The Renormalisation Group and Self Avoiding Walk

David Brydges, John Imbrie and Gordon Slade

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The problem is to find the asymptotic growth as $N \to \infty$ of the expected end-to-end-distance $\langle |\omega_N| \rangle$.

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In dimension $d = 3$ nothing is known rigorously. Simulations and other methods indicate that $\langle \|\omega_N\| \rangle \sim DN^{\alpha}$ for some $\alpha > \frac{1}{2}$

Four Dimensions

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Small parameter: Walk need not be self-avoiding but weighted so as to suppress self intersections.

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$$\prod_{\text{step } xy \in \omega} (A^{-1})_{x,y}$$

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Key step: Prove that

$$\chi_\beta \sim \left(\hat{\beta}|\log^{1/4}\hat{\beta}|\right)^{-1} \text{ where } \hat{\beta} = (\beta - \beta_c)$$

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$(2\pi)^\Lambda$ absorbed into Lebesgue measure.

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With right choice of F this says “add a step to the walk”.

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$$I_x = \begin{cases} (1+\beta\lambda\tau_x)e^{-\lambda\tau} & \text{for } x \neq a, b \\ \varphi_a, \overline{\varphi}_b & \text{for } x = a, b \end{cases}$$

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$$\chi_\beta = \sum_b \mathbb{E}\left[I^\Lambda\right]$$

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Consequence:

$$\mathbb{E}\big(I^{\Lambda}\big) = \mathbb{E}_{n} \ldots \mathbb{E}_{2}\mathbb{E}_{1}\big(I^{\Lambda}\big)$$

where, in right hand side, $\varphi = \sum_{j} \varphi_{j}$ and likewise $d\varphi$

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where X is summed over all subsets of Λ which are unions of scale $j+1$ disjoint cubes partitioning Λ.

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$I_{j,x}$ depends only on φ_y for y nearest neighbours of x.

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The representation of Z_j by (I_j, K_j) is not unique but can be made unique by imposing a normalisation condition on K_j. Then we have proved, in the hierarchical case, that as $j \to \infty$,

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If $\beta = \beta_c$ for $j = 0$, for $x \neq a, b$,

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Finite range property of decomposition and cubes of side $>$ range implies

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