Exact Solution of the Six-Vertex Model with Domain Wall Boundary Condition. Disordered Phase

Pavel Bleher

Indiana University-Purdue University
Indianapolis

Joint work with **Vladimir Fokin**

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Introduction

The six-vertex model, or the model of two-dimensional ice, is stated on a square lattice with arrows on edges. The arrows obey the rule that at every vertex there are two arrows pointing in and two arrows pointing out. Such rule is sometimes called the ice-rule. There are only six possible configurations of arrows at each vertex, hence the name of the model.

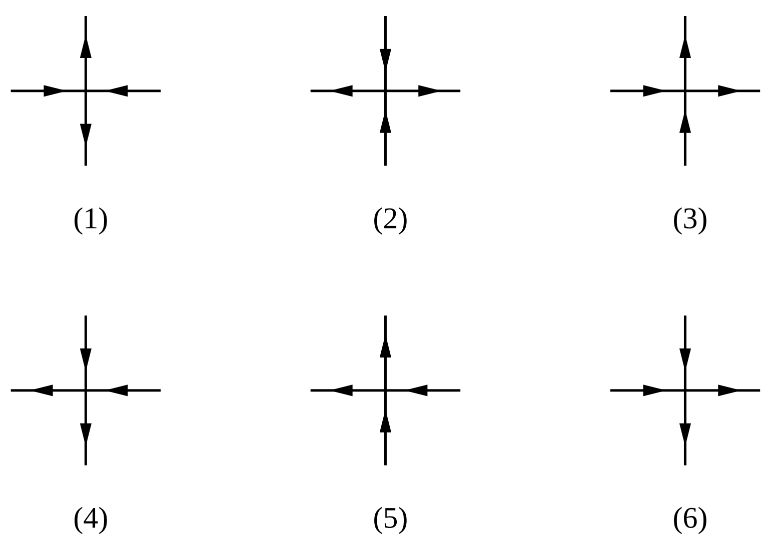


Fig. 1. The six possible configurations of arrows at each vertex.

We will consider the domain wall boundary condition (DWBC), in which the arrows on the upper and lower boundaries point in the square, and the ones on the left and right boundaries point out. One possible configuration with DWBC on the 4×4 lattice is shown on Fig. 2.

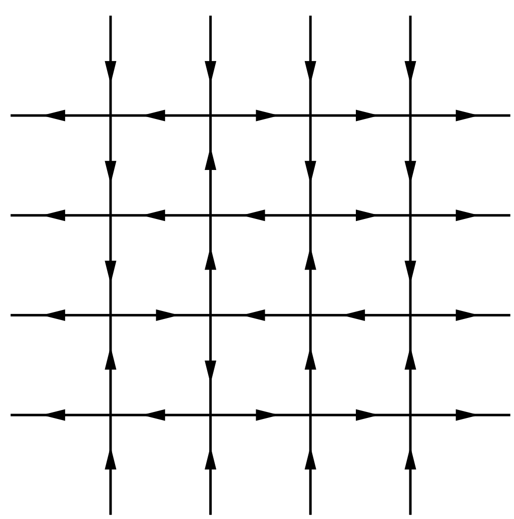


Fig. 2. An example of 4×4 configuration.

The name of square ice comes from the two-dimensional arrangement of water molecules, H_2O, with oxygen atoms at the vertices of a lattice and one hydrogen atom between each pair of adjacent oxygen atoms. We place an arrow in the direction from a hydrogen atom toward an oxygen atom if there is a bond between them.

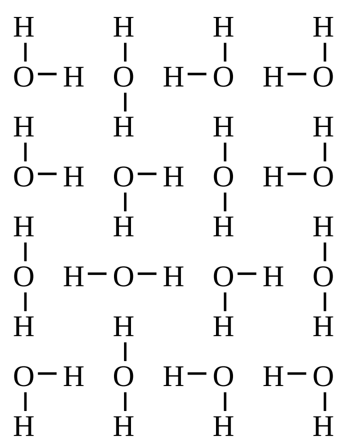


Fig. 3. The corresponding ice model.

For each possible vertex state we assign the weight w_i, $i = 1, \ldots, 6$, and define, as usual, the partition function as a sum over all possible arrow configurations, given as the product of all the corresponding vertex weights

$$Z_N = \sum_{\text{arrow configurations}} \prod_{i=1}^{6} w_i^{n_i},$$

where n_i is the number of vertices in the state i in each arrow configuration. We will consider the case, when the weights are invariant under the simultaneous reversal of all arrows, i.e.,

$$a := w_3 = w_4, \quad b := w_5 = w_6, \quad c := w_1 = w_2.$$

Define the parameter Δ as

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}.$$

There are three physical phases for the six-vertex model: the ferroelectric phase, $\Delta > 1$; the anti-ferroelectric phase, $\Delta < -1$; and, the disordered phase, $-1 < \Delta < 1$. The phase diagram of the model is given on Fig. 4.

