

# QUANTUM FIELD THEORY IN GENERAL CURVED SPACETIMES

Stefan Hollands  
Georg-August-Universität Göttingen  
Germany

Based on papers : [gr-qc/0404074]  
[gr-qc/0209029]  
[gr-qc/0111108]  
[gr-qc/0103074] (with RM Wald)

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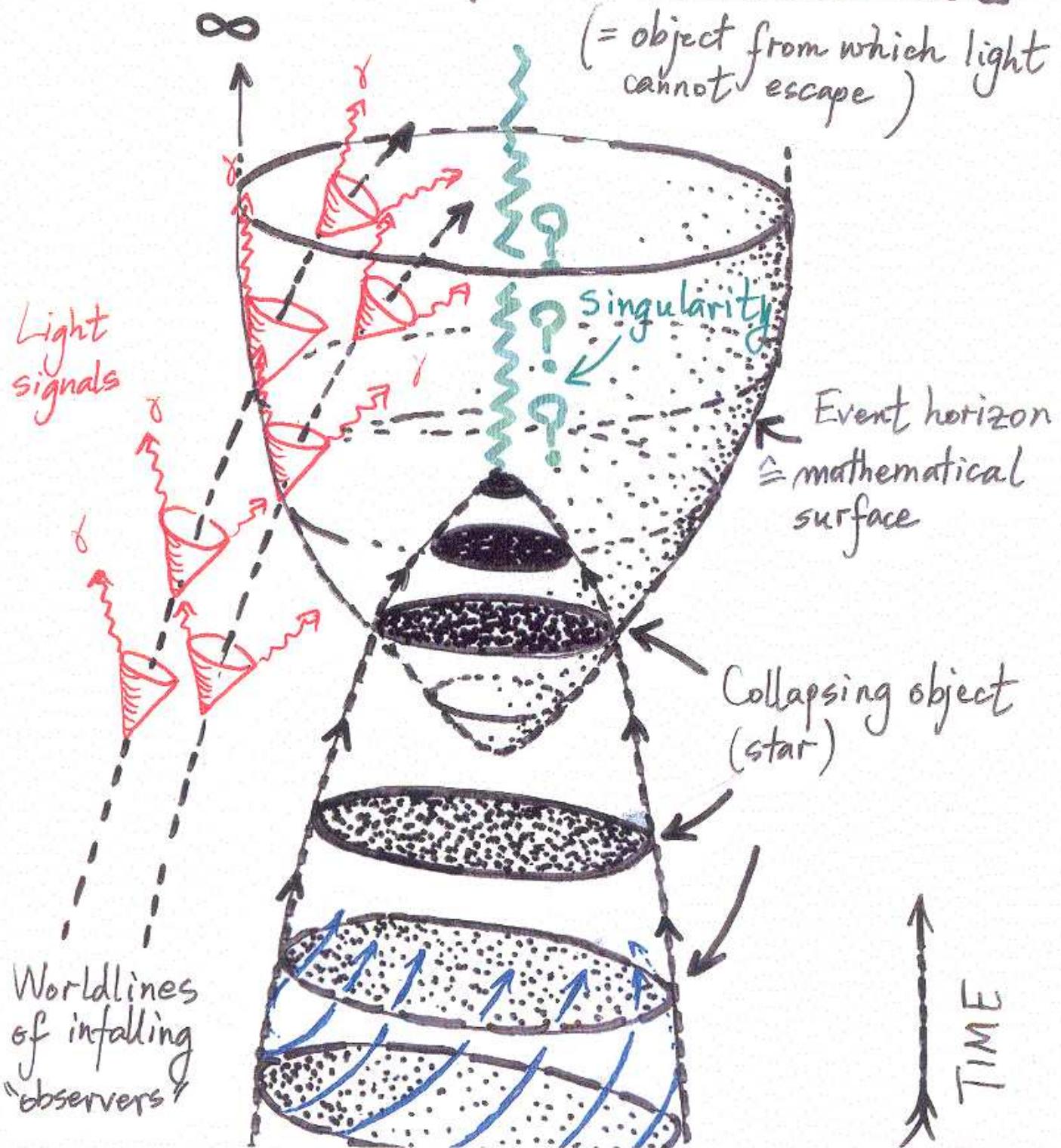
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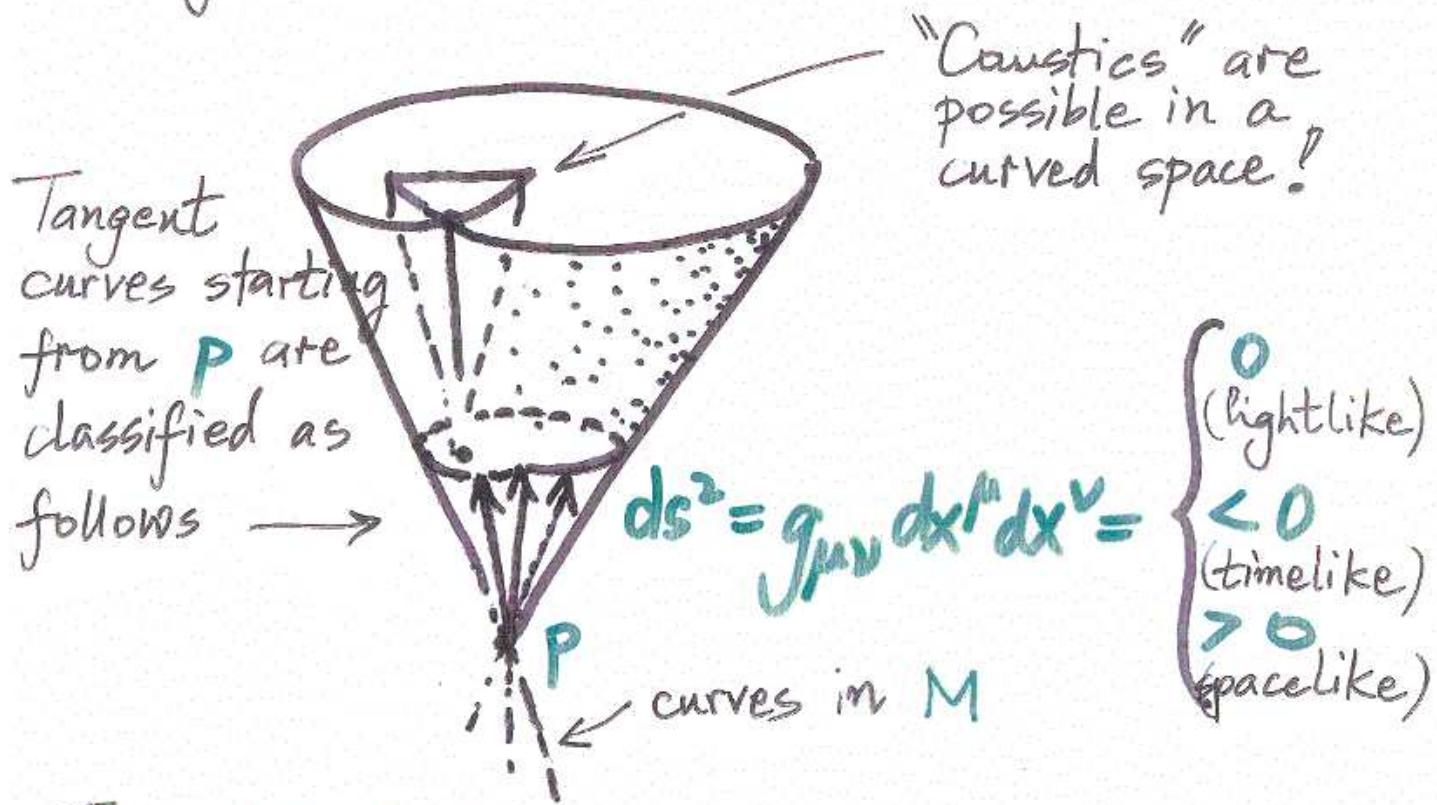
(by Brunetti, Fredenhagen, Köhler, Radzikowski, Verch ...)

