Unimodularity and Stochastic Processes

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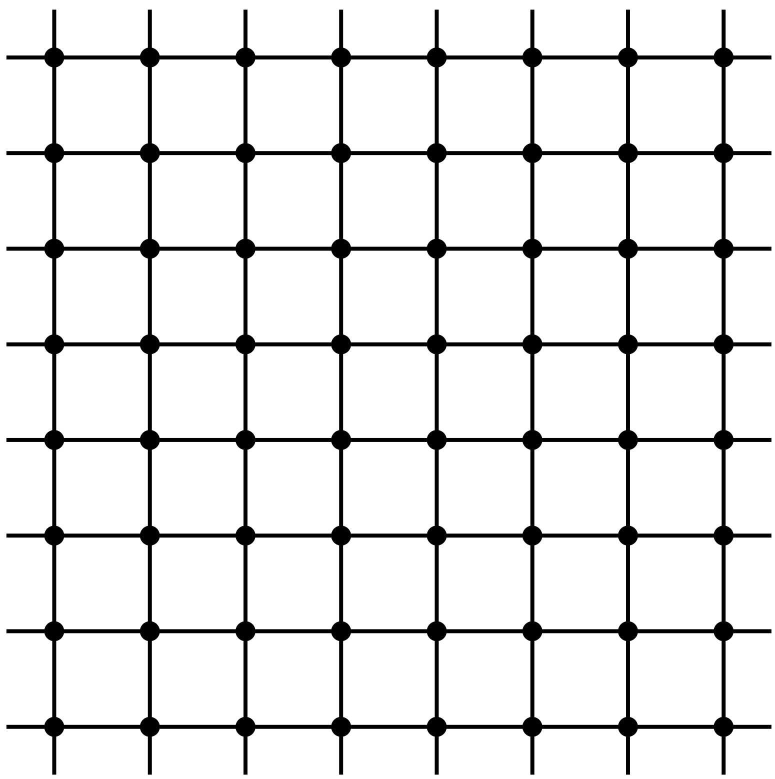
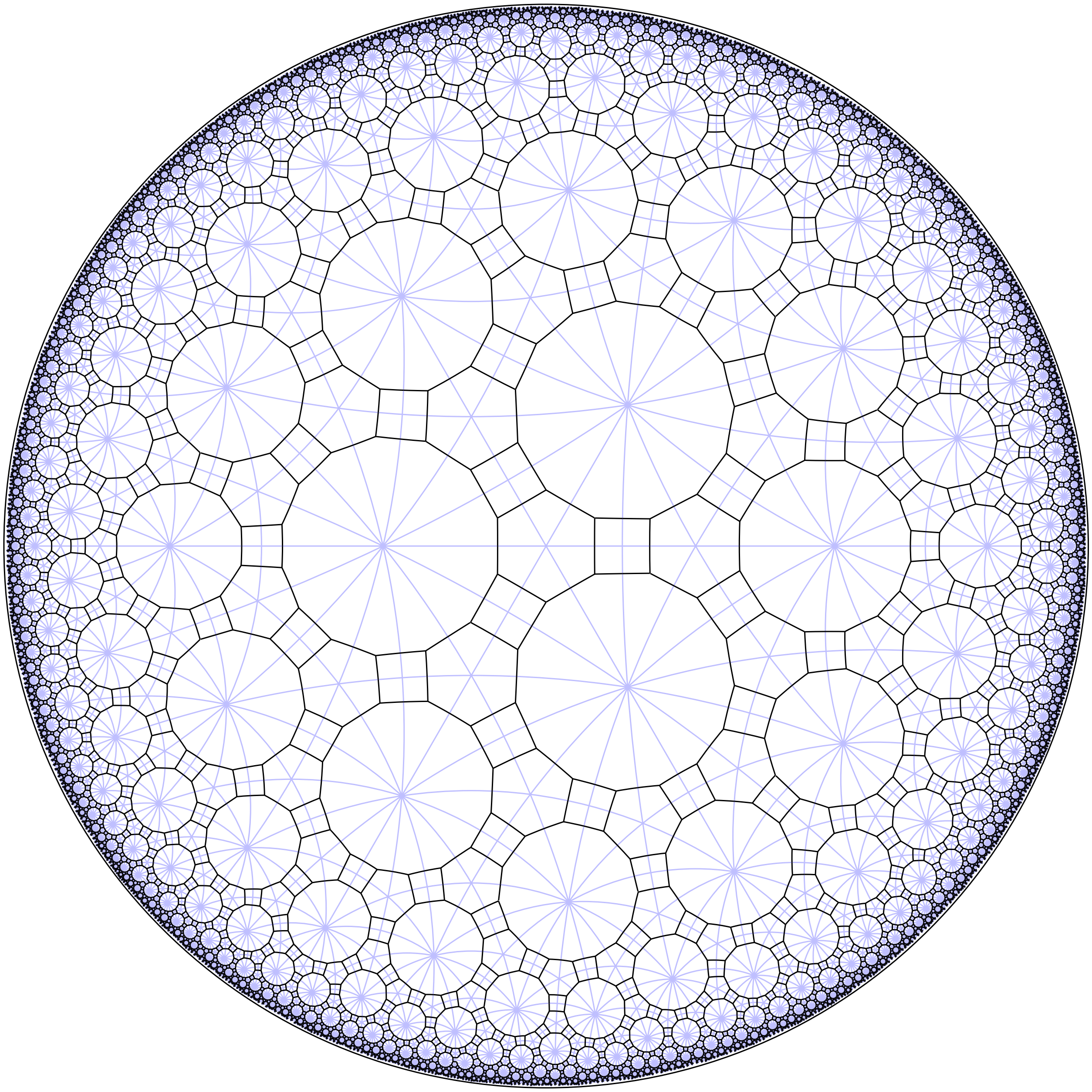
We might explain unimodularity as a non-obvious use of group-invariance. Simplest setting: transitive graphs. A **graph** is a pair $G = (\mathsf{V}, \mathsf{E})$ with E a symmetric subset of $\mathsf{V} \times \mathsf{V}$. An **automorphism** of G is a permutation of V that induces a permutation of E. The set of all automorphisms of G forms a group, $\mathrm{Aut}(G)$. We call G **transitive** if $\mathrm{Aut}(G)$ acts transitively on V (i.e., there is only one orbit).