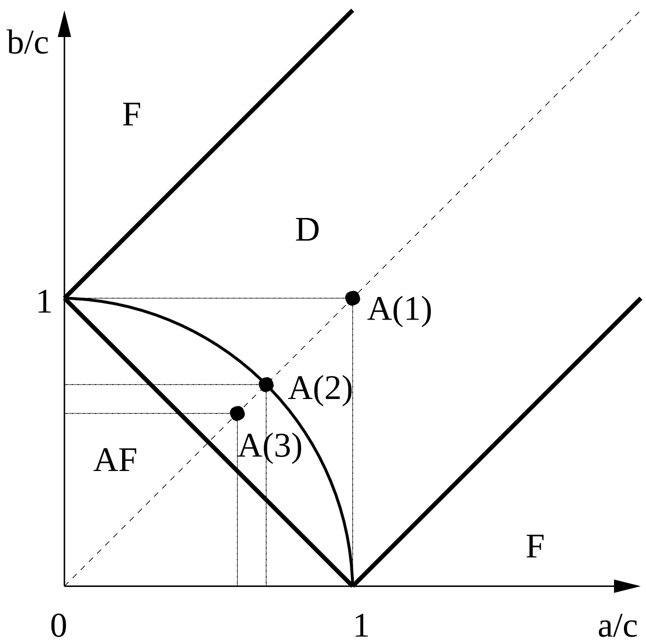


Fig. 4. The phase diagram of the model, where **F**, **AF** and **D** mark ferroelectric, antiferroelectric, and disordered phases, respectively. The circular arc corresponds to the so-called "free fermion" line, when $\Delta = 0$, and the three dots correspond to 1-, 2-, and 3-enumeration of alternating sign matrices.

In these phases we parametrize the weights in the standard way: for the ferroelectric phase,

$$a = \mathsf{sinh}(t-\gamma), \quad b = \mathsf{sinh}(t+\gamma), \quad c = \mathsf{sinh}(2\gamma),$$
$$|\gamma| < t,$$

for the anti-ferroelectric phase,

$$a = \mathsf{sinh}(\gamma-t), \quad b = \mathsf{sinh}(\gamma+t), \quad c = \mathsf{sinh}(2\gamma),$$
$$|t| < \gamma,$$

and for the disordered phase

$$a = \mathsf{sin}(\gamma-t), \quad b = \mathsf{sin}(\gamma+t), \quad c = \mathsf{sin}(2\gamma),$$
$$|t| < \gamma.$$

Here we will discuss the disordered phase, and we will use the last parametrization. There are two parameters in the model: t and γ.

A solution for the free energy of the six-vertex model with periodic boundary conditions (PBC) was found by Lieb by means of Bethe Ansatz.

E. H. Lieb, *Phys. Rev. Lett.* **18** (1967) 692; *Phys. Rev. Lett.* **18** (1967) 1046-1048; *Phys. Rev. Lett.* **19** (1967) 108-110; *Phys. Rev.* **162** (1967) 162.

In the most general form of the six-vertex model the Bethe Ansatz solution with PBC was obtained by Sutherland.

B. Sutherland, *Phys. Rev. Lett.* **19** (1967) 103-104.

A detailed classification of the phases of the six-vertex model is given in the book of Baxter.

R. Baxter, *Exactly solved models in statistical mechanics,* Academic Press, San Diego, CA.

The six-vertex model with antiperiodic boundary conditions is solved in the paper

M. T. Batchelor, R. J. Baxter, M. J. O'Rourke, and C. M. Yung, *J. Phys.* **A 28** (1995) 2759–2770.

The six-vertex model with DWBC was introduced by Korepin,

V. E. Korepin, *Commun. Math. Phys.* **86** (1982), 391-418,

who derived an important recursion relation for the partition function of the model. This lead to a beautiful determinantal formula for the partition function of Izergin,

A. G. Izergin, *Sov. Phys. Dokl.* **32** (1987), 878.

A detailed proof of this formula and its generalizations are given in the paper of Izergin, Coker, and Korepin,