Gravity curves the geometry of space and time:

## Extreme example : BLACK HOLE



Mathematically, a curved spacetime is described by a manifold  $M$  together with a metric  $g|_P \in T_P^*M \otimes T_P^*M$  at every point.



The KLEIN GORDON EQUATION

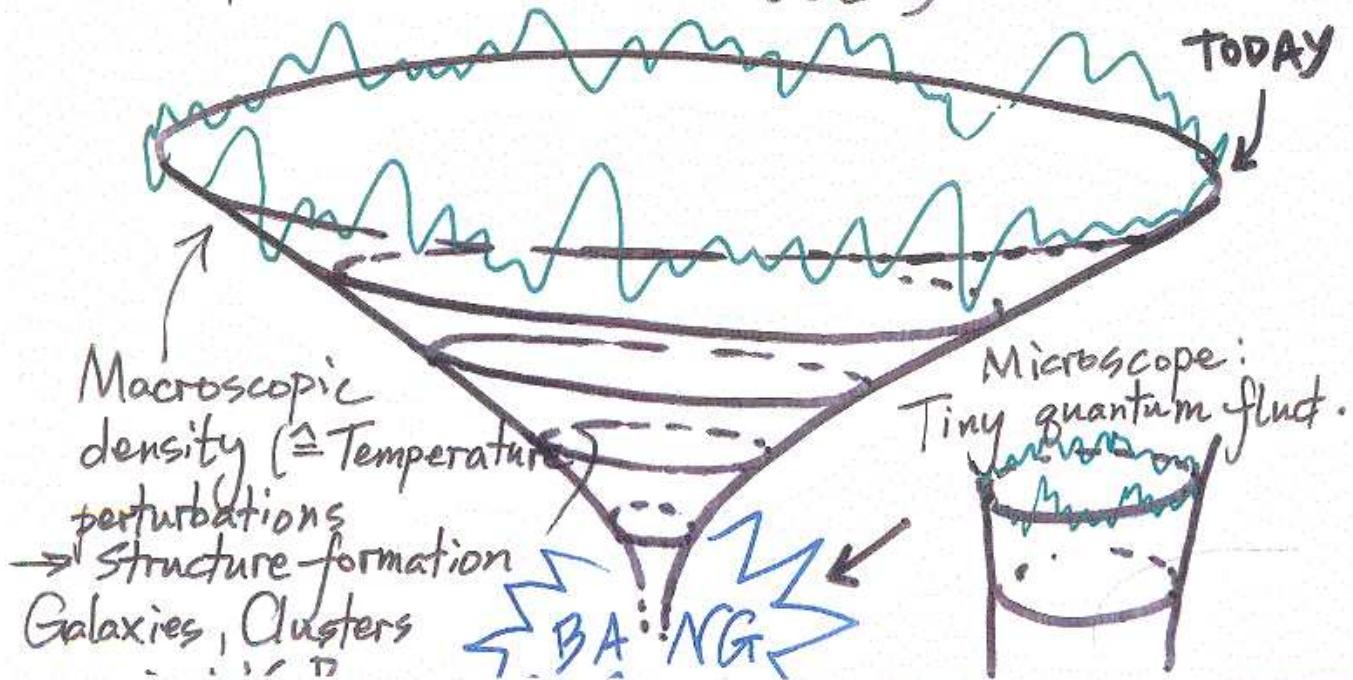
$$0 = (\nabla_\mu \nabla^\mu - m^2) \phi(x)$$

describes the propagation of waves on a curved background, where  $\phi(x)$  is the wave amplitude at point  $x$ .

One not only wants to study classical fields, but also would like to know their **quantum properties!**

To an excellent approximation (essentially all the way back to the big bang), one can treat metric as **classical** background field, while **quantizing** the other matter fields (e.g. Klein-Gordon field).

- Quantum properties of Black Holes: They are not completely black, but emit a thermal radiation. (HAWKING EFFECT)
- Early universe: Quantum field fluctuations generate macroscopic density perturbations ( $\rightarrow$  CMB)



Standard Formalism to describe quantum properties of fields:

## QUANTUM FIELD THEORY

Formalism in flat space relies heavily on special properties of flat space, notably spacetime symmetries  
( $\rightarrow$  preferred "vacuum" state, particle interpretation...)

QFT in curved space is very different from QFT in flat space:

- There is no preferred vacuum state.
- There is no preferred Hilbert space representation of quantum field operators.
- There is no  $S$  (scattering) - Matrix.
- No momentum space techniques (Fourier transformation).
- There are no spacetime symmetries, so how to define "energy" or "momentum" of system.
- No Euclidean methods (analytic continuation is not possible), no Euclidean path-integral.

...

→ forced to new formalism, which emphasizes the local properties of field observables, but is not based on global concept such as vacuum state, particle interpretation, S-matrix, ...

- "Algebraic" Formulation.
- "Local, covariant fields".
- "Microlocal spectrum condition".

Try to explain in this talk.

[SH & Wald CMP 223 ('01), CMP 231 ('02),  
CMP 237 ('03)] [Brunetti et al. CMP 208,  
CMP 180 ('97), CMP 237 ('03)]  
[Junker et al. Ann. Poincaré 3 ('02)] ...

## RESULTS SO FAR:

- Perturbative construction of interacting QFT's in arbitrary glob. hyperbolic curved spacetimes.
- Renormalization group.
- Analysis of interacting energy-momentum tensor.
- Operator Product Expansion.
- PCT-Theorem, spin-statistics. [SH, CMP 244 ('04)  
 Verch, CMP 223 ('01)]
- "Quantum Inequalities". [Fenster CQG 17 ('00)]

## ALGEBRAIC FORMALISM

Idea: Avoid problems related to absence of preferred vacuum state by working instead with "algebraic relations" between quantum fields.