Consider the following examples: Let G be an infinite transitive graph and let $\mathbf{P}$ be an invariant percolation, i.e., an $\mathrm{Aut}(G)$-invariant measure on 2^{V}, on 2^{E}, or even on $2^{\mathsf{V} \cup \mathsf{E}}$. Let ω be a configuration with distribution $\mathbf{P}$.

Example: Could it be that ω is a single vertex? I.e., is there an invariant way to pick a vertex at random?

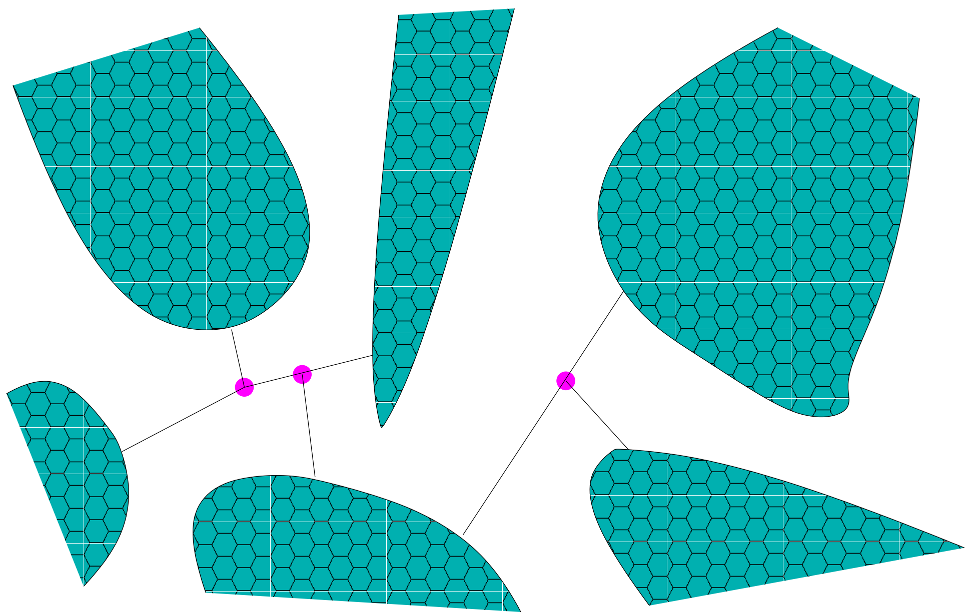
No: If there were, the assumptions would imply that the probability p that $\omega = \{x\}$ is the same for all x, whence an infinite sum of p would equal 1, an impossibility.

