LOCAL SETS

of the

GAUSSIAN FREE FIELD

PART TWO

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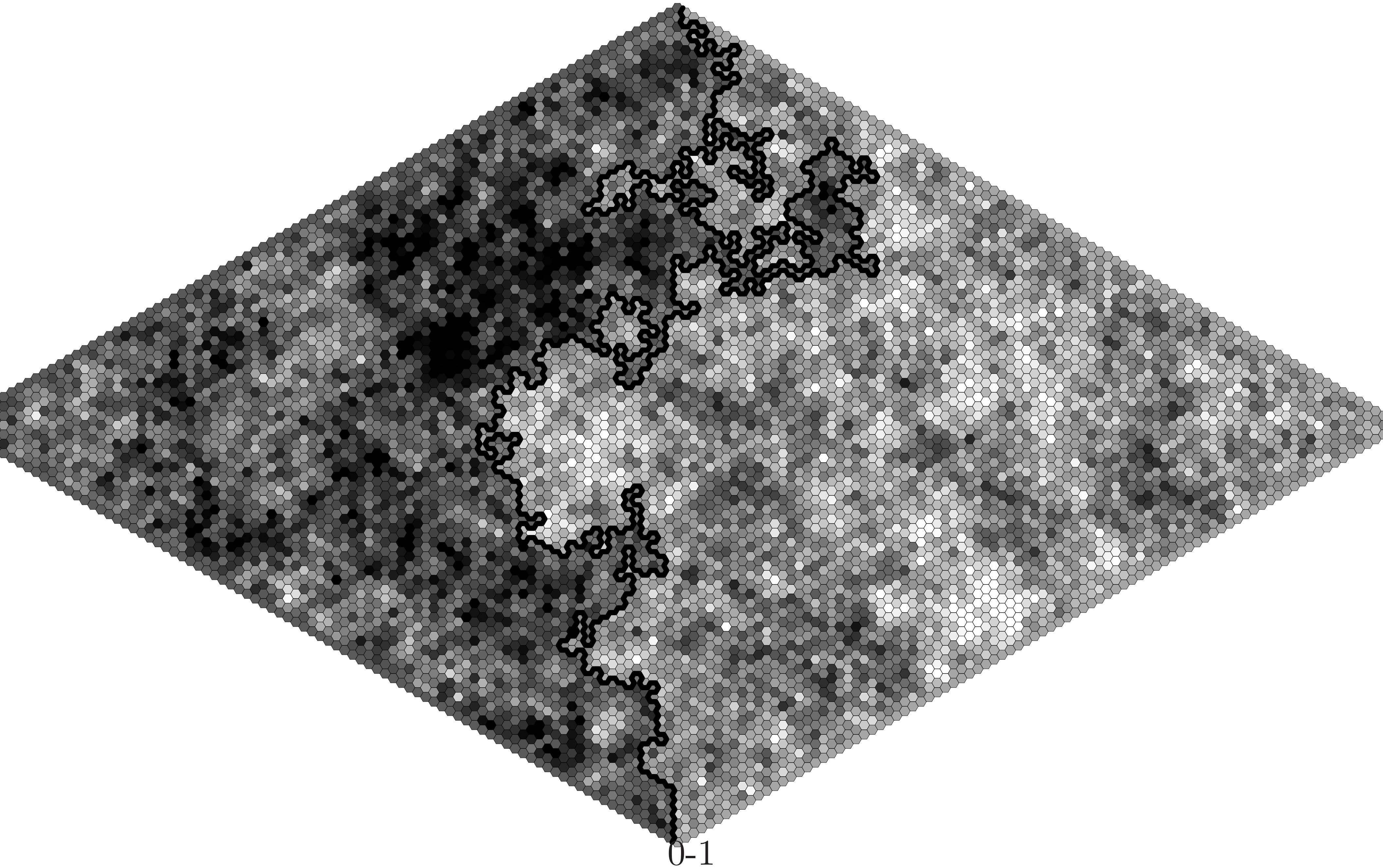
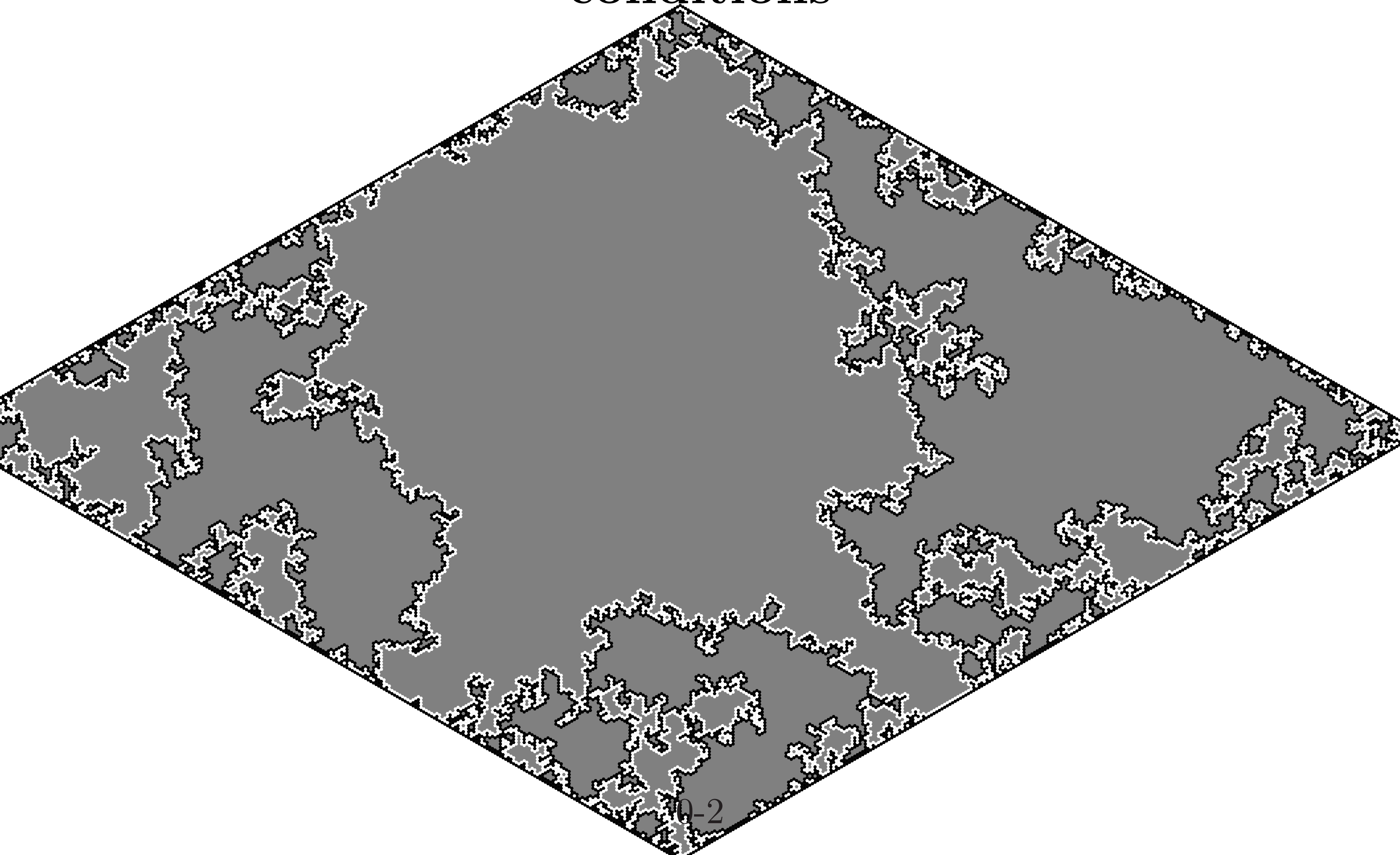
based on work with Schramm; Schramm and Wilson; and Werner

DGFF with $\pm\lambda$ boundary conditions



Zero contour lines, zero boundary conditions



Discrete deterministic local sets

A vertex-subset valued function A defined on the set of possible instances h of the GFF (i.e., set of real-valued functions on the vertices of G) is called **local** if $A(h_1) = A(h_2)$ whenever h_1 and h_2 agree on A. Such an A is called a **deterministic local set** (i.e., given h, it is a deterministic function of h).