Example: Free KG quantum field

Define algebra  $\mathcal{A}(M, g)$  of quantum fields to be free \*-algebra generated by symbols  $\phi(f)$  [ $f$  a testfunction] subject to relations:

(i) Linearity:  $\phi(f)$  complex linear in  $f$

(ii) KG-eqn.:  $\phi((\nabla_\mu \nabla^\mu - m^2)f) = 0$

(iii) Hermitian:  $\phi(f)^* = \phi(\bar{f})$

(iv) Commutator:  $[\phi(f), \phi(h)] = i \Delta(f, h) \mathbb{1}$

retarded — advanced  
fundamental solution

(Informally,  $\phi(f) = \int_M \phi(x) f(x) d\mu$ )

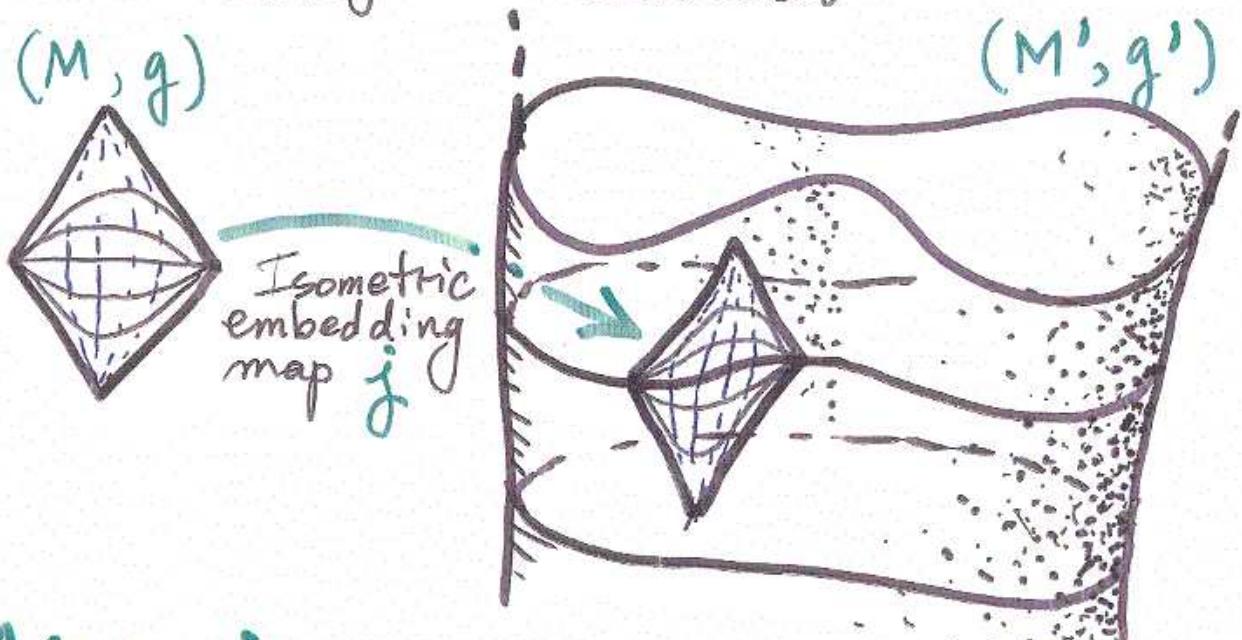
Quantum states are linear, positive, normalized functionals  $\langle - \rangle_\Psi : \mathcal{A} \rightarrow \mathbb{C}$

Normalized:  $\langle \mathbb{1} \rangle_\Psi = 1$

Positive:  $\langle A^* A \rangle_\Psi \geq 0$

# LOCAL, COVARIANT FIELDS

Idea : Quantum fields should depend locally and covariantly on metric:



$$A(M, g) \xrightarrow{\alpha_j} A(M', g')$$

injective homomorphism  
between algebras  
(w/ functorial properties)

A field  $\circlearrowleft$  is called a "local & covariant field" if (a) it is defined for all spacetimes and an element of  $\mathcal{A}$   
 (b)  $\alpha_j(\circlearrowleft) = \circlearrowleft'$  in the above situation.

Free KG-field is an example of local covariant field, but vacuum state is not local and covariant in any sense.

## APPLICATION / ILLUSTRATION of local covariance property :

Construction of composite operators in free Klein-Gordon theory. Simplest example:  $\hat{O}(x) = \phi^2(x)$

If we form this field naively, we get meaningless infinite answer (infinite correlation fct's).

Therefore, try

$$\phi^2(x) \doteq \lim_{x' \rightarrow x} \left\{ \underset{\text{"point-split" expression}}{\underset{\uparrow}{\phi(x)\phi(x')}} - \underset{\text{"subtraction piece"}}$$

The field  $\phi^2$  is then an element of a larger algebra,

$$\mathcal{W}(M,g) \supset \mathcal{A}(M,g)$$

(distribution algebra  $\hat{=}$  deformation quantization of classical Poisson algebra)  
 $\rightarrow \mu$ -local Analysis

BUT: What is "correct" subtraction piece?