Example: Could it be that ω is a finite nonempty vertex set? I.e., is there an invariant way to pick a finite set of vertices at random?

No: If there were, then we could pick one of the vertices of the finite set at random (uniformly), thereby obtaining an invariant probability measure on singletons.

Cluster means connected component of ω.

A vertex x is a **furcation** of a configuration ω if removing x would split the cluster containing x into at least 3 infinite clusters.



EXAMPLE: The number of furcations is $\mathbf{P}$-a.s. 0 or ∞. For the set of furcations has an invariant distribution on 2^{V}.

EXAMPLE: $\mathbf{P}$-a.s. each cluster has 0 or ∞ furcations.

This does not follow from elementary considerations as the previous examples do (we can prove this).

But suppose we have the following kind of conservation of mass.

We call $f : \mathsf{V} \times \mathsf{V} \to [0, \infty]$ **diagonally invariant** if $f(\gamma x, \gamma y) = f(x, y)$ for all $x, y \in \mathsf{V}$ and $\gamma \in \mathrm{Aut}(G)$.

THE MASS-TRANSPORT PRINCIPLE. *For all diagonally invariant f, we have*

$$\sum_{x \in \mathsf{V}} f(o, x) = \sum_{x \in \mathsf{V}} f(x, o) \,,$$

where o is any fixed vertex of G.

Suppose this holds.

Write $K(x)$ for the cluster containing x.

Now, given the configuration ω, define $F(x, y; \omega)$ to be 0 if $K(x)$ has 0 or ∞ furcations, but to be $1/N$ if y is one of N furcations of $K(x)$ and $1 \le N < \infty$. Then F is diagonally invariant, whence the Mass-Transport Principle applies to $f(x, y) := \mathbf{E}F(x, y; \omega)$. Since $\sum_y F(x, y; \omega) \le 1$, we have

$$\sum_x f(o, x) \le 1 \,. \tag{1}$$

If any cluster has a finite positive number of furcations, then each of them receives infinite mass. More precisely, if o is one of a finite number of furcations of $K(o)$, then $\sum_x F(x, o; \omega) = \infty$. Therefore, if with positive probability some cluster has a finite positive number of furcations, then with positive probability o is one of a finite number of furcations of $K(o)$, and therefore $\mathbf{E}\Big[\sum_x F(x, o; \omega)\Big] = \infty$. That is, $\sum_x f(x, o) = \infty$, which contradicts the Mass-Transport Principle and (1).

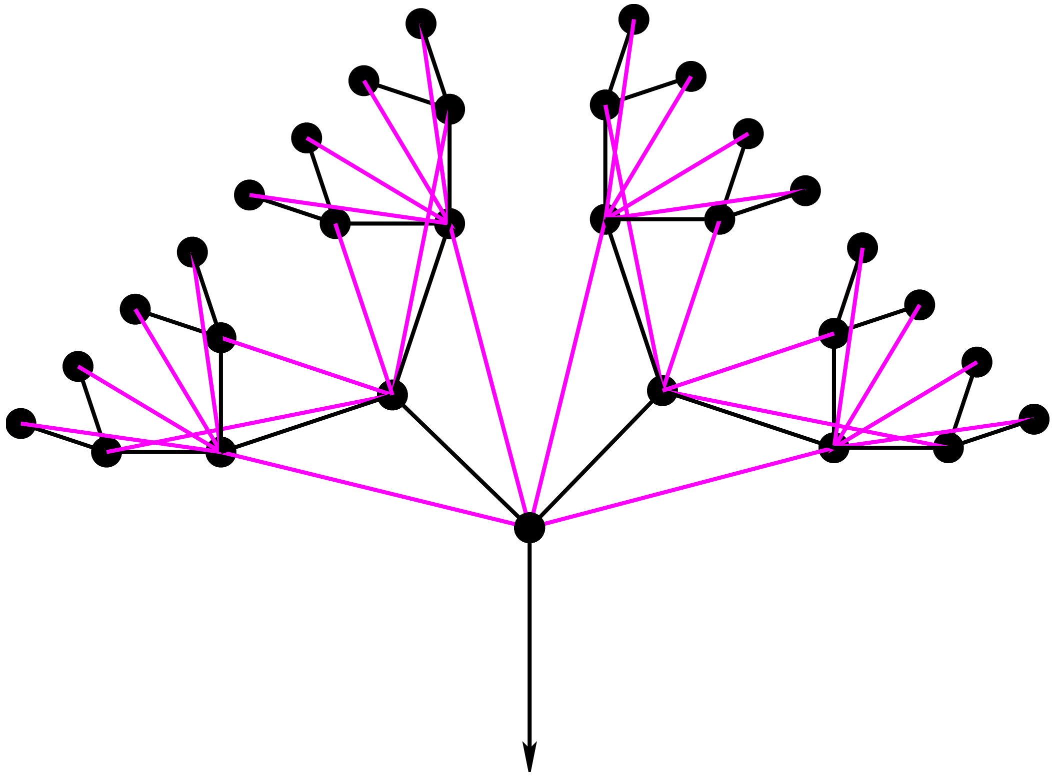
Call G **unimodular** if the Mass-Transport Principle holds for G. Which graphs enjoy this wonderful property? All graphs do that are properly embedded in euclidean or hyperbolic space with a transitive action of isometries of the space. All Cayley graphs do:

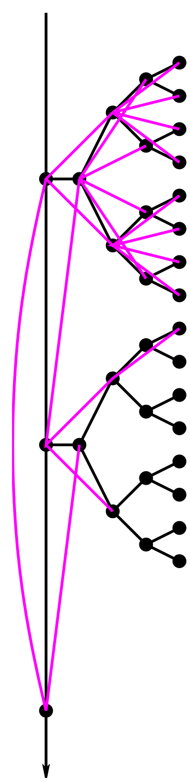
We say that a group Γ is **generated** by a subset S of its elements if the smallest subgroup containing S is all of Γ. In other words, every element of Γ can be written as a product of elements of the form s or s^{-1} with $s \in S$. If Γ is generated by S, then we form the associated **Cayley graph** G with vertices Γ and (unoriented) edges $\big\{(x, xs)\,;\; x \in G,\; s \in S \cup S^{-1}\big\}$. Because S generates Γ, the graph is connected. Cayley graphs are transitive since left multiplication by yx^{-1} is an automorphism of G that carries x to y.

Now if $f : \Gamma^2 \to [0, \infty]$ is diagonally invariant, then for o the identity of Γ and any $x \in \Gamma$, we have $f(o, x) = f(x^{-1}, o)$. Since inversion preserves counting measure on Γ, we obtain the Mass-Transport Principle.

(For a general transitive graph, the Mass-Transport Principle is equivalent to unimodularity of Haar measure on $\mathrm{Aut}(G)$. History: Adams (1990), van den Berg and Meester (1991), Häggström (1997), Benjamini, L., Peres, Schramm (1999). I ignore other uses of unimodularity in probability that go back considerably longer.)

Non-example: the "grandparent" graph of Trofimov:





The grandparent graph is not unimodular: let $f(x, y)$ be the indicator that y is the grandparent of x. Then

$$\sum_x f(o, x) = 1$$

while

$$\sum_x f(x, o) = 4 .$$

Another definition: G is **amenable** if there is a sequence K_n of finite vertex sets in G such that the number of neighbors of K_n divided by the size of K_n tends to 0.

EXAMPLE: $\mathbb{Z}^d$

Non-examples: regular trees of degree at least 3; hyperbolic tessellations.

All amenable transitive graphs are unimodular (Soardi and Woess).

A selection of theorems:

Bernoulli(p) percolation on G puts each edge in ω independently with probability p. The probability of an infinite cluster in ω is 0 or 1 by Kolmogorov's 0-1 Law. It increases in p, so there is a **critical value** p_c where it changes. What is the probability of an infinite cluster at p_c? Benjamini and Schramm conjectured it is 0 on any transitive graph with $p_c < 1$. It was known for $\mathbb{Z}^d$ for $d = 2$ (Kesten) and $d \geq 19$ (Hara and Slade).

THEOREM (BLPS 1999). *This is true for all non-amenable transitive unimodular graphs.*

It is unknown whether this holds for non-unimodular graphs.

Theorem (Häggström; Häggström and Peres; L. and Peres; L. and Schramm). *Let G be a transitive unimodular graph. Given invariant random transition probabilities $p_\omega(x, y)$ and an invariant p-stationary measure $\nu_\omega(x)$, biasing ω by $\nu_\omega(o)$ gives a measure that is invariant from the point of view of the walker.*

Example: Degree-biasing for simple random walk on the clusters.

This is false on non-unimodular graphs.

Theorem (Aldous and L.). *Let G be a transitive unimodular graph. Given invariant random symmetric rates $r_\omega(x, y)$ such that $\mathbf{E}\big[\sum_x r(o, x)\big] < \infty$, the associated continuous-time random walk has no explosions a.s.*

This is false on non-unimodular graphs.

