Practicalities of renormalizing quantum field theories

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Outline

- Background - basic theory ϕ^4; statement of problem (renormalizable quantum field theories)
- Calculational techniques - integration methods
- Automatic computations - use of computer algebra
- Checks - internal consistency checks, relation to large N_f critical exponents at Wilson-Fisher fixed points
- Recent results - scheme issues, renormalization of quantum chromodynamics (QCD)
- Renormalization in relation to non-locality

Statement of problem

- ϕ^4 theory

$$L = \frac{1}{2}(\partial\phi_o)^2 + \frac{g_o}{4!}\phi_o^4 \quad , \quad L = \frac{1}{2}Z_\phi(\partial\phi)^2 + \frac{g}{4!}Z_\phi^2 Z_g \phi^4$$

- Since ϕ^4 theory is not finite, require renormalization constants Z_ϕ and Z_g which relate bare and renormalized objects; $\phi_o = \sqrt{Z_\phi}\phi$ and $g_o = Z_g g$
- Renormalization constants are encoded in the renormalization group functions $\gamma_\phi(g)$, $\beta(g), \ldots$
- These lead to understanding of the structure of the quantum theory such as asymptotic freedom in QCD from running of the coupling constant or determination of critical exponents at fixed points, defined as zeroes of the β-function
- As theory is not finite one needs a mathematical way of handling the infinities
- Moreover this procedure must be consistent with the symmetries of the theory

- For gauge theories and theories with supersymmetry the consequences for the renormalization of the field theories can be determined by methods such as algebraic renormalization [Sorella et al]
- For example alegbraic renormalization determines the form of the renormalization constants which are consistent with the Slavnov-Taylor identities
- Infinities require mathematical procedure or regularization to quantify them
- Dimensional regularization: set space-time dimension to be d where $d = 4 - 2\epsilon$, $\epsilon > 0$ and $g_o = Z_g g \mu^{2\epsilon}$ with μ an arbitrary mass scale
- Lattice regularization: discretize space-time which introduces a scale a which is the size of the lattice spacing
- Physics independent of the arbitrary scale μ or a which leads to the renormalization group
- For a Green's function $\Gamma^{(n)}(\mu, g, \ldots)$ one requires

$$\mu \frac{d}{d\mu} \Gamma^{(n)}(\mu, g, \ldots) \;=\; 0$$

- Will primarily concentrate on ultraviolet infinities here; infrared infinities are regularized when one has a non-zero mass

Dimensional regularization

- Renormalization constants depend on ϵ and involve simple poles in ϵ
- To have a finite theory need to remove these poles systematically and consistently
- Requires specification of method of removal or subtraction - various standard schemes to achieve this
- Minimal Subtraction ($\overline{\text{MS}}$) - remove only the poles in ϵ (and common factor of $4\pi e^{-\gamma}$) corresponds to a mass independent renormalization scheme
- Mass dependent renormalization schemes - subtraction removes a finite part in addition to the poles where the particular finite part derives from say putting external particles on their mass shell; MOM or on-shell schemes
- RI$'$ (modified regularization invariant) scheme which is used in lattice computations and is a cross between $\overline{\text{MS}}$ and MOM but is a mass dependent scheme
- Whatever regularization and scheme one uses they must be compatible with the symmetries of the underlying quantum field theory such as gauge symmetry and supersymmetry
- Ultimately physical quantities are scheme independent

Calculational methods

- Need methods to extract regularized divergences from Feynman integrals at one loop and higher
- One loop standard tool is to use Feynman parameters based on

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + (1-x)b]^2}$$

where $a = k^2 - m_1^2$ and $b = (k-p)^2 - m_2^2$

- Not practical at higher loops as presence of one or more mass scales complicates integration
- Develop different approach primarily tailored to the problem on interest

Massive integrals

- Can renormalize massive field theories and extract divergences by expanding in vacuum bubbles (ie no external momenta)

$$\int_k \frac{1}{[k^2 - m^2][(k-p)^2 - m^2]} = \int_k \frac{1}{[k^2 - m^2]^2} - \int_k \frac{p^2}{[k^2 - m^2]^3} + \frac{4}{d}\int_k \frac{k^2 p^2}{[k^2 - m^2]^4} + \ldots$$

- Truncation from renormalizability criterion
- For bosonic 2-point function $O\left((p^2)^2\right)$ terms will be finite in renormalizable theories
- In four dimensions three loop single mass scale vacuum bubbles known to their finite part; two loop three mass scale vacuum bubbles known to the finite part
- More recently new technique for two loop four-point boxes has been developed by Smirnov et al based on representing the propagator by a Mellin-Barnes integral and using contour integration to evaluate the Feynman integral
- No infrared problems

Massless integrals

- Extend vacuum bubble expansion technique to massless case, [Chetyrkin et al], by introducing fictitous mass $\bar{\mu}$ which acts as an infrared regulator
- Based on the iteration of the identity

$$\frac{1}{(k-p)^2} = \frac{1}{[k^2-\bar{\mu}^2]} + \frac{[2kp - p^2 - \bar{\mu}^2]}{(k-p)^2[k^2-\bar{\mu}^2]}$$

- In a particular Feynman graph truncate expansion by applying Weinberg's criterion for finiteness of an integral
- Another technique is integration by parts based on the identity

$$0 = \int \frac{d^d k}{(2\pi)^d} \frac{\partial}{\partial k^\mu} \left[k^\mu I(p, k, ...) \right]$$

where I is the integrand derived from the propagators and vertices

- For massless theories the rule produces powers of k^2 which cancel denominator propagators and reduces Feynman diagrams to a simpler integration topology
- MINCER algorithm constructed on this principle [Chetyrkin et al]