A. G. Izergin, D. A. Coker, and V. E. Korepin, *J. Phys. A*, **25** (1992), 4315.

The formula of Izergin is

$$Z_N = \frac{[\sin(\gamma + t)\sin(\gamma - t)]^{N^2}}{\left(\prod_{n=0}^{N-1} n!\right)^2}\,\tau_N\,,$$

where τ_N is the Hankel determinant,

$$\tau_N = \det\left(\frac{d^{i+k-2}\phi}{dt^{i+k-2}}\right)_{1\le i,k\le N},$$

and

$$\phi(t) = \frac{\sin(2\gamma)}{\sin(\gamma + t)\sin(\gamma - t)}\,.$$

An elegant derivation of the Izergin formula from the Yang-Baxter equation is given in the papers of Korepin and Zinn-Justin,

V. Korepin and P. Zinn-Justin, *J. Phys. A* **33** No. 40 (2000), 7053

and Kuperberg,

G. Kuperberg, *Intern. Math. Res. Notes* (1996), 139-150.

One of the applications of the determinantal formula is that it implies that the partition function τ_N solves the Toda equation,

$$\tau_N \tau_N'' - {\tau_N'}^2 = \tau_{N+1}\tau_{N-1}, \qquad (') = \frac{\partial}{\partial t},$$

This was used by Korepin and Zinn-Justin to derive the free energy of the six-vertex model with DWBC, assuming some Ansatz on the behavior of subdominant terms in the large N asymptotics of the free energy.

Another application of the Izergin determinantal formula is that τ_N can be expressed in terms of a partition function of a random matrix model. The relation to the random matrix model was obtained and used by Zinn-Justin,

P. Zinn-Justin, *Phys. Rev. E* **62** (2000), 3411-3418,

This relation will be very important for us. It can be described as follows.

For the evaluation of the Hankel determinant, it is convenient to use the integral representation of $\phi(t)$, namely, to write it in the form of the Laplace transform,

$$\phi(t) = \int_{-\infty}^{\infty} e^{t\lambda} m(\lambda) d\lambda,$$

where

$$m(\lambda) = \frac{\sinh \frac{\lambda}{2}(\pi - 2\gamma)}{\sinh \frac{\lambda}{2}\pi}.$$

Then, after some manipulations, we arrive at the formula,

$$\tau_N = \frac{\prod_{n=0}^{N-1} n!}{\pi^{N(N-1)/2}} \int dM e^{\mathrm{Tr}\ [tM - V(M)]},$$

where the integration is over the space of $N \times N$ Hermitian matrices, and $m(x) = e^{-V(x)}$. The matrix model integral can be solved, furthermore, in terms of orthogonal polynomials.

Introduce monic polynomials $P_n(x) = x^n + \dots$, orthogonal on the line with respect to the weight $e^{tx}m(x)$, so that

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)e^{tx}m(x)dx = h_n\delta_{nm}.$$

Then it follows from the matrix integral that

$$\tau_N = \prod_{n=0}^{N-1} h_n.$$

(the Dyson formula). The orthogonal polynomials satisfy the three term recurrent relation,

$$xP_n(x) = P_{n+1}(x) + Q_nP_n(x) + R_nP_{n-1}(x),$$

where R_n can be found as $R_n = \frac{h_n}{h_{n-1}}$, This gives that $h_n = h_0 \prod_{j=1}^{n} R_j$, where

$$h_0 = \int_{-\infty}^{\infty} e^{tx}m(x)dx = \frac{\sin(2\gamma)}{\sin(\gamma+t)\sin(\gamma-t)}.$$

Thus,

$$\tau_N = h_0^N \prod_{n=1}^{N-1} R_n^{N-n}.$$

The main technical result of our work is the asymptotics of R_n as $n \to \infty$.

Theorem 1. *(Asymptotics of the recurrent coefficient). As $n \to \infty$,*

$$R_n = \frac{n^2}{\gamma^2}[R + \cos(n\omega) \sum_{j:\, \kappa_j \le 2} c_j n^{-\kappa_j} + cn^{-2} + O(n^{-2-\varepsilon})], \qquad \varepsilon > 0,$$

where

$$R = \left(\frac{\pi}{2\cos\frac{\pi\zeta}{2}} \right)^2, \quad \zeta \equiv \frac{t}{\gamma}; \qquad \omega = \pi(1+\zeta);$$

$$\kappa_j = 1 + \frac{2j}{\frac{\pi}{2\gamma} - 1},$$

and c_j, c are some explicit numbers.