Theorem (Fontes and Mathieu; Aldous and L.). *Let G be a transitive unimodular graph. Given invariant random pairs of symmetric rates* (r_ω, R_ω) *such that*

$$r_\omega(x, y) \le R_\omega(x, y)$$

for all x, y *and almost all* ω, *let* $p_t(o, o)$ *and* $P_t(o, o)$ *be the expected [annealed] return probabilities for the associated continuous-time (minimal) random walks. Then for all* $t > 0$, *we have*

$$p_t(o, o) \ge P_t(o, o) \,.$$

It is unknown whether this holds for non-unimodular graphs.

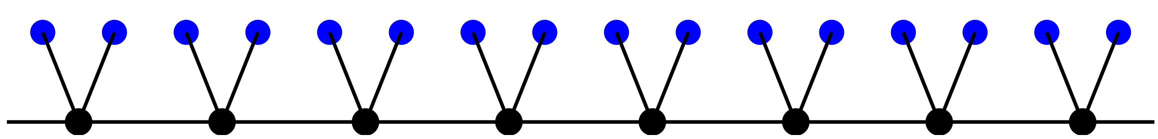
Extensions of unimodularity:

On finite graphs, the Mass-Transport Principle is obvious if we take o to be a uniform random "root" and average over o:

$$\mathbf{E}\Big[\sum_x f(o,x)\Big] = \mathbf{E}\Big[\sum_x f(x,o)\Big]\,. \tag{2}$$

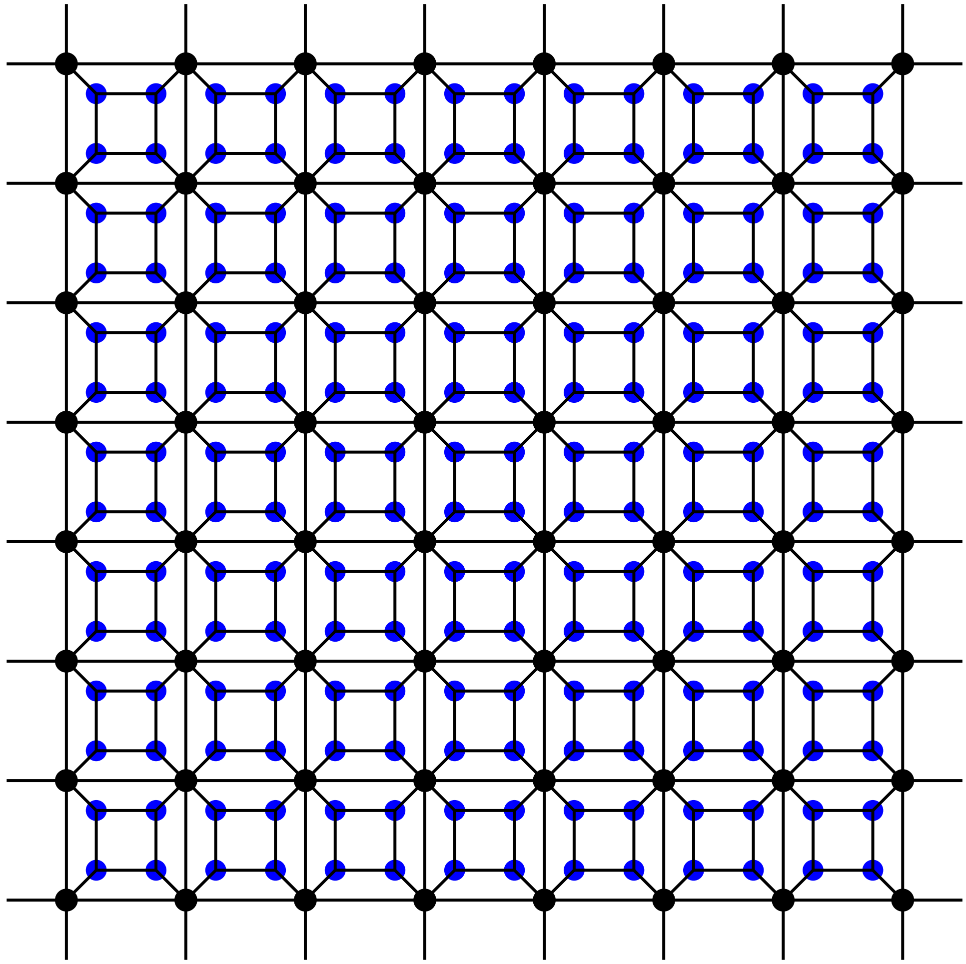
This is just interchanging the order of summation. But it is crucial that the root be chosen uniformly. Indeed, (2) characterizes the uniform measure.