MINCER

- Package for evaluating massless 2-point functions in dimensional regularization to their finite parts [Chetyrkin et al]
- Encoded in symbolic manipulation language FORM and freely available
- At three loops there are 14 basic independent integration topologies each with their own integration by parts routine
- Can be used for higher point functions but only in cases where external momenta are nullified to reduce to a 2-point function **and** where this does not introduce spurious infrared singularities

$$\int_k \frac{1}{(k^2)^2}$$

- Nullification can be systematically dealt with by using infrared rearrangement [Vladimirov et al]
- MINCER widely used for renormalization of four dimensional gauge theories including QCD

Automatic calculations

- Higher loop calculations involve huge number of Feynman diagrams
- For example, three loop QCD fixed in the maximal abelian gauge involves 37322 Feynman diagrams to extract all wave function anomalous dimensions and the β-function; four loop QCD β-function in linear covariant gauge requires $O(50000)$ diagrams [Vermaseren et al]
- Such calculations require computer tools involving MINCER or new packages for four loops
- Diagrams automatically generated by QGRAF package and converted to MINCER input form
- Extraction of renormalization constants is the final stage
- Traditional approach is by method of subtractions where the absolute divergence of a diagram is determined by subtracting all subgraph divergences
- Alternative is to calculate completely as a function of *bare* parameters
- Counterterms introduced naturally (and equivalently to subtraction method) by rescaling with the renormalization constants $g_o = Z_g g$ etc
- Amenable to automatic computer algebra programmes

Qgraf

- Example of QGRAF diagram - three loop gluon 2-point function of the benz topology

```
*--#[ dthree276:
*
     1/2
    *vx(AA(-1),AA(1),AA(2))
    *vx(AA(-3),AA(3),AA(4))
    *vx(AA(1),AA(3),AA(5))
    *vx(AA(2),AA(5),AA(6))
    *vx(AA(4),AA(7),AA(8))
    *vx(AA(6),AA(7),AA(8));
#define CHOICE "7"
#define TOPOLOGY "be"
*
*--#] dthree276:
```

Massless integrals - continued

- Recently a new strategy has been developed for n-point functions at two loop for cases where the external momenta are non-zero
- Differential equation method of Remiddi el al
- Basic idea to derive the complete set of differential equations from the loop integrals at a particular loop order and solve them using master integrals as boundary conditions
- Another approach is that of conformal integration in d-dimensions called *uniqueness*
- When sum of the powers of the propagators round a momentum loop equals the spacetime dimension d then the loop integration can be performed to give the answer as the product of Γ-functions whose arguments depend on the exponents of the massless propagators
- More appropriate for large N_f methods when one computes in the neighbourhood of a fixed point of the β-function and where there is a scaling symmetry

Checks

- Usual internal consistency checks from the renormalization group structure of renormalization constants

$$
\begin{aligned}
\beta(g) &= (d-4)g + Ag^2 + Bg^3 + Cg^4 + O(g^5) \\
\gamma_\phi(g) &= ag + bg^2 + cg^3 + O(g^4)
\end{aligned}
$$

where $\{A, B, C, \ldots\}$ and $\{a, b, c, \ldots\}$ are independent of d

$$
\begin{aligned}
Z_g &= 1 - \frac{Ag}{(d-4)} + \left(\frac{A^2}{(d-4)^2} - \frac{B}{2(d-4)} \right) g^2 \\
&+ \left[-\frac{A^3}{(d-4)^3} + \frac{7AB}{6(d-4)^2} - \frac{C}{3(d-4)} \right] g^3 + O(g^4) \\
Z_\phi &= 1 + \frac{ag}{(d-4)} + \left(\frac{(a^2 - aA)}{2(d-4)^2} + \frac{b}{2(d-4)} \right) g^2 \\
&+ \left[\frac{(2aA^2 - 3a^2A + a^3)}{6(d-4)^3} + \frac{(3ab - 2aB - 2bA)}{6(d-4)^2} - \frac{c}{3(d-4)} \right] g^3 \\
&+ O(g^4)
\end{aligned}
$$

- This check is important for automatic calculations
- Renormalization group functions in mass independent ($\overline{\mathrm{MS}}$) renormalization scheme depend only on a_{n1} - the residue of the simple poles in ϵ
- This leaves a_{n1} unchecked
- Other checks potentially available from symmetries of underlying field theory - supersymmetry and/or gauge symmetry
- For example quantities of a physical nature have gauge independent renormalization group functions in mass independent schemes
- Partial checks available from large N critical point method [Vasil'ev et al]

$O(N)\ \phi^4$

- $\overline{\mathrm{MS}}$ β-function in d-dimensions defines non-trivial d-dimensional Wilson-Fisher fixed point g_c given by $\beta(g_c) = 0$ where

$$\begin{aligned}\beta(g) &= (d-4)\frac{g}{2} + [N+8]\frac{g^2}{6} - [3N+14]\frac{g^3}{6} \\ &\quad + \left[33N^2 + 922N + 2960 + 96(5N+22)\zeta(3)\right] \frac{g^4}{432} + O(g^5)\end{aligned}$$

- In $d = 4 - 2\epsilon$ the critical coupling g_c depends on ϵ and N
- Expanding in powers of $1/N$ as $N \to \infty$

$$g_c = \frac{6\epsilon}{N} + \left[-48\epsilon + 108\epsilon^2 - 99\epsilon^3 + O(\epsilon^4)\right] \frac{1}{N^2} + O\left(\frac{1}{N^3}\right)$$

- Critical exponent $\beta'(g_c)$ is scheme independent object

$$\beta'(g_c) = -\epsilon + \left[18\epsilon^2 - 33\epsilon^3 - \frac{5}{2}\epsilon^4 + O(\epsilon^5)\right] \frac{1}{N} + O\left(\frac{1}{N^2}\right)$$