ANSWER: Invoke local covariance principle for  $\phi^2$ !  $\Rightarrow H$  must be constructed locally & covariantly from metric.

$\Rightarrow H$  is not expectation value of point-split product in any state!

"Normal ordering" is incorrect.

The imposition of local covariance property reduces ambiguity in defining  $\phi^2$  to making a redefinition of the form

$$\phi^2 \rightarrow \phi^2 + \mathcal{E} \leftarrow = \text{local curvature expression might be operator.}$$

$\Rightarrow$  Need to impose further conditions

- SCALING CONDITIONS
- COMMUTATOR

$$[\phi^2, \phi] = 2i\Delta \phi$$

- ANALYTICITY

$\Rightarrow$  Ambiguity now reduced to

$$\phi^2 \rightarrow \phi^2 + (c_1 m^2 + c_2 R) \mathbf{1}$$

from scaling & analyticity

from commut. relation

Similar analysis can be performed for any composite field.

HOW ABOUT INTERACTING FIELDS ?

## Interacting KG-field:

Classical action from which eqn's of motion  
are derived (Classically) :

$$I = \int_M (\underbrace{\nabla_\mu \phi \nabla^\mu \phi + m^2 \phi}_{\hat{=} \text{Action for free KG field}} + \underbrace{\sum f_A \partial^A}_{\hat{=} \text{Interaction}}) d\mu$$

$O^A$  = Monomial in basic field  $\phi$ , its covariant derivatives, curvature terms.

# Quantum interacting field observables

$$F_S \in \mathcal{A}_S(M, g) \subset \underline{\mathcal{W}[[f]]}$$

↑ interacting field algebra      = formal power series  
 local field observable      in couplings  $f_A$ ,  
 $\hat{=} \int_M h_A O^A d\mu$  with coefficients  
 in a distribution algebra  $\mathcal{W}$

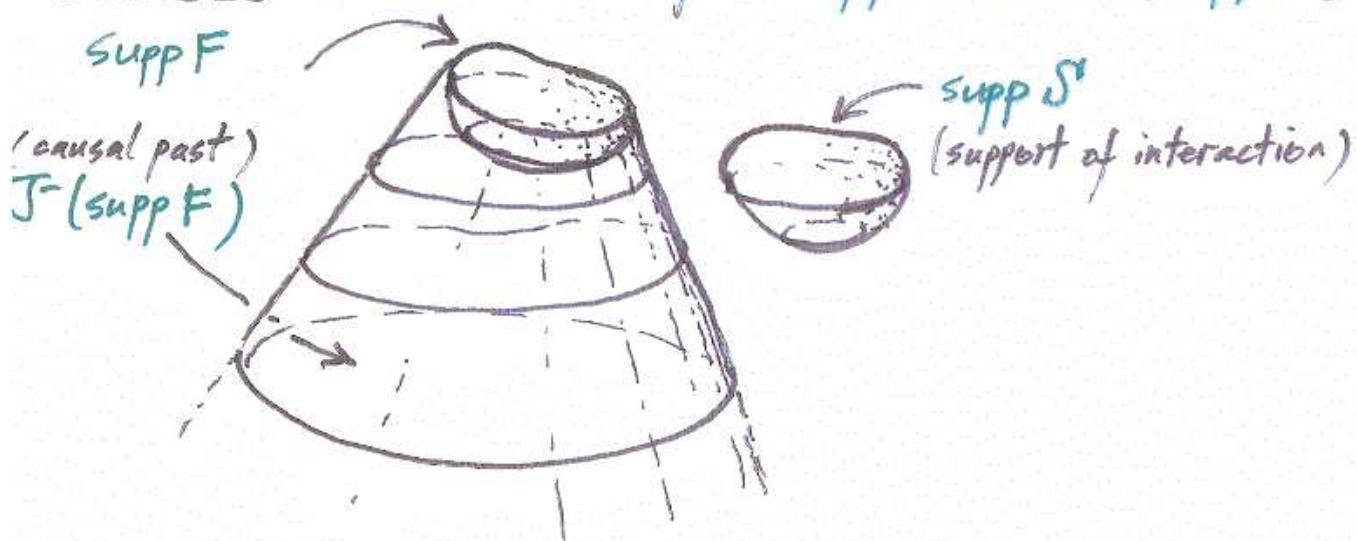
$$F_S = \sum_{n \geq 0} \frac{i^n}{n!} R_n \left( \bigotimes^n S; F \right)$$

$R_n$  = "Retarded products":  $\text{elements of } \mathcal{W} \xrightarrow{\text{(Classical Action Functionals)}} \mathcal{W}^{\otimes(n+1)}$

Retarded products are characterized by functional relations of alg. nature!