Applications

Define

$$F_N = \frac{1}{N^2} \ln \frac{\tau_N}{\left(\prod_{n=0}^{N-1} n!\right)^2}.$$

Theorem 2. *(Leading asymptotics). As $N \to \infty$,*

$$F_N = F + O(N^{-1}),$$

where

$$F = \frac{1}{2} \ln \frac{R}{\gamma^2} = \ln \left(\frac{\pi}{2\gamma \cos \frac{\pi\zeta}{2}} \right) .$$

This coincides with the formula of Zinn-Justin, obtained in the saddle-point approximation. Earlier it was derived by Korepin and Zinn-Justin, from an Ansatz for the free energy asymptotics.

We have the identity,

$$\frac{\partial^2 F_N}{\partial t^2} = \frac{R_N}{N^2},$$

which is equivalent to the Toda equation. By using this identity, we obtain the following asymptotics.

Theorem 3. *(Subdominant asymptotics). As $N \to \infty$,*

$$\frac{\partial^2 (F_N - F)}{\partial t^2} = \frac{1}{\gamma^2} \cos(N\omega) \sum_{j:\, \kappa_j \le 2} c_j N^{-\kappa_j} + cN^{-2} + O(N^{-2-\varepsilon}).$$

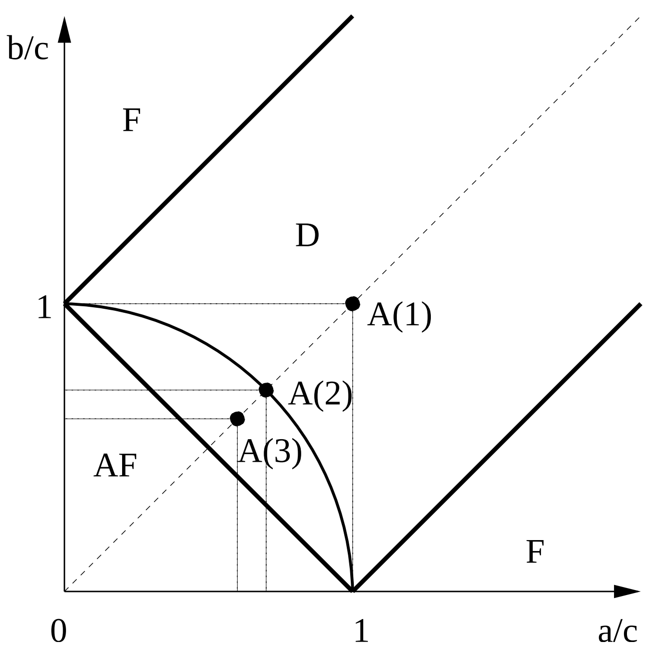
This gives a quasiperiodic behavior, as $N \to \infty$, of the second derivative of the free energy subdominant terms.

For the partition function Z_N we obtain the leading asymptotics,

$$\frac{1}{N^2} \ln Z_N = f + O(N^{-1}),$$

$$f = \ln \left(\frac{\pi[\cos(2t) - \cos(2\gamma)]}{4\gamma \cos \frac{\pi t}{2\gamma}} \right).$$

Let us compare this asymptotics with known exact results. There are cases, for which the model has been solved earlier by different methods: the free fermion line and $A(1)$,$A(2)$,$A(3)$.



Phase diagram.

The free fermion line, $\gamma = \frac{\pi}{4}$, $|t| < \frac{\pi}{4}$.

In this case the exact result is

$$Z_N = 1,$$

see

F. Colomo and A. G. Pronko, Square ice, alternating sign matrices, and classical orthogonal polynomials. Preprint (arXiv:math-ph/0411076)

It implies $f = 0$. This agrees with our formula, which also gives $f = 0$ when $\gamma = \frac{\pi}{4}$.

The ASM (ice) point, $\gamma = \frac{\pi}{3}$, $t = 0$. In this case the weights are $a = b = c = \frac{\sqrt{3}}{2}$, hence $Z_N = \left(\frac{\sqrt{3}}{2}\right)^{N^2} A(N)$, where $A(N)$ is the number of configurations in the six-vertex model with DWBC. There is a one-to-one correspondence between the set of configurations in the six-vertex model with DWBC and the set of $N \times N$ alternating sign matrices. By definition, an alternating sign matrix (ASM) is a matrix with the following properties:

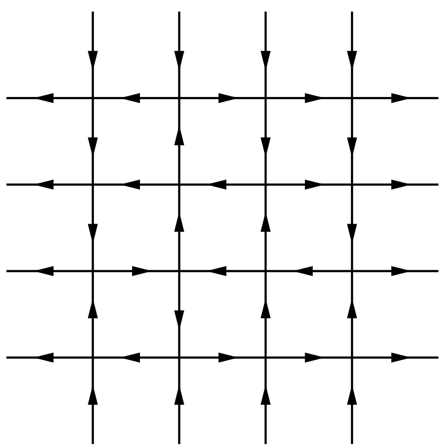
- all entries of the matrix are $-1, 0, 1$;

- if we look at the sequence of (-1)'s and 1's, they are alternating along any row and any column;

- the sum of entries is equal to 1 along any row and any column.

