conformal field theory, see [1, 9, 8] and also Chapter XVIII of [4]. We can arrive to small temperatures from zero temperature by conformal mapping  $\exp[2\pi T z/v]$ . It maps the whole complex plane of  $z$  without the origin to a strip of the width  $1/T$ . This replaces zero temperature by positive temperature  $T$ . The conformal mapping results in a replacement of variable  $x$  by  $[v/\pi T] \sinh[\pi T x/v]$ . So the entropy of the subsystem at temperature  $T$  is given by the formula:

$$S(x) = f\left(\frac{v}{\pi T} \sinh\left[\frac{\pi T x}{v}\right]\right), \quad \text{at } T > 0. \quad (9)$$

In order to match this to formula (7) we have to consider asymp-

totic of large  $x$ . The formula simplifies:

$$S(x) = f \left( \exp \left[ \frac{\pi T(x - x_0)}{v} \right] \right), \quad Tx \rightarrow \infty. \quad (10)$$

Here  $\pi T x_0 / v = -\ln(v/2\pi T)$ .

Formulae (7) and (10) should coincide. Both represent the entropy of the subsystem for small positive temperatures. This provides an equation for  $f$ :

$$f \left( \exp \left[ \frac{\pi T(x - x_0)}{v} \right] \right) = \frac{\pi T c}{3v} (x - x_0). \quad (11)$$

This formula describes asymptotic for large  $x$ , so we added  $-x_0$  to the right hand side. We are considering the region  $x > 1/T$  and  $x_0 \sim \ln(1/T)$ , so  $x >> x_0$  at  $T \rightarrow 0$ . In order to solve the equation for  $f$ , let us introduce a new variable  $y = \exp[\pi T(x - x_0)/v]$ . Then the equation (11) reads:

$$f(y) := \frac{c}{3} \ln y \quad (12)$$

So we found the function  $f$  in (8). Now we know that at zero temperature entropy of the gas containing on the interval  $(0, x)$

is:

$$S(x) = \frac{c}{3} \ln x \quad \text{as } x \rightarrow \infty \quad (13)$$

Let us remind that for Bose gas  $c = 1$ , see [4]. Our result agrees with the third law of thermodynamics. Specific entropy is a limit of the ratio  $S(x)/x$  as  $x \rightarrow \infty$ . The limit is zero.

Now we can go back to our formula (9) and substitute the expression for  $f$ , which we found:

$$S(x) = \frac{c}{3} \ln \left( \frac{v}{\pi T} \sinh \left[ \frac{\pi T x}{v} \right] \right)$$

This formula describes crossover between zero and positive temperature. It is important that we have here hyperbolic sinh not trigonometric.

*The proof presented here is universal.* It is also applicable to quantum spin chains. For example to XXZ spin chain. The Hamiltonian of the model is:

$$H_{XXZ} = - \sum_m \left\{ \sigma_x^{(m)} \sigma_x^{(m+1)} + c \sigma_y^{(m)} \sigma_y^{(m+1)} + \Delta (\sigma_z^{(m)} \sigma_z^{(m+1)} - 1) \right\}. \quad (14)$$

# Universality at large $x$

$$S(x) \rightarrow \frac{c}{3} \ln x \quad \text{von Neumann entropy}$$

$c$  - central charge of Virasoro algebra

Holzhey, Larsen, Wilczek, 1994

Asymptotics can be derived from the second law of thermodynamics

Y. Affleck considered spin chains with arbitrary spin  $s$ :

$$c = \frac{3s}{s+1}$$

$$S(x) = \left( \frac{s}{s+1} \right) \ln x$$

for spin 1

$$f = \sum_j \vec{S}_j \vec{S}_{j+1} - (\vec{S}_j \vec{S}_{j+1})^2$$

Spin  $1/2$

$$H = \sum_j \vec{S}_j \vec{S}_{j+1}$$

Asymptotic of Tsallis entropy is given by a power law:

$$(1 - \alpha)S_{\text{Tsallis}} \sim x^{\frac{\alpha}{6}(\alpha^{-1} - \alpha)} - 1$$