Partial list of functional relations satisfied by  $R_n$

- CAUSALITY:  $R_n = 0$  if  $\text{supp } S \cap J^-(\text{supp } F) = \emptyset$



- GLZ-RELATION:

$$\sum_{n+m=N} [R_n(F; S^{\otimes n}), R_m(H; S^{\otimes m})] \\ = R_{N+1}(F; H \otimes S^{\otimes N}) - (F \leftrightarrow H)$$

- SCALING PROPERTIES:  $R_n$  scales homogeneously up to a polynomial in  $\ln \lambda$  under rescalings of the metric  $g_{\mu\nu} \rightarrow \lambda^2 g_{\mu\nu}$  by const. conformal factor, and analogous rescaling of dimensionful "couplings" in  $S$ .

- $R_n(O_1(z_1); \bigotimes_j O_j(z_j))$  are LOCAL, COVARIANT FIELDS  
 field monomials  
 (including arbitrary derivatives) (multilocal, of course)

The construction of  $R_n$  implies construction of interacting fields  $\mathcal{O}_S(x)$  to all orders in perturbation theory. Ambig. similar to free fields.

The nature of the  $R_n$ 's imply that interacting fields satisfy following key properties to all orders:

## ① RENORMALIZATION GROUP:

Let  $\lambda \in \mathbb{R}_+$  and  $g'_{\mu\nu} = \lambda^2 g_{\mu\nu}$ . Then there exists a map  $S \rightarrow S(\lambda) = S'$  on the space of local functionals so that

$$\mathcal{A}(M, g)_S \xrightarrow{\sim} \mathcal{A}(M, g')_{S'}$$

with  $\mathcal{O}_S^A(x) \xrightarrow{\text{* - isomorphic}} Z^A_B(\lambda) \mathcal{O}_{S'}^B(x)$

"Mixing matrix", is a formal power series in couplings, whose coefficients are formal power series in  $\ln \lambda$ .

Furthermore, we have  $S(\lambda) = \lambda^{B_A \beta_{g_A}} S$  where  $g_A$  are the couplings and where  $B_A$  are the beta-fncts.

## ② CONSERVATION OF INTERACTING STRESS-TENSOR

- WARD-LIKE IDENTITIES for ret. products containing factor of stress-tensor
- MICROLOCAL SPECTRUM CONDITION  
(see below)

## MAIN THEOREM

Construction of ret. products with desired functional relations is possible to all orders in perturbation theory.

Basic steps in algorithm:

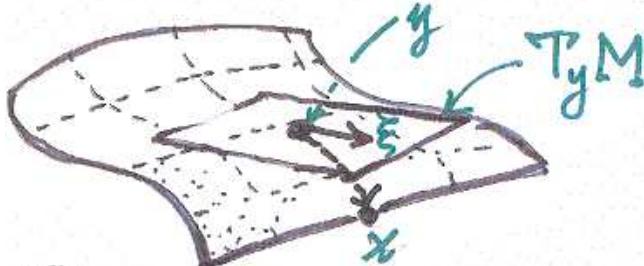
- ① Construct  $R_n$  inductively in perturbation order  $n$ .  
Use causality and GLB-relation to pass from  $n \rightarrow n+1$
- ② This construction fixes  $R_{n+1}$  uniquely as a distribution on  $M^{n+1} \setminus \Delta_{n+1}$  "total diagonal"  $\{(x, x, \dots, x)\}$
- ③ Now extend  $R_{n+1}$  to total diagonal (Renormalization).  
For this, use "scaling expansion" ( $\rightarrow$  Covariance relation)

$$R_{n+1}(y; x_1, \dots, x_{n+1}) \sim \sum' G(y) \cdot u(\xi_1, \dots, \xi_n)$$

Curvature  
polynomials  
& local operators

"almost" homog.  
Minkowski distrib.  
on  $\mathbb{R}^4 \times \dots \times \mathbb{R}^4$