Discrete non-deterministic local sets

A coupling (h, A) of a subset A of the vertices with a DGFF h is is called **local** if for every deterministic set A_0, the conditional probability $P(A \subset A_0|h)$ is a measurable function of the values of h in A_0.

Examples of discrete local sets

1. Any deterministic set that does not depend on h.
2. Union of all negative-height hexagon clusters that include hexagons adjacent to the boundary.
3. A coupling of h with a random set A that is equal to (1) with probability $\frac{1}{3}$ and (2) with probability $\frac{2}{3}$.

Equivalent definition of local

LEMMA: A random subset A of the vertices of D, coupled with an instance h of the discrete Gaussian free field on G with boundary conditions h_∂, is **local** if and only if for every deterministic subset A_0 of the vertices of G and function ϕ on the vertices of G that vanishes outside of A_0, the event $A \subset A_0$ is independent of the random variable $(h, \phi)_\nabla$.

Space of closed subsets of $\overline{\mathbb{H}}$

Let Γ be the space all closed subsets of $\overline{\mathbb{H}} \cup \{\infty\}$ (with respect to the d_* metric). Then Γ is a compact metric space when it is endowed with the **Hausdorff** metric induced by d_*, i.e., the distance between sets $S_1, S_2 \in \Gamma$ is

$$\max\{\sup_{x \in S_1} d_*(x, S_2), \sup_{y \in S_2} d_*(y, S_1)\}.$$

Let $\mathcal{G}$ be the Borel σ-algebra on Γ induced by this metric.

Continuum local sets

Following the discrete definitions, we say a random closed set A (with law given by a measure on $(\Gamma, \mathcal{G})$), coupled with the GFF h, is **local** if for every deterministic open $B \subset D$ and function $\phi \in H(B)$ (which vanishes in $D \backslash B$), the event $B \cap A \neq \emptyset$ is independent of the random variable $(h, \phi)_\nabla$.

Equivalently, for every deterministic closed $A_0 \subset D$, the conditional probability $P(A \subset A_0 | h)$ is a measurable function of the projection of h onto the space of functions that are harmonic off of A_0—i.e., it does not depend on the projection of h onto the orthogonal space of functions supported on A_0.

Denote by η_A the expectation of h in the complement of A conditioned on the heights on (an infinitesimal neighborhood of) A. This η_A is harmonic off of A.

Unions of local sets

Given two local sets A_1 and A_2 (coupled with GFF) we define a coupling of the triple (A_1, A_2, h) in a way that preserves the marginal laws of (h, A_1) and (h, A_2) and such that *conditioned* on h, the conditional laws of A_1 and A_2 are almost surely independent of one another.

LEMMA: If A_1 and A_2 are boundary connected local sets coupled with h, then their union $A_1 \cup A_2$ (with the coupling described above) is also local. Moreover, $\eta_{A_1 \cup A_2}$ almost surely tends to η_{A_1} on paths in $D \backslash (A_1 \cup A_2)$ approaching points in $A_1 \backslash A_2$.