# 1 Hubbard Model

The Hamiltonian for the Hubbard model  $H$  can be represented as

$$H = - \sum_{\substack{j=1 \\ \sigma=\uparrow,\downarrow}} (c_{j,\sigma}^\dagger c_{j+1,\sigma} + c_{j+1,\sigma}^\dagger c_{j,\sigma}) + u \sum_{j=1} n_{j,\uparrow} n_{j,\downarrow} \quad (16)$$

Here  $c_{j,\sigma}^\dagger$  is a canonical Fermi operator on the lattice [operator of creation of an electron] and operator  $n_{j,\sigma} = c_{j,\sigma}^\dagger c_{j,\sigma}$  is an operator on number of electrons in site number  $j$  with spin  $\sigma$ . Summation in the Hamiltonian goes through the whole infinite lattice. We are considering repulsive case  $u > 0$  below half filling [less than one electron per lattice site]. The model was solved in [11]. A collection of important papers on the model can be found in [12]. Charge and spin separate in the model. The model is gapless. Both charge and spin degrees of freedom can be described

For the Hubbard model the expression for the entropy  $S(x)$  depends on the phase [Korepin, PRL vol 92, 096402, 2004]. In the repulsive case  $U > 0$  below half-filling  $D < 1$  [less than one electron per lattice site] the entanglement is:

$$S(x) = \frac{2}{3} \ln x \quad \text{as} \quad x \rightarrow \infty \quad (9)$$

The factor of 2 is related to the fact that there are two gap-less degrees of freedom: both spin and charge.

At half-filled band  $D = 1$  [one electron per lattice site] spin degrees of freedom are gap-less, but there is a gap for charge degrees of freedom, so

$$S(x) = \frac{1}{3} \ln x \quad (10)$$

In the attractive case  $U < 0$  at zero magnetic field:

$$S(x) = \frac{1}{3} \ln x \quad (11)$$

charge degrees of freedom are gap-less, but there is a gap for spin degrees of freedom.

Counterexamples : gapless models with  
 $E(p) \sim p^2$  for small  $p$  momentum

Ferrimagnet spin 1/2

$$H = - \sum_{j=-\infty}^{\infty} \vec{S}_j \cdot \vec{S}_{j+1}$$

$$(S_-)^M |1\rangle : S_- = \sum_j S_j^-$$

$$M \sim x \rightarrow \infty$$

$$S(x) = \frac{1}{2} \ln x \quad \text{Salerno, Popkov}$$

Eta pairing as a model for superconductivity

$$\hat{c}^+ = \sum_j (-1)^j c_{j\uparrow}^+ c_{j\downarrow}^+$$

$$(\hat{c}^+)^M |0\rangle \quad \text{Vedral, Lloyd, Fan}$$

$$S(x) = \frac{1}{2} \ln x \quad ; \quad c_j |0\rangle = 0$$

# XY Spin Chain

$$H_{XY} = - \sum_{j=-\infty}^{\infty} (1+\gamma) \sigma_j^x \sigma_{j+1}^x + (1-\gamma) \sigma_j^y \sigma_{j+1}^y + h \sigma_j^z$$

$\gamma$ -anisotropy parameter  $0 < \gamma < 1$

Physics depends on a case

$$1a: \quad 2\sqrt{1-\gamma^2} < h < 2$$

$$1b: \quad 0 < h < 2\sqrt{1-\gamma^2}$$

$$2: \quad 2 \cancel{>} h ; \quad h > 2$$

The Hamiltonian has a gap for infinite chain.

$$\frac{B}{A} \quad \underset{x}{\longrightarrow} \quad B \quad \text{gs}$$

$$P(x) = P_A = \text{tr}_B |gs\rangle \langle gs|$$

$$S_{vN}(x) = -\text{tr}_A P(x) \ln P(x)$$

$$S = \lim_{x \rightarrow \infty} S_{vN}(x)$$

Elliptic functions

$$I(\kappa) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - \kappa^2 \sin^2 \alpha}} \quad ; \quad \kappa' = \sqrt{1 - \kappa^2}$$