- If one can compute $\beta'(g_c)$ in d-dimensions via $1/N$ expansion methods then coefficients of the polynomial in N in original β-function can be read off directly
- Vasil'ev et al provided technique of large N d-dimensional critical point analysis
- Exploits d-dimensional Wilson-Fisher fixed point equivalence of $O(N)$ ϕ^4 in $d = 4 - 2\epsilon$ and $O(N)$ non-linear σ model in $d = 2 + \varepsilon$
- First three terms in $1/N$ series for exponents η, ν and ω known
- Allows one to predict higher order coefficients

- For wave function renormalization set ($e_1 \equiv 0$)
  $\gamma(g) = \sum_{r=1}^{\infty}(c_r N^2 + d_r N + e_r) N^{r-2} g^{r+1}$
- Hence with $U_{6,2} = \sum_{n>m>0}^{\infty} \frac{(-1)^{n-m}}{n^6 m^2}$

$$
\begin{aligned}
e_9 \quad = \quad & [1560674304\zeta(10) \;-\; 12534896640\zeta(9) \;+\; 11070010560\zeta(8) \\
& + 1732018176\zeta(7)\zeta(3) \;+\; 581961984\zeta(7) \;-\; 3411394560\zeta(6)\zeta(3) \\
& - 2684240640\zeta(6) \;+\; 209534976\zeta^2(5) \;-\; 1567752192\zeta(5)\zeta(4) \\
& + 1754664960\zeta(5)\zeta(3) \;-\; 975533568\zeta(5) \;-\; 9289728\zeta(4)\zeta^2(3) \\
& + 1310201856\zeta(4)\zeta(3) \;+\; 1636615872\zeta(4) \;-\; 137158656\zeta^3(3) \\
& - 1708996608\zeta^2(3) \;+\; 294403968\zeta(3) \\
& - 89800704U_{62} \;-\; 341350433]/1950396973056
\end{aligned}
$$

- $U_{6,2}$ is knot number associated with $(3,4)$ torus knot

Large N_f QCD

- Similar method but exponents are known to fewer orders in $1/N_f$
- ω at $O(1/N_f)$ and quark wave function and mass anomalous dimensions at $O(1/N_f^2)$ in d-dimensions
- N_f is number of quark flavours; N_c is number of colours
- At analogous d-dimensional Wilson fixed point QCD is equivalent to non-abelian Thirring model (NATM) $1 \leq i \leq N_f$, $1 \leq I \leq N_F$, $1 \leq a \leq N_A$

$$L = i\bar{\psi}^{iI}\partial\!\!\!/\psi^{iI} + \frac{\lambda^2}{2}\left(\bar{\psi}^{iI}T^a_{IJ}\gamma^\mu\psi^{iJ}\right)^2$$

- Or

$$L = i\bar{\psi}^{iI}\partial\!\!\!/\psi^{iI} + A^a_\mu\bar{\psi}^{iI}T^a_{IJ}\gamma^\mu\psi^{iJ} - \frac{A^{a\,2}_\mu}{2\lambda}$$

$$L^{\mathrm{QCD}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu} - \frac{1}{2\alpha}(\partial^\mu A^a_\mu)^2 - \bar{c}^a\partial^\mu D_\mu c^a + i\bar{\psi}^{iI}D\!\!\!\!/\psi^{iI}$$

- Quark loops in NATM reproduce the triple and quartic gluon interactions at criticality [Hasenfratz and Hasenfratz]

QCD β-function in large N_f

- Compute related critical exponent $\omega = -\beta'(g_c)$ at d-dimensional Wilson-Fisher fixed point
- Insert (gauge invariant) operator $\mathcal{O} = G^a_{\mu\nu} G^{a\,\mu\nu}$ in gluon 2-point function using NATM Lagrangian
- Anomalous dimension of this operator relates to critical renormalization group function of the associated coupling of the operator
- Massless propagators in critical region are

$$\psi(k) \sim \frac{A\not{k}}{(k^2)^{\mu-\alpha}} \quad , \quad A_{\mu\nu}(k) \sim \frac{B}{(k^2)^{\mu-\beta}} \left[\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right]$$

where $\alpha = \mu + \frac{1}{2}\eta$, $\beta = 1 - \eta - \chi$ and $d = 2\mu$

- Canonical dimensions derived from d-dimensional Lagrangian

- Relevant Feynman diagrams at each order in large N_f perturbation theory derived by noting that quark loop is $O(N_f)$, $A = O(1)$ and $B = O(1/N_f)$
- Regularization is introduced by shifting the quark-gluon anomalous dimension $\beta \to \beta - \Delta$
- Poles in Δ are subtracted in a minimal way into a renormalization constant which determines the critical exponent
- Requires the computation of massless critical Feynman diagrams using uniqueness

$$\begin{aligned}\omega &= (\mu - 2) \; - \; \left[(2\mu - 3)(\mu - 3) C_2(R) \right. \\ &\quad \left. - \; \frac{(4\mu^4 - 18\mu^3 + 44\mu^2 - 45\mu + 14) C_2(G)}{4(2\mu - 1)(\mu - 1)} \right] \frac{\eta_1^{\mathrm{o}}}{T(R) N_f}\end{aligned}$$

where $\eta^{\mathrm{o}} = (2\mu - 1)(\mu - 2)\Gamma(2\mu)/[4\Gamma^2(\mu)\Gamma(\mu + 1)\Gamma(2 - \mu)]$

- New coefficients

$$\begin{aligned}a_4 &= -\,[154 C_F + 53 C_A]/3888 \\ a_5 &= [(288\zeta(3) + 214) C_F + (480\zeta(3) - 229) C_A]/31104\end{aligned}$$

Large N_f NATM

- Repeat exercise for NATM with analogous operator $\frac{1}{2} A^a_\mu A^{a\,\mu}$
- Insert operator into gluon 2-point function
- Gauge variant operator but compute in Landau gauge

$$\eta_{A^2} ~=~ - ~ \frac{C_A \eta^0_1}{4(\mu-2) T_F N_f} ~+~ O \left(\frac{1}{N^2_f} \right)$$