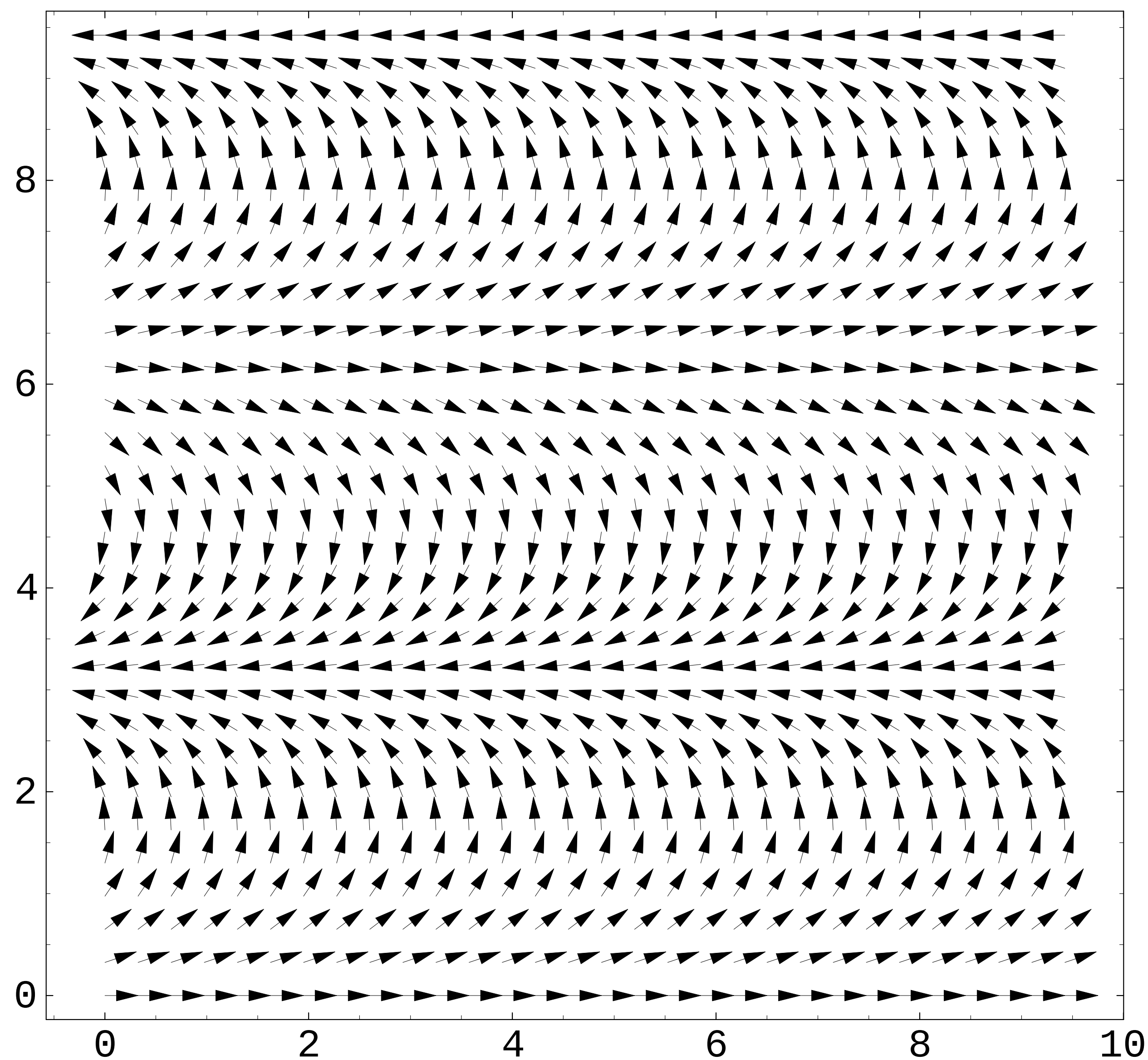
Examples of discrete local set

1. Any deterministic set that does not depend on h.
2. What else?

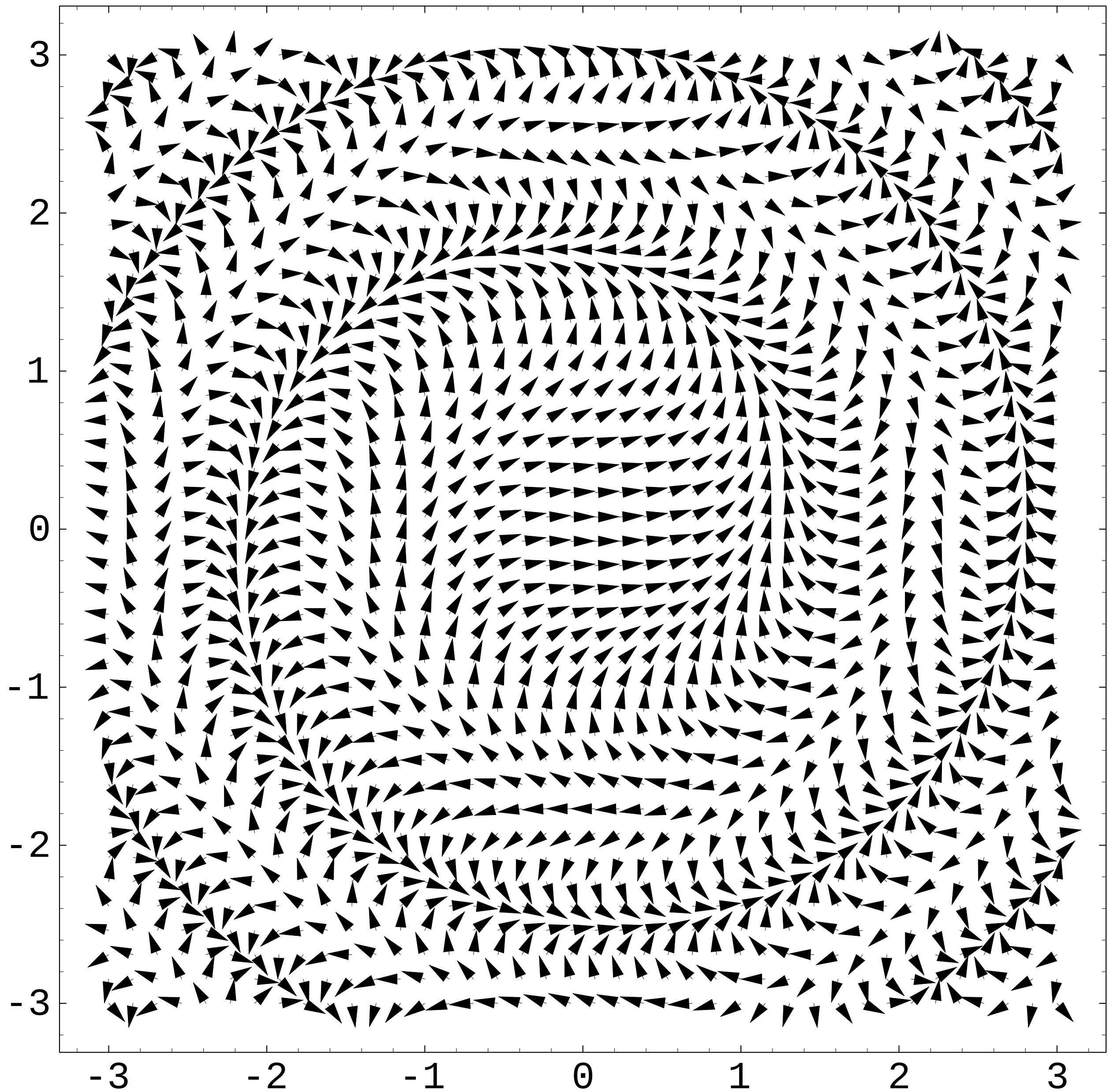
Limits of discrete local sets are local

LEMMA: Let D_n be a sequence of TG-domains with maps $\phi_n : D_n \to \mathbb{H}$ such that $r_D \to \infty$ as $n \to \infty$, and let A_n be a sequence of discrete local subsets of $D_n \cap \mathrm{T}G$. Then there is a subsequence along which $(h, \phi_n A_n)$ converges weakly to a limiting coupling (h, A). In any such limit, A is local.

Vector Field e^{ih} **where** $h(x,y) = \pi/2 - y$



Vector Field e^{ih} where $h(x,y) = x^2 + y^2$



Some time derivatives of SLE

We will construct