Consider this graph:



We should choose o to be a blue vertex with probability twice that of a black vertex in order that (2) hold.

With this graph:



we should choose o to be a blue vertex with probability four times that of a black vertex in order that (2) hold.

What about the hyperbolic triangle tessellation?

We call G **quasi-transitive** if $\mathrm{Aut}(G)$ acts quasi-transitively on V (i.e., there are only finitely many orbits). If G is quasi-transitive and amenable, then each orbit has a natural frequency (BLPS), which should be used for the probability of choosing a representative from that orbit for o in (2).

If there are probabilities α_i for the orbit representatives $o_1, \ldots, o_L$ such that choosing o_i with probability α_i makes (2) true, then we call G **unimodular**.

How do we tell? The following is necessary and sufficient: if x is in the orbit of o_i and y is in the orbit of o_j, then

$$\frac{|S(x)y|}{|S(y)x|} = \frac{\alpha_j}{\alpha_i}\,,$$

where $S(x) := \big\{\gamma \in \mathrm{Aut}(G)\,;\; \gamma x = x\big\}$.

Consider now the space of rooted graphs or networks. In fact, consider only rooted-isomorphism classes of networks. A probability measure on this space is **unimodular** if the Mass-Transport Principle holds:

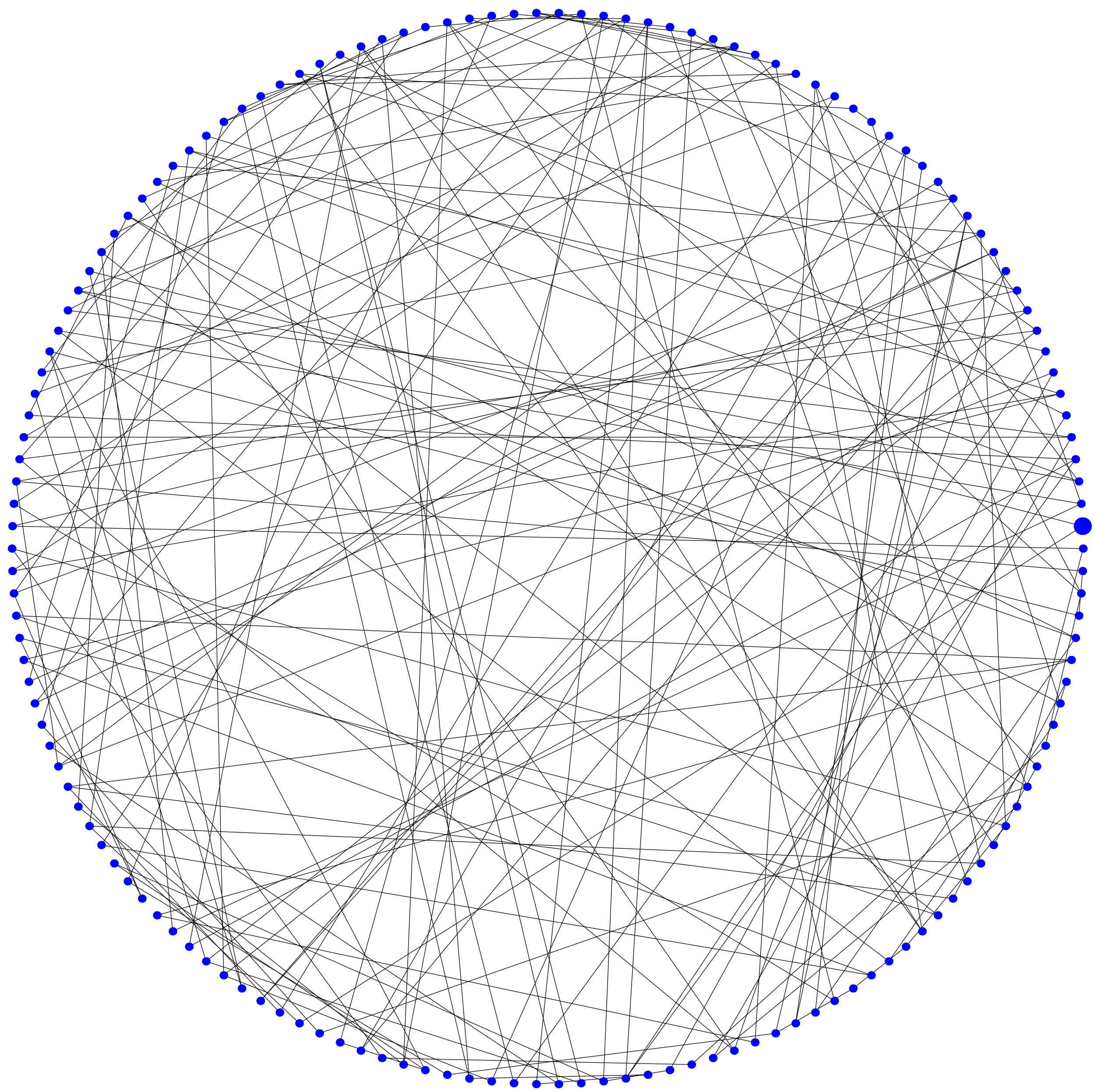
$$\mathbf{E}\Big[\sum_{x \in \mathsf{V}(G)} f(G; o, x)\Big] = \mathbf{E}\Big[\sum_{x \in \mathsf{V}(G)} f(G; x, o)\Big] \tag{3}$$