- Interesting structure

$$\gamma_{A^2}(g_c) ~=~ \gamma_A(g_c) ~+~ \gamma_c(g_c)$$

  in all dimensions at $O(1/N_f)$
- In four dimensions $\gamma_{A^2}(g) = \gamma_A(g) + \gamma_c(g)$ to all orders in perturbation theory in Landau gauge
- Shown first explicitly at three loops in $\overline{\mathrm{MS}}$ computation

- Properties of $\frac{1}{2} A^a_\mu A^{a\,\mu}$ subject of intense study for potential relation to an effective gluon mass in various gauges
- In maximal abelian gauge (MAG) there is a similar renormalization structure for analogous dimension two operator involving off-diagonal fields

$$A^A_\mu T^A ~=~ A^a_\mu T^a ~+~ A^i_\mu T^i$$

where $[T^i, T^j] = 0$, $T^i \in \{\text{centre}\}$

$$\mathcal{O} ~=~ \frac{1}{2} A^a_\mu A^{a\,\mu} ~+~ \alpha \bar{c}^a c^a$$

- Comparable all orders result is

$$\gamma_{\mathcal{O}}(g) ~=~ \gamma_{A^i}(g) ~-\gamma_{c^i}(g)$$

MAG

- Nonlinear gauge fixing where gauge field is split into diagonal (centre) and off-diagonal components $A^A_\mu T^A = A^a_\mu T^a + A^i_\mu T^i$ with $1 \leq a \leq N^o_A$ (off-diagonal), $1 \leq i \leq N^d_A$ (centre)
- Gauge fixing is via a modification of covariant gauge fixing procedure

$$
\begin{aligned}
L^{\mathrm{Landau}}_{\mathrm{gf}} &= \delta\bar{\delta}\left[\frac{1}{2}A^A_\mu A^{A\,\mu} + \frac{1}{2}\alpha\bar{c}^A c^A\right] \\
L^{\mathrm{MAG}}_{\mathrm{gf}} &= \delta\bar{\delta}\left[\frac{1}{2}A^a_\mu A^{a\,\mu} + \frac{1}{2}\alpha\bar{c}^a c^a\right] + \delta\left[\bar{c}^i \partial^\mu A^i_\mu\right]
\end{aligned}
$$

- Algebraic renormalization defines structure of renormalization constants

$$
\begin{aligned}
A^{a\,\mu}_{\mathrm{o}} &= \sqrt{Z_A}\, A^{a\,\mu} \;,\; A^{i\,\mu}_{\mathrm{o}} = \sqrt{Z_{A^i}}\, A^{i\,\mu} \\
c^a_{\mathrm{o}} &= \sqrt{Z_c}\, c^a \;,\; \bar{c}^a_{\mathrm{o}} = \sqrt{Z_c}\, \bar{c}^a \\
c^i_{\mathrm{o}} &= \sqrt{Z_{c^i}}\, c^i \;,\; \bar{c}^i_{\mathrm{o}} = \frac{\bar{c}^i}{\sqrt{Z_{c^i}}} \;,\; \psi_{\mathrm{o}} = \sqrt{Z_\psi}\psi \;, \\
g_{\mathrm{o}} &= \mu^\epsilon Z_g\, g \;,\; \alpha_{\mathrm{o}} = Z^{-1}_\alpha Z_A\, \alpha \,,\; \bar{\alpha}_{\mathrm{o}} = Z^{-1}_{\alpha^i} Z_{A^i}\, \bar{\alpha}
\end{aligned}
$$

Gauge fixed Lagrangian

$$
\begin{aligned}
L_{\mathrm{gf}} \quad = \quad & -\frac{1}{2\alpha}\left(\partial^\mu A^a_\mu\right)^2 - \frac{1}{2\bar{\alpha}}\left(\partial^\mu A^i_\mu\right)^2 + \bar{c}^a \partial^\mu \partial_\mu c^a + \bar{c}^i \partial^\mu \partial_\mu c^i \\
& + g\Big[f^{abk} A^a_\mu \bar{c}^k \partial^\mu c^b - f^{abc} A^a_\mu \bar{c}^b \partial^\mu c^c \\
& \quad - \frac{1}{\alpha} f^{abk} \partial^\mu A^a_\mu A^b_\nu A^{k\,\nu} - f^{abk} \partial^\mu A^a_\mu c^b \bar{c}^k \\
& \quad - \frac{1}{2} f^{abc} \partial^\mu A^a_\mu \bar{c}^b c^c - 2 f^{abk} A^k_\mu \bar{c}^a \partial^\mu \bar{c}^b - f^{abk} \partial^\mu A^k_\mu \bar{c}^b c^c \Big] \\
& + g^2 \Big[f^{acbd}_d A^a_\mu A^{b\,\mu} \bar{c}^c c^d - \frac{1}{2\alpha} f^{akbl}_o A^a_\mu A^{b\,\mu} A^k_\nu A^{l\,\nu} \\
& \quad + f^{adcj}_o A^a_\mu A^{j\,\mu} \bar{c}^c c^d - \frac{1}{2} f^{ajcd}_o A^a_\mu A^{j\,\mu} \bar{c}^c c^d + f^{ajcl}_o A^a_\mu A^{j\,\mu} \bar{c}^c c^l \\
& \quad + f^{alcj}_o A^a_\mu A^{j\,\mu} \bar{c}^c c^l - f^{cjdi}_o A^i_\mu A^{j\,\mu} \bar{c}^c c^d - \frac{\alpha}{4} f^{abcd}_d \bar{c}^a \bar{c}^b c^c c^d \\
& \quad - \frac{\alpha}{8} f^{abcd}_o \bar{c}^a \bar{c}^b c^c c^d + \frac{\alpha}{8} f^{acbd}_o \bar{c}^a \bar{c}^b c^c c^d - \frac{\alpha}{4} f^{abcl}_o \bar{c}^a \bar{c}^b c^c c^l \\
& \quad + \frac{\alpha}{4} f^{acbl}_o \bar{c}^a \bar{c}^b c^c c^l - \frac{\alpha}{4} f^{albc}_o \bar{c}^a \bar{c}^b c^c c^l + \frac{\alpha}{2} f^{akbl}_o \bar{c}^a \bar{c}^b c^k c^l \Big]
\end{aligned}
$$