The above correspondence is established as follows: given a configuration of arrows on edges, we assign (-1) to any vertex of type (1), 1 to any vertex of type (2), and 0 to any vertex of other types.

Example:



$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

A 4×4 configuration and the corresponding ASM.

For the number of ASMs there is an exact formula:

$$A(N) = \prod_{n=0}^{N-1} \frac{(3n+1)!n!}{(2n)!(2n+1)!}.$$

This formula was conjectured by Mills, Robbins, and Rumsey, and proved by Zeilberger by combinatorial arguments. Another proof was given by Kuperberg, who used the Izergin formula, and also by Colomo and Pronko, who used orthogonal polynomials. From the formula for $A(N)$ we find that

$$Z_N = C\left(\frac{9}{8}\right)^{N^2} N^{-\frac{5}{36}}\left(1+O(N^{-2})\right).$$

This gives $f = \ln\frac{9}{8}$, which agrees with our asymptotic formula.

The $x = 3$ ASM point, $\gamma = \frac{\pi}{6}$, $t = 0$. Here the exact result is

$$Z_N = \frac{3^{N/2}}{2^{N^2}} A(N;3),$$

where

$$\begin{cases} A(2m+1;3) = 3^{m(m+1)} \prod_{k=1}^{m} \left[\frac{(3k-1)!}{(m+k)!}\right]^2, \\ A(2m+2;3) = 3^m \frac{(3m+2)!m!}{[(2m+1)!]^2} A(2m+1;3)\,. \end{cases}$$

In this case $A(N;3)$ counts the number of alternating sign matrices with weight 3^k, where k is the number of (-1) entries. The formulae for $A(N;3)$ were conjectured by Mills, Robbins, and Rumsey, and proved by Kuperberg, and also by Colomo and Pronko. From these formulae we obtain that

$$Z_N = C \left(\frac{3}{4}\right)^{N^2} N^{\frac{1}{18}} \left(1 + O(N^{-2})\right).$$

This gives $f = \ln \frac{3}{4}$, which agrees with our asymptotic formula.

Zinn-Justin's Conjecture

Zinn-Justin conjectured the asymptotics,

$$Z_N \sim C N^{\kappa} e^{N^2 f}\,,$$

i.e.,

$$\lim_{N\to\infty} \frac{Z_N}{C N^{\kappa} e^{N^2 f}} = 1\,.$$

The equivalent form of the Zinn-Justin conjecture is

$$\frac{1}{N^2}\ln Z_N = f + \frac{\kappa \ln N}{N^2} + \frac{\ln C}{N^2} + o(N^{-2}).$$

The exact formulae on the free fermion line and at $A(1)$, $A(3)$ support this conjecture, with the value of κ given as

$$\kappa = \begin{cases} 0, \quad \gamma = \frac{\pi}{4}, \quad |t| < \frac{\pi}{4}\,; \\ -\frac{5}{36}, \quad \gamma = \frac{\pi}{3}, \quad t = 0; \\ \frac{1}{18}, \quad \gamma = \frac{\pi}{6}, \quad t = 0. \end{cases}$$

We prove the Zinn-Justin conjecture and we obtain an exact formula for the exponent κ.

Theorem 4.

$$Z_N = CN^{\kappa}e^{N^2 f}\Big(1+O(N^{-\varepsilon})\Big), \qquad \varepsilon > 0\,,$$

where

$$\kappa = \frac{2\gamma^2}{3\pi(\pi-2\gamma)} - \frac{1}{12}\,.$$

Sketch of the Proof

- Zinn-Justin's rescaling of the matrix integral.

- Large N asymptotics of the equilibrium measure for the rescaled matrix integral.

- The Riemann-Hilbert problem for the rescaled orthogonal polynomials.

- Undressing of the RH problem and the Deift-Zhou nonlinear steepest descent method.

- Deformation of contours in the undressed RH problem and the proof of the large N asymptotics of the recurrent coefficient.

- Exact solution of the six-vertex model.

Reference

Pavel Bleher and Vladimir Fokin, Exact solution of the six-vertex model with domain wall boundary condition. Disordered phase. arXiv: math-ph/0510033 (submitted to CMP).