for all Borel non-negative f that are invariant under isomorphisms.

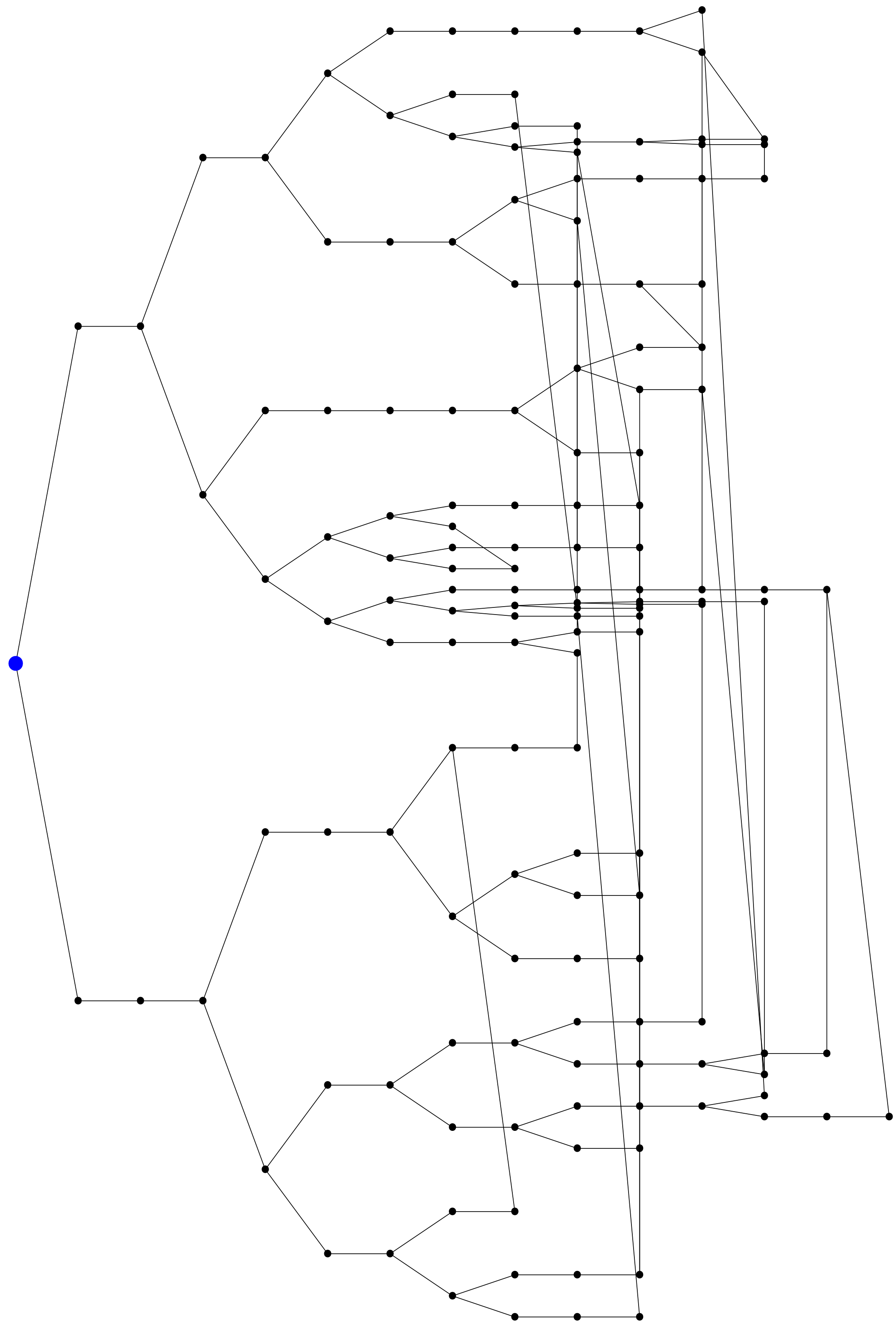
For example, as observed by Benjamini and Schramm and by Aldous and Steele, all weak limits of uniformly rooted finite networks are unimodular.