- $f^{ABCD}_d = f^{ABi} f^{CDi}$, $f^{ABCD}_o = f^{ABe} f^{CDe}$

- Renormalize 2-point functions to deduce wave function anomalous dimensions, except for c^i
- β-function emerges from A^i 2-point renormalization; similar to background field gauge renormalization
- Deduce Z_{c^i} from renormalization of $A^a_\mu \bar{c}^i c^b$ vertex
- Deduce $\gamma_{\mathcal{O}}(a)$ from these anomalous dimensions and Slavnov-Taylor identity
- Extensive use of computer algebra using MINCER algorithm
- Feynman rules generated *automatically* from the gauge fixing term
- The off-diagonal sector of the MAG corresponds to QCD fixed in the (nonlinear) Curci-Ferrari gauge
- Group theory based on

$$f^{ijk} = 0 \, , \; f^{ijc} = 0 \, , \; f^{ibc} \neq 0 \, , \; f^{abc} \neq 0$$

Diagram count

Green's function	One loop	Two loop	Three loop	Total
$A^a_\mu \, A^b_\nu$	6	131	6590	6727
$A^i_\mu \, A^j_\nu$	3	54	2527	2584
$c^a \, \bar{c}^b$	3	81	4006	4090
$\psi \, \bar{\psi}$	2	27	979	1008
$A^a_\mu \, \bar{c}^i \, c^b$	5	287	22621	22913
Total	19	580	36723	37322

Number of Feynman diagrams for each Green's function for the MAG renormalization.

Green's function	One loop	Two loop	Three loop	Total
$A^A_\mu \, A^B_\nu$	3	19	282	304
$c^A \, \bar{c}^B$	1	9	124	134
$\psi \, \bar{\psi}$	1	6	79	86
$A^A_\mu \bar{\psi} \, \psi$	2	33	697	732
Total	7	67	1182	1256

Number of Feynman diagrams for each Green's function for the CF gauge renormalization.

$\overline{\text{MS}}$ results

$$
\begin{aligned}
\gamma_{c^i}(a) \quad = \quad & \frac{1}{4N_A^o}\left[N_A^o\left((-\alpha-3)C_A\right)+N_A^d\left((-2\alpha-6)C_A\right)\right] a \\
& + \frac{1}{96{N_A^o}^2}\left[{N_A^o}^2\left((-6\alpha^2-66\alpha-190)C_A^2+80C_AT_FN_f\right)\right. \\
& + N_A^oN_A^d\left((-54\alpha^2-354\alpha-323)C_A^2+160C_AT_FN_f\right) \\
& \left. + {N_A^d}^2\left((-60\alpha^2-372\alpha+510)C_A^2\right)\right] a^2 \\
& + \frac{1}{6912{N_A^o}^3}\left[{N_A^o}^3((-162\alpha^3-2727\alpha^2-2592\zeta_3\alpha-18036\alpha\right. \\
& - 1944\zeta_3-63268)C_A^3+(6912\alpha+62208\zeta_3+6208)C_A^2T_FN_f \\
& + (-82944\zeta_3+77760)C_AC_FT_FN_f+8960C_AT_F^2N_f^2) \\
& + {N_A^o}^2N_A^d((-2754\alpha^3+648\zeta_3\alpha^2-28917\alpha^2-4212\zeta_3\alpha-69309\alpha \\
& + 37260\zeta_3-64544)C_A^3+(25488\alpha+103680\zeta_3-13072)C_A^2T_FN_f \\
& + (-165888\zeta_3+155520)C_AC_FT_FN_f+17920C_AT_F^2N_f^2) \\
& + N_A^o{N_A^d}^2((-7884\alpha^3+22680\zeta_3\alpha^2-84564\alpha^2+97524\zeta_3\alpha-47142\alpha \\
& + 433836\zeta_3-56430)C_A^3+(25056\alpha-124416\zeta_3-18144)C_A^2T_FN_f) \\
& + {N_A^d}^3((-6480\alpha^3+34992\zeta_3\alpha^2-70092\alpha^2+8424\zeta_3\alpha \\
& \left. + 114912\alpha+77112\zeta_3-161028)C_A^3)\right] a^3 \; + \; O(a^4)
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\mathcal{O}}(a) \quad = \quad & \frac{1}{12N_A^o}\left[N_A^o\left((-3\alpha+35)C_A-16T_fN_f\right)+N_A^d\left((-6\alpha-18)C_A\right)\right]a \\
& + \frac{1}{96{N_A^o}^2}\left[{N_A^o}^2\left((-6\alpha^2-66\alpha+898)C_A^2-560C_AT_fN_f\right.\right. \\
& \left. -384C_FT_fN_f\right)+\ N_A^oN_A^d\left((-54\alpha^2-354\alpha-323)C_A^2+160C_AT_fN_f\right) \\
& \left. +{N_A^d}^2\left((-60\alpha^2-372\alpha+510)C_A^2\right)\right]a^2 \\
& + \frac{1}{6912{N_A^o}^3}\left[{N_A^o}^3((-162\alpha^3-2727\alpha^2-2592\zeta_3\alpha-18036\alpha-1944\zeta_3\right. \\
& +302428)C_A^3+(6912\alpha+62208\zeta_3-356032)C_A^2T_FN_f+(-82944\zeta_3 \\
& -79680)C_AC_FT_FN_f+49408C_AT_F^2N_f^2+13824C_F^2T_FN_f \\
& +33792C_FT_F^2N_f^2)+{N_A^o}^2N_A^d((-2754\alpha^3+648\alpha^2\zeta_3-28917\alpha^2 \\
& -4212\alpha\zeta_3-69309\alpha+37260\zeta_3-64544)C_A^3+(25488\alpha+103680\zeta_3 \\
& -13072)C_A^2T_FN_f+(-165888\zeta_3+155520)C_AC_FT_FN_f \\
& +17920C_AT_F^2N_f^2)+N_A^o{N_A^d}^2((-7884\alpha^3+22680\alpha^2\zeta_3-84564\alpha^2 \\
& +97524\alpha\zeta_3-47142\alpha+433836\zeta_3-56430)C_A^3+(25056\alpha-124416\zeta_3 \\
& -18144)C_A^2T_FN_f)+{N_A^d}^3((-6480\alpha^3+34992\alpha^2\zeta_3-70092\alpha^2 \\
& \left.+8424\alpha\zeta_3+114912\alpha+77112\zeta_3-161028)C_A^3)\right]a^3\ +\ O(a^4)
\end{aligned}
$$