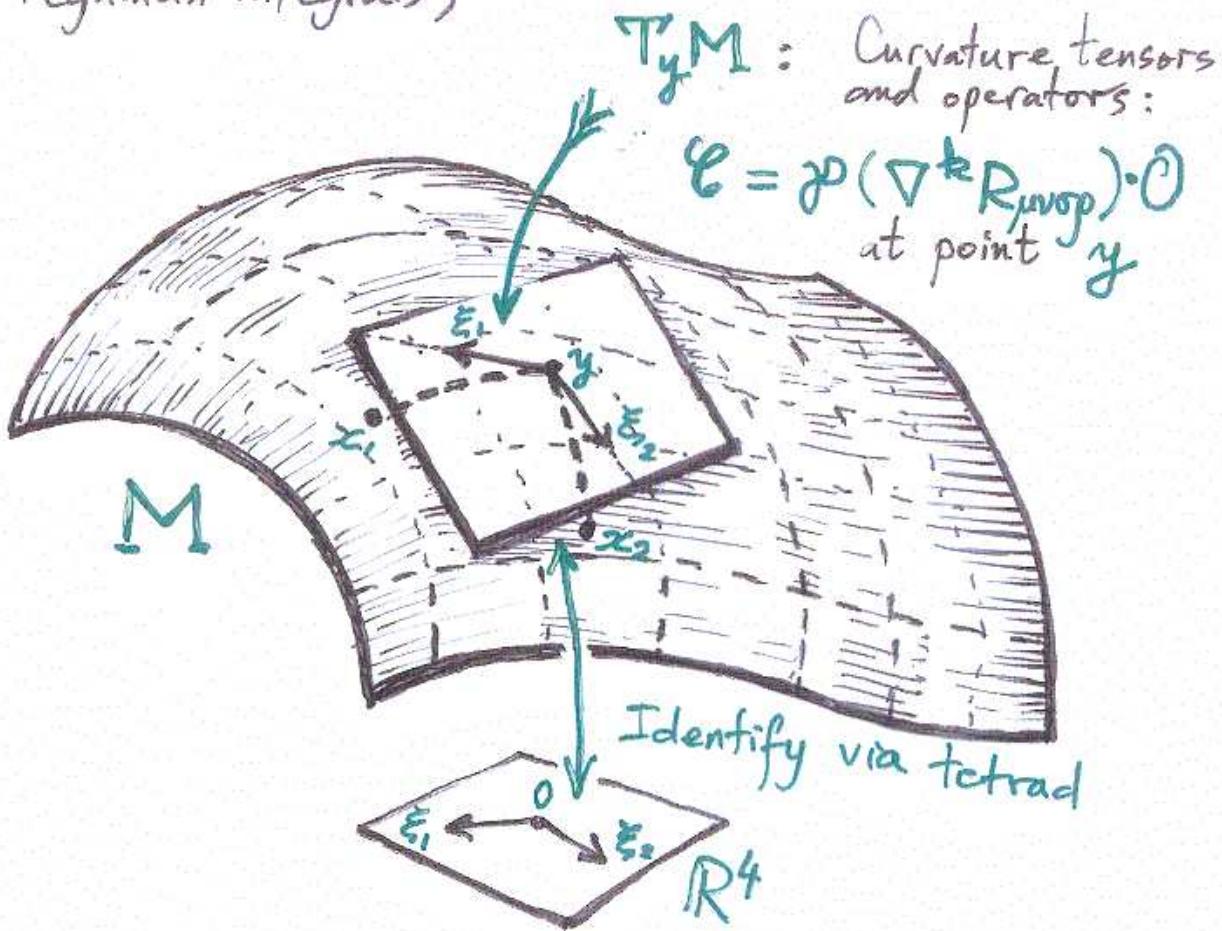
("Scale" towards diagonal  
in Tangent space at  $y \in M$   
via exponential map )



- ④ Perform extension (renormalization) of  $u$  using well known methods. But must preserve relations!

# DETAILED ILLUSTRATION of distributions

$u(\xi_1, \dots, \xi_n)$  appearing in the scaling expansion  
 ( $\xi$  can be viewed as generalized Minkowski Feynman integrals)



The distribution  $u(\xi_1, \xi_2)$  lives in  $R^4$ . Tetrad rotation corresponds to Lorentz transfo, so should be Lorentz invariant. Scaling properties of  $u$  is

$$u(\lambda \xi_1, \lambda \xi_2) = \lambda^s \left\{ u(\xi_1, \xi_2) \right\}$$

$\hookrightarrow$  logarithms  
 renormalization  
 $\rightarrow$  RG

$s$ : total scaling power +  $\sum_{m=1}^N v_m(\xi_1, \xi_2) \ln^m \lambda$

$\hookrightarrow$  dimensional analysis

### ③ OPERATOR-PRODUCT-EXPANSION

There exist local covariant expressions  $C^{AB}_c(x_1, x_2)$  that are distributions w/ wave-front-set property of the form

$$C^{AB}{}_C = \sum P^{AB}{}_C (\sigma) \cdot \sigma_+^{-p} \cdot (\ln \sigma_+)^q$$

↑                      ↑                      ↑  
 polynomials      geodesic distance      suitable  
 in curvature      between pt.'s  $x_1, x_2$       iE-prescription  
 tensors & couplings

such that OPE holds at short distances

$$O^A(x_1)_{S'} \cdot O^B(x_2)_{S'} \underset{\substack{\uparrow \\ \text{asymptotic expansion for} \\ \text{coinciding pt's}}}{\sim} \sum_G C^{AB}_G(x_1, x_2) O^G(x_2)$$

Generalization to multi-point products is also possible, but coefficients have more complicated form.