EXAMPLE: If we want the offspring distribution $\langle p_k \rangle$ for a unimodular version UGW of Galton-Watson trees, let $r_k := c^{-1} p_{k-1}/k$ for $k \geq 1$ and $r_0 := 0$, where $c := \sum_{k \geq 0} p_k/(k+1)$. With the sequence $\langle r_k \rangle$ and n vertices, give each vertex k balls with probability r_k, independently. Then pair the balls at random and place an edge for each pair between the corresponding vertices. There may be one ball left over; if so, ignore it. In the limit, we get a random tree where the root has degree k with probability r_k and each neighbor of the root has an independent Galton-Watson($\langle p_k \rangle$) tree.

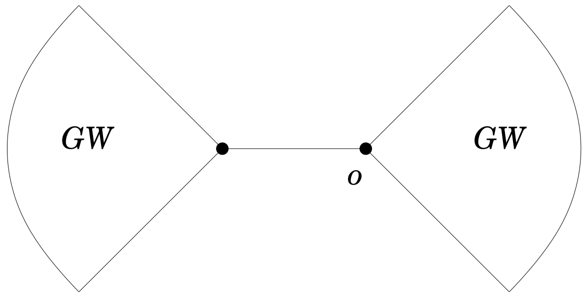
All the theorems given for transitive unimodular graphs hold for unimodular random rooted networks (Aldous-L.).



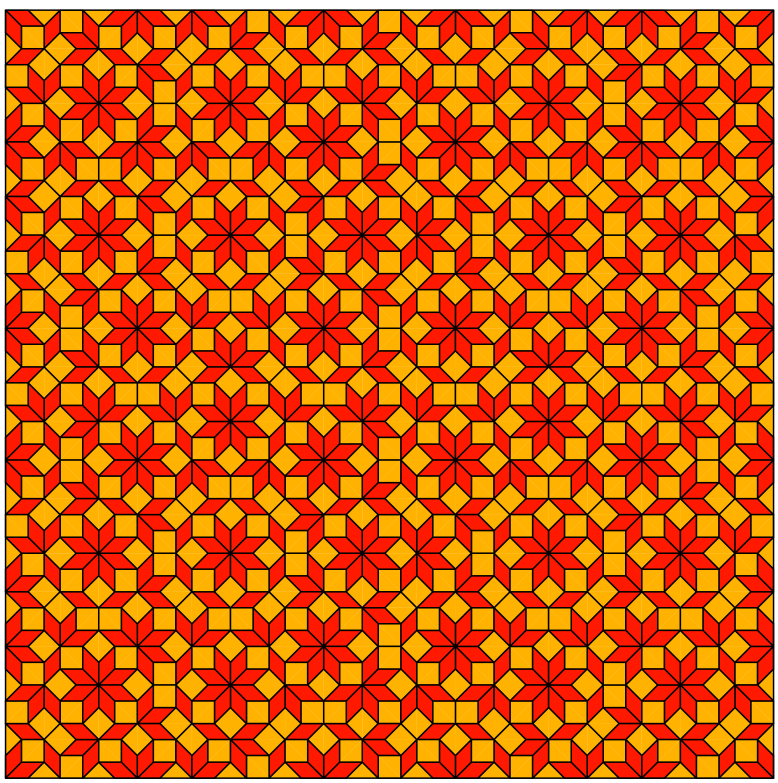
(150 vertices with $p_1 = p_2 = 1/2$)



EXAMPLE: Biasing UGW by the degree of the root gives a stationary measure for simple random walk (L., Pemantle and Peres):



EXAMPLE: Aperiodic tessellations:



Like Palm measure.