- Structure of renormalization constants not inconsistent with renormalization group
- Repeated Landau gauge results with split algebra code prior to renormalizing MAG
- Correct $\overline{\text{MS}}$ β-function emerges at three loops from $\gamma_{A^i}(a)$ - independent of α, N^d_A and N^o_A
- Curci-Ferrari anomalous dimensions correctly emerge for the off-diagonal sector in the limit $N^d_A/N^o_A \rightarrow 0$; difficult to see in $SU(N_c)$
- Correct abelian limit emerges $\forall$ N^d_A and N^o_A
- MAG anomalous dimensions cannot be deduced from Landau or CF gauge results

Gribov problem and QCD

- In non-abelian gauge theory there is a problem fixing the gauge globally [Gribov]

$$Z \; = \; \int \mathcal{D}A \, \delta(\partial^\mu A^a_\mu) \, \det\left(-\partial^\nu D^a_\nu\right) \; e^{-S}$$

- Gribov introduced a dimensional parameter γ such that γ^4 is effectively the volume of the region defined by the first zero of the Faddeev-Popov operator
- γ is not independent and satisfies the (Gribov) mass gap equation
- Zwanziger localized Gribov's non-local formulation of the problem into a renormalizable Lagrangian

$$\begin{aligned} L^Z \quad &= \quad L^{\text{QCD}} \, + \, \bar{\phi}^{ab\,\mu} \partial^\nu \left(D_\nu \phi_\mu \right)^{ab} \, - \, \bar{\omega}^{ab\,\mu} \partial^\nu \left(D_\nu \omega_\mu \right)^{ab} \\ & \quad - g f^{abc} \partial^\nu \bar{\omega}^{ae} \left(D_\nu c \right)^b \phi^{ec\,\mu} \\ & \quad - \frac{\gamma^2}{\sqrt{2}} \left(f^{abc} A^{a\,\mu} \phi^{bc}_\mu \, + \, f^{abc} A^{a\,\mu} \bar{\phi}^{bc}_\mu \right) \, - \, \frac{d N_A \gamma^4}{2g^2} \end{aligned}$$

- Involves new ghost fields: $\{\phi, \bar{\phi}\}$ and $\{\omega, \bar{\omega}\}$

Gribov-Zwanziger renormalization

- The renormalization properties of the Zwanziger's extra (ghost) fields, ϕ and ω, have been constructed and have interesting properties
- $Z_\phi = Z_\omega = Z_c$ in the Landau gauge
- Moreover $Z_\phi = Z_\omega = Z_c$ in arbitrary linear covariant gauge by explicit calculation at two loops

$$
\begin{aligned}
\gamma_\gamma(a) \quad &= \quad (16 T_F N_f - (35 + 3\alpha) C_A) \frac{a}{48} \\
&\quad + (192 C_F T_F N_f + 280 C_A T_F N_f - (449 - 3\alpha)) \frac{a^2}{192} \; + \; O(a^3)
\end{aligned}
$$

- $\gamma_\gamma(a)$ in the Landau gauge is not independent since

$$
\gamma_\gamma(a) \; = \; \frac{1}{4} \left(\gamma_A(a) \; + \; \gamma_c(a) \right)
$$

- Similar to the renormalization of the dimension two operator $\frac{1}{2} A^{a\,\mu} A^a_\mu$

$$
\gamma_{A^2}(a) \; = \; \gamma_A(a) \; + \; \gamma_c(a)
$$

Gribov mass gap

- As Zwanziger's Lagrangian is local and renormalizable, it can be used to study problems involving a non-zero γ
- For example, the correction to Gribov's one loop mass gap can be computed
- Zwanziger demonstrated that the Gribov horizon condition leading to the mass gap equation was equivalent to

$$f^{abc}\langle A^{a\,\mu}(x)\phi^{bc}_{\mu}(x)\rangle \;=\; \frac{dN_A\gamma^2}{\sqrt{2}g^2}$$