A. Its, B.-Q. Min, V. Korepin, 2004

Determinant representation, block Toeplitz

Case 1a

$$S = \ln 2 + \frac{1}{6} \left\{ \ln \left( \frac{K^2}{16K'} \right) + \left( 1 - \frac{K^2}{2} \right) \frac{4}{\pi} I(K) I(K') \right\}$$

$$K = \frac{1}{\pi} \sqrt{\left(\frac{h}{2}\right)^2 + r^2 - 1}$$

Case 1b

$$K = \sqrt{\frac{1 - (h/2)^2 - r^2}{1 - (h/2)^2}} ; \text{ same expression for } S$$

Case 2

$$S = \frac{1}{12} \left\{ \ln \frac{16}{(K \cdot K')^2} + (K^2 - K'^2) \frac{4}{\pi} I(K) I(K') \right\}$$

$$K = \frac{r}{\sqrt{(h/2)^2 + r^2 - 1}}$$

These are limiting expressions

Singularities are on the boundaries  
between Cases: phase transitions.

quant-ph/0409027

Singularities: Critical magnetic field:

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1. Cases 1a & Case 2  $h > 2\sqrt{1-\gamma^2}$

$$h \rightarrow 2 \quad S \rightarrow \frac{1}{6} \ln |2-h| + \frac{2}{3} \ln 4\gamma ; \quad \gamma \neq 0$$

$$\text{correction } O(|2-h| \ln^2 |2-h|)$$

2. Isotropic limit  $\gamma \rightarrow 0$ ;  $h < 2$  case 1b

$$S \rightarrow \left( \frac{1}{3} \right) \ln \gamma + \frac{1}{6} \ln (4-h^2) + \frac{1}{3} \ln 2 + O(\gamma \ln^2 \gamma)$$

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## Other example of a gapped model

Spin chain with spin = 1

VBS  $S = AKLT$

$$H_{VBS} = \sum_{j=-\infty}^{\infty} \left( \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3} (\vec{S}_j \cdot \vec{S}_{j+1})^2 \right)$$

$$\vec{S}_j \cdot \vec{S}_{j+1} = S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + S_j^z S_{j+1}^z$$

$S^{x,y,z}$  generators of  $SU(2)$  algebra with  $S=1$   
spin = 1

$\frac{1}{2} \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{6} (\vec{S}_j \cdot \vec{S}_{j+1})^2 + \frac{1}{2}$  is a projector on a state with spin 2



quant-ph/0406067

$$AKLT = VBS$$

Haldane conjectured a gap for integer spin

1 Dimension Spin 1

$$H_{VBS} = \sum_{j=-\infty}^{\infty} \left( \vec{S}_j \vec{S}_{j+1} + \frac{1}{3} (\vec{S}_j \vec{S}_{j+1})^2 \right)$$

$$\vec{S}_j \vec{S}_{j+1} = S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + S_j^z S_{j+1}^z$$

$$\frac{1}{2} (\vec{S}_j \vec{S}_{j+1}) + \frac{1}{6} (\vec{S}_j \vec{S}_{j+1})^2 + \frac{1}{2} = \text{projection on spin } \frac{1}{2}$$

Fleck, Kennedy, Lieb, Tasaki: Commun. Math. Phys. v115, 198

Experimental realization in optical lattices

Ranjan-Ripoll; Martin-Delgado, Cirac cond-mat/0404566

Laughlin ansatz for fractional quantum Hall

Universal quantum computation by local measurements

F. Verstraete, J.I. Cirac quant-ph/0311130

The ground state is unique. Two neighboring spins are never parallel

$$\text{Spin } 1 = \text{symm}(\text{spin } \frac{1}{2}) \otimes (\text{spin } \frac{1}{2}) = \dots$$



Here  $\longrightarrow$  antisymmetrization of product of two spin  $1/2$

VBS can be generalized to arbitrary graph

Entropy of a large subsystem is a constant equal to the number of links necessary to cut to isolate the subsystem