# MICROLOCAL STRUCTURE

## CONDITION:

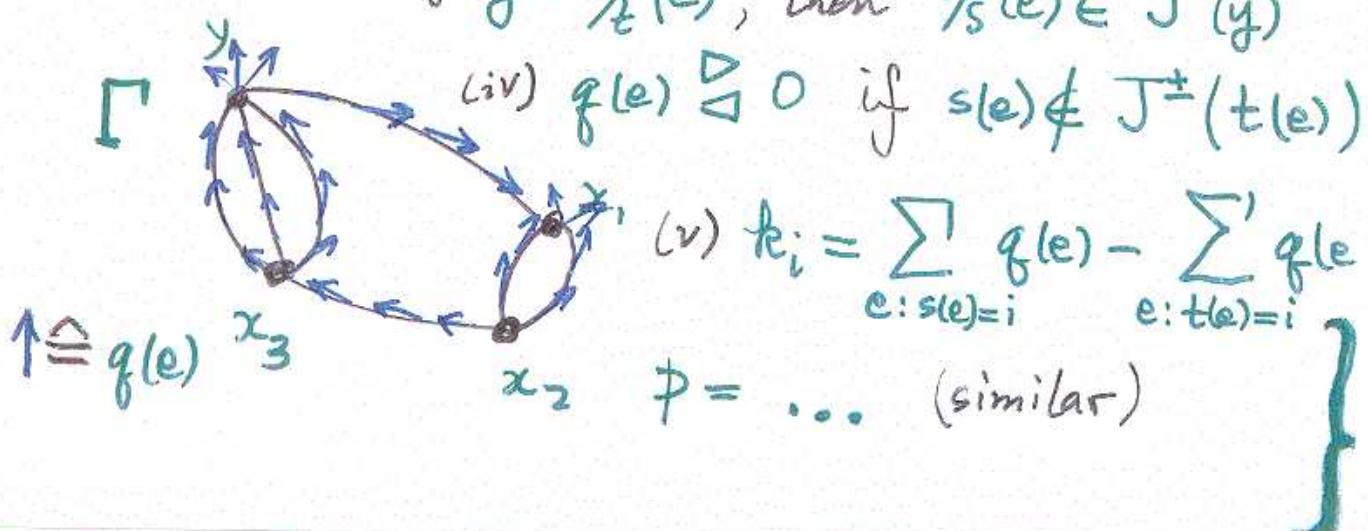
In quantum field theory, operator products are singular, but singularities have specific general "character". For example, retarded products  $R_n(O^{A_1}(x_1) \cdots O^{A_n}(x_n); O^B(y))$  must have following singularities:

$$WF(R_n) \subseteq \left\{ (x_1, \dots, x_n, y; k_1, \dots, k_n, p) \in T(M^{n+1}) \right|$$

Exists associated Feynman graph  $\Gamma \subset M$

- s.t.
- (i) edges  $e$  are null curves
  - (ii) edges carry coparallel covector fields  $g(e)$  [w.r.t. metric]
  - (iii) If  $y = s_t(e)$ , then  $t/s(e) \in J^+(y)$

(iv)  $g(e) \neq 0$  if  $s(e) \notin J^\pm(t(e))$



# BIGGEST OPEN PROBLEM :

How to go beyond perturbation theory

(formal power series  $\mathbb{C}[[g_i]] \rightsquigarrow$   
Convergent power series )

Might perturbation series converge?

Usual arguments against convergence:

- # Feynman diagrams grows too rapidly.
- Explicit counter examples (anharmonic oscillator in QM).
- Perturbation series defined irresp. of sign of potential  $\nmid$  Spectrum condition.

All these issues are related to convergence of quantities related to states as opposed to algebraic relations. Those might still converge.

Conjecture: Perturbation series for

$c_C^{AB}(x) \in \mathcal{D}(M)[[g_i]]$  converges in asymptotically free gauge theories.

# CONCLUSIONS

- Perturbative construction of interacting field theories has been achieved
  - Renormalization Group
  - Operator - Product - Expansion
  - Energy momentum tensor is conserved
- } have been constructed

## In the future :

- Apply formalism to quantum gauge theories in early Universe.
- More "algebraic" computational schemes  
"perturbative Vertex Algebras"
- Mathematics behind perturbation theory :  
(Fourier Integral operators, Number Theory...)