- With non-zero γ gluon and ghost propagtors are modified $(P_{\mu\nu}(p) = \eta_{\mu\nu} - p_\mu p_\nu/p^2)$

$$\begin{aligned}
\langle A^a_\mu(p)A^b_\nu(-p)\rangle &= -\frac{\delta^{ab}p^2}{[(p^2)^2+C_A\gamma^4]}P_{\mu\nu}(p) \\
\langle A^a_\mu(p)\bar{\phi}^{bc}_\nu(-p)\rangle &= -\frac{f^{abc}\gamma^2}{\sqrt{2}[(p^2)^2+C_A\gamma^4]}P_{\mu\nu}(p) \\
\langle \phi^{ab}_\mu(p)\bar{\phi}^{cd}_\nu(-p)\rangle &= -\frac{\delta^{ac}\delta^{bd}}{p^2}\eta_{\mu\nu} + \frac{f^{abe}f^{cde}\gamma^4}{p^2[(p^2)^2+C_A\gamma^4]}P_{\mu\nu}(p)
\end{aligned}$$

- The presence of the non-zero γ leads to a gluon propagator which is suppressed in the infrared
- Evaluating Zwanziger's condition at two loops leads to the (finite) gap equation where $s_2 = (2\sqrt{3}/9)\mathrm{Cl}_2(2\pi/3)$

$$
\begin{aligned}
1 \quad = \quad & C_A \left[\frac{5}{8} - \frac{3}{8} \ln \left(\frac{C_A \gamma^4}{\mu^4} \right) \right] a \\
& + \left[C_A^2 \left(\frac{2017}{768} - \frac{11097}{2048} s_2 + \frac{95}{256} \zeta(2) - \frac{65}{48} \ln \left(\frac{C_A \gamma^4}{\mu^4} \right) \right. \right. \\
& \qquad \left. + \frac{35}{128} \left(\ln \left(\frac{C_A \gamma^4}{\mu^4} \right) \right)^2 + \frac{1137}{2560} \sqrt{5} \zeta(2) - \frac{205\pi^2}{512} \right) \\
& \quad + C_A T_F N_f \left(- \frac{25}{24} - \zeta(2) + \frac{7}{12} \ln \left(\frac{C_A \gamma^4}{\mu^4} \right) \right. \\
& \quad \left. \left. - \frac{1}{8} \left(\ln \left(\frac{C_A \gamma^4}{\mu^4} \right) \right)^2 + \frac{\pi^2}{8} \right) \right] a^2 + O(a^3)
\end{aligned}
$$

- One loop gap equation also ensures Faddeev-Popov ghost propagator is enhanced in the infrared [Gribov]

- Full ghost propagator is of the form

$$G_c(p^2) \ = \ \frac{\delta^{ab}}{p^2[1 + u(p^2)]}$$

- Kugo-Ojima confinement condition requires ghost enhancement in the infrared which corresponds to $1/(p^2)^2$ behaviour as $p^2 \to 0$ which is equivalent to $u(0) = -\,1$
- Can compute the ghost propagator in the Gribov-Zwanziger Lagrangian and expand to $O(p^2)$ at two loops
- The Kugo-Ojima condition is satisfied at two loops provided the Gribov mass gap condition is used
- Hence ghost enhancement at two loops in the Gribov-Zwanziger Lagrangian
- Gap equation and ghost propagator calculations required use of multi-scale two loop vacuum bubble integrals such as

$$I(m_1^2, m_2^2, m_3^2) \ = \ \int_{kl} \frac{1}{[k^2 - m_1^2][l^2 - m_2^2][(k-l)^2 - m_3^2]}$$

- The quantities s_2 and $\zeta(2)\sqrt{5}$ arise in the finite parts of $I(i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2)$ and $I(i\sqrt{C_A}\gamma^2, i\sqrt{C_A}\gamma^2, -i\sqrt{C_A}\gamma^2)$ respectively

Non-local dimension two operator

- Problem with study of dimension two operator $\frac{1}{2} A^a_\mu A^{a\,\mu}$ is lack of gauge invariance
- First term of two gauge invariant but non-local operators

$$\tilde{A}^2_\mu \;= \min_{\{U\}} \int \left(A^U_\mu\right)^2 \quad \text{and} \quad G^a_{\mu\nu} \frac{1}{D^2} G^{a\,\mu\nu}$$

- Concentrate on latter and repeat Zwanziger localization of operator
- Rewrite

$$L \;=\; L^{\mathrm{gf}} - \frac{m^2}{4} G^a_{\mu\nu} \frac{1}{D^2} G^{a\,\mu\nu}$$

as

$$L \;=\; L^{\mathrm{gf}} + \frac{im}{4} (B^a_{\mu\nu} - \bar{B}^a_{\mu\nu}) G^{a\,\mu\nu} + \frac{1}{4} \bar{B}^a_{\mu\nu} \left(D^\sigma D_\sigma B^{\mu\nu}\right)^a - \frac{1}{4} \bar{H}^a_{\mu\nu} \left(D^\sigma D_\sigma H^{\mu\nu}\right)^a$$

where $\{B^a_{\mu\nu}, \bar{B}^a_{\mu\nu}\}$ and $\{H^a_{\mu\nu}, \bar{H}^a_{\mu\nu}\}$ are localizing ghosts; $H^a_{\mu\nu}$ are anti-commuting

- Lagrangian is renormalizable but not multiplicatively
- Analysing symmetry and stability conditions using algebraic renormalization, new interactions generated

- New quartic interaction $\lambda^{abcd} \left(\bar{B}^a_{\mu\nu} B^{b\,\mu\nu} - \bar{H}^a_{\mu\nu} H^{b\,\mu\nu} \right) \left(\bar{B}^c_{\sigma\rho} B^{d\,\sigma\rho} - \bar{H}^c_{\sigma\rho} H^{d\,\sigma\rho} \right)$
- Mass operator mixes into dimension two operators $\left(\bar{B}^a_{\mu\nu} B^{a\,\mu\nu} - \bar{H}^a_{\mu\nu} H^{a\,\mu\nu} \right)$ and $\left(\bar{B}^a_{\mu\nu} - B^a_{\mu\nu} \right)^2$
- Renormalize massless theory at one loop using MINCER algorithm
- Renormalization group functions of A^a, c^a and ψ and β-function are unchanged

$$\gamma_B(a) \;=\; \gamma_H(a) \;=\; (\alpha - 3) C_A a \;+\; O(a^2)$$

- Compute anomalous dimension of non-local operator $G^a_{\mu\nu} \frac{1}{D^2} G^{a\,\mu\nu}$ by substituting gauge invariant equivalent operator $B^a_{\mu\nu} G^{a\,\mu\nu}$ into B-gluon two-point function
- Compute in theory with massless fields which means there is no mixing; similar to how one computes the quark mass anomalous dimension

$$\gamma_{BG}(a) \;=\; - \left(\frac{11}{6} C_A - \frac{2}{3} T_F N_f \right) a^2 \;+\; O(a^3)$$