In 1 dimensional case (spin chain)  
 $S=2$  - limiting entropy

H. Fan, V. Korepin, V. Roychowdhury  
PRL, 2004

quant-ph/0309188

$$\langle S_z^a S_L^b \rangle = g_0^2 \left(\frac{4}{3}\right) p(L) ; \quad p(L) = \left(\frac{-1}{3}\right)^L = \left(\frac{1}{3}\right)^L$$

Entropy of a block of  $L$  spins

$$S_L = S(p_L) = 2 + \frac{3}{4} (1-p(L)) \log(1-p(L)) - \\ - \frac{1+3p(L)}{4} \log(1+3p(L))$$

Entropy of two spins-1's separated by  $M$  sites

$$S(M) = 2 \log 3 - \frac{5}{9} (1-p(M)) \log(1-p(M)) - \frac{3}{9} (1+p(M)) \log(1+p(M)) \\ - \frac{1}{9} (1+2p(M)) \log(1+2p(M))$$

incurrence  $S_b \Rightarrow C_L = 1 - p^2(L) = 1 - \left(\frac{1}{3}\right)^L$

$$S(M) = 1 - \frac{1}{6} p^2(M)$$

Arbitrary graph  $Z_e$  = coordination number (valence)

$$S_e = \frac{1}{2} Z_e$$

$$\hat{\mathcal{P}}_j = \prod_{j \neq y} \left( \frac{\hat{S}^2 - j(j+1)}{j(j+1) - j(j+1)} \right) = \hat{\mathcal{P}}_y (k, e)$$

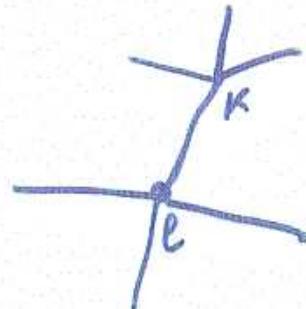
$$|S_k - S_e| \leq j \leq S_k + S_e$$

$$\hat{S}^2 = (\hat{S}_k + \hat{S}_e)^2$$

Consider the projector on highest spin  $\gamma = S_k + S_e$

$$\hat{\mathcal{P}}_I (k, e) = \prod_{j=|S_k - S_e|}^{S_k + S_e - 1} \left( \frac{\hat{S}^2 - j(j+1)}{(S_k + S_e)(S_k + S_e + 1) - j(j+1)} \right)$$

$$H = \sum_{\langle k \ell \rangle} \hat{\mathcal{P}}_{k \ell}$$



Unique ground state

Entropy in 1 D Hubbard

$$\frac{2}{3} \ln X$$

$$\mathcal{H}_j(\ell, K) = \prod_{j \neq j} \frac{\hat{S}^z - j(j+1)}{j(j+1) - j(j+1)}$$

$$|s_k - s_\ell| \leq j \leq s_k + s_\ell$$

$$\hat{S}^z = (\hat{\vec{S}}_k + \hat{\vec{S}}_\ell)^z$$

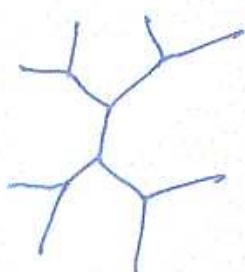
$$\mathcal{H}_2(\ell, K) = \frac{1}{2} (\vec{S}_k \cdot \vec{S}_\ell) + \frac{1}{6} (\vec{S}_k \cdot \vec{S}_\ell)^2 + \frac{1}{3}$$

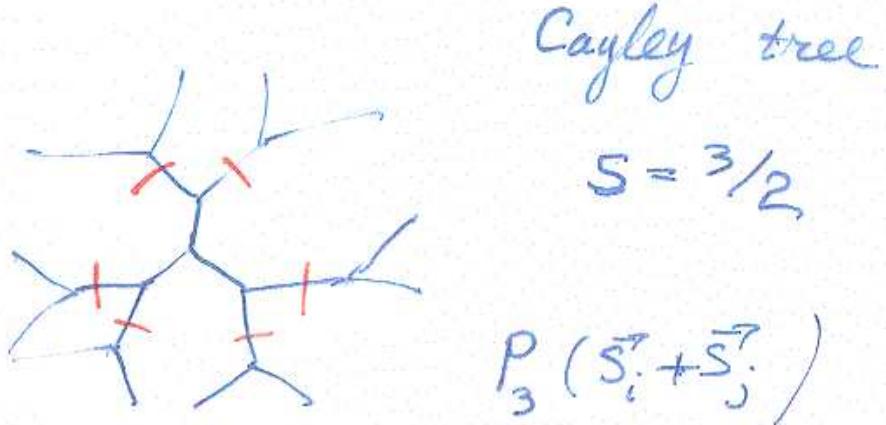
$$[\hat{S}_\ell^a, \hat{S}_k^b] = i \delta_\ell^k \epsilon^{abc} \hat{S}_\ell^c$$

Bethe lattice   $s = \frac{3}{2}$

$$H = \sum_{\langle \ell, K \rangle} \mathcal{H}_3(\ell, K)$$

$$\mathcal{H}_3(\ell, K) = \vec{S}_\ell \cdot \vec{S}_K + \frac{116}{243} (\vec{S}_\ell \cdot \vec{S}_K)^2 + \frac{16}{243} (\vec{S}_\ell \cdot \vec{S}_K)^2 + \text{const}$$





$$H = \sum_{(i,j)} P_3 (s_i + s_j)$$

Entropy  $S = \text{number of links necessary to cut to isolate the system}$

Fan, Korepin, Roychowdhury, Verstraete  
coefficient = 1

agrees with M.B. Plenio, J. Eisert, J. Preissig,  
M. Cramer, PRL, 2005

Take two copies of Bose operators in each vertex  
 $[a_k, a_e^+] = \delta_{ke}^k$  ;  $[b_k, b_e^+] = \delta_{ke}^k$  ;  $[a_k, b_e^+] = 0 = [a_k, b_e]$

$Z_L = a_e^+ a_e + b_e^+ b_e \rightarrow$  fix number of particles in vertex  $L$

$$\hat{S}_e^+ = a_e^+ b_e ; \hat{S}_e^- = a_e b_e^+ ; 2\hat{S}_e^z = a_e^+ a_e - b_e^+ b_e$$

representation of  $SU(2)$  algebra

VBS Hamiltonian

$$H = \sum_{\langle k \ell \rangle} \tilde{\tau}_{k \ell}$$

has a unique ground state

$$|gs\rangle = \prod_{\langle k \ell \rangle} (a_k^+ b_\ell^+ - a_\ell^+ b_k^+) |0\rangle$$

$a_k |0\rangle = 0 = b_k |0\rangle$       Laughlin ansatz for fractional quantum Hall

Korepin, Kinnillor, 1989

Algebra & Analise  $\Rightarrow$  Steklov Math. Inst

Entanglement is a resource for quantum control

$$|A, B\rangle = \sum_{j=1}^n |A\rangle_j \cdot |B\rangle_j$$

We can use one subsystem to control another

If we measure A the state of B will change

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# Localizable Ent. in Spin Chains

$$1 \ 1 \ 1 \ 1 \ 1 \underset{i}{0} \ 1 \ 1 \ 1 \ 1 \underset{j}{0} \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1$$

$E_{ij}$  - concurrence

J.I. Cirac, M. Popp, F. Verstraete 2003

$$E_{ij} \geq \max_a (\langle \sigma_i^a \sigma_j^a \rangle - \langle \sigma_j^a \rangle^2)$$

$$a = x, y, z$$

$$H_{xxx} = \sum_j \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z \right)$$

$$E_{i,i+1} \geq \frac{4}{3} \ln 2 - \frac{1}{3} \approx 0.6$$

$$E_{i,i+2} \geq \frac{1}{3} - \frac{16}{3} \ln 2 + 3\beta(3) \approx 0.24$$

Asymptotics  $|i-j| \rightarrow \infty$

$$E_{ij} \geq \frac{2 \sqrt{\ln |i-j|}}{\pi^{3/2} |i-j|}$$

B.-Q. Yin., V.E. Korepin quant-ph/10309188