- Same one loop anomalous dimension emerges for each of the operators in the set $G^a_{\mu\nu} G^{a\,\mu\nu}$, $D_\mu G^a_{\nu\sigma} D^\mu G^{a\,\nu\sigma}$, $D_\mu D_\nu G^a_{\sigma\rho} D^\mu D^\nu G^{a\,\sigma\rho}$ and $D_\mu D_\nu D_\sigma G^a_{\rho\theta} D^\mu D^\nu D^\sigma G^{a\,\rho\theta}$

Scheme issues

- Lattice regularization of gauge theories allows one to study non-perturbative structure - hadron masses, confinement mechanism, instantons etc
- Requires large numerical calculations using supercomputers
- Hence it is necessary to have an optimal computational strategy, one of which is the choice of renormalization scheme
- One illustrative problem is the extraction of structure function moments underlying deep inelastic scattering and in particular the coupling of the twist-2 non-singlet gauge invariant operators

$$\bar{\psi}\gamma^{\{\mu_1}D^{\mu_2}\ldots D^{\mu_n\}}\psi \quad , \quad D_\mu \;=\; \partial_\mu \;+\; igT^a A^a_\mu$$

  for low moment, n

- Dealing with space-time derivatives on the lattice is computationally complex and adds significantly to the calculation time
- Define new renormalization scheme, RI$'$, regularization invariant
- Compute quantities on the lattice in RI$'$ and then match same quantity computed in continuum in dimensional regularization

RI$'$

- Quark propagator is renormalized as in a mass dependent scheme

$$\lim_{\epsilon \to 0} \left[Z_\psi^{\mathrm{RI}'} \Sigma_\psi(p) \right] \Big|_{p^2 = \mu^2} = \not{p}$$

where $\Sigma_\psi(p)$ is the bare (massless) quark two-point function

- In $\overline{\mathrm{MS}}$ right hand side would be non-unit
- Gluon and ghost wave function renormalization constants defined in similar way
- Coupling constant renormalized as in $\overline{\mathrm{MS}}$
- Need to relate variables of the theory as defined in RI$'$ to those in the standard reference scheme $\overline{\mathrm{MS}}$
- Clearly

$$\alpha' = \frac{Z'_A}{Z_A} \alpha \quad , \quad a' = \frac{Z'_g}{Z_g} a$$

- Leading to following relationship between the variables (known to three loops)

$$\begin{aligned} a' &= a + O(a^5) \\ \alpha' &= \left[1 + \left(\left(-9\alpha^2 - 18\alpha - 97\right) C_A + 80 T_F N_f\right) \frac{a}{36}\right] \alpha + O(a^2) \end{aligned}$$

- The anomalous dimensions have a different form

$$\begin{aligned} \gamma_\psi(a) &= \alpha C_F a + \frac{1}{4}\left[(\alpha^2 + 8\alpha + 25) C_A C_F - 6C_F^2 - 8 C_F T_F N_f\right] a^2 \\ \gamma'_\psi(a) &= \alpha C_F a + \left[\left(9\alpha^3 + 45\alpha^2 + 223\alpha + 225\right) C_A \right. \\ &\qquad \left. - 54 C_F - (80\alpha + 72)\, T_F N_f\right] \frac{C_F a^2}{36} \end{aligned}$$

$n = 2$ moment

- Renormalize operator $\bar{\psi}\gamma^{\{\mu}D^{\nu\}}\psi$ ($n = 2$) in both $\overline{\text{MS}}$ and RI$'$

$$\gamma_{\bar{\psi}\gamma^\mu D^\nu \psi}(a) \;=\; \frac{8}{3}C_F a \;+\; \frac{8C_F}{27}\left[47C_A - 14C_F - 16T_F N_f\right] a^2 \;+\; O(a^3)$$

$$\gamma'_{\bar{\psi}\gamma^\mu D^\nu \psi}(a) \;=\; \frac{8}{3}C_F a \;+\; \frac{C_F}{54}\left[\left(27\alpha^2 + 81\alpha + 1434\right) C_A - 224C_F - 504T_F N_f\right] a^2 \;+\; O(a^3)$$

- Scheme dependence of renormalization group function appears at two loops
- In mass independent renormalization scheme the anomalous dimensions are independent of the gauge parameter
- Lattice matching of finite parts of the Green's functions such as

$$G^{\mu\nu}_{\bar{\psi}\gamma^{\{\mu}D^{\nu\}}\psi}(p) \;=\; \langle\psi(p)\,[\mathcal{S}\bar{\psi}\gamma^{\{\mu}D^{\nu\}}\psi](0)\,\bar{\psi}(-p)\rangle$$

carried out in the Landau gauge ($\alpha = 0$)

Conclusions

- Have given an overview of technical problems with renormalizing quantum field theories
- These include exploiting the properties of the renormalization group equation and critical point theory to probe the higher order structure of the renormalization group functions
- Practical computations require algorithms which can be implemented in symbolic manipulation programmes
- Possibility of studying problems where there is non-locality in non-abelian theories
- Also scheme issues for interfacing with the lattice computations lead to improved extraction of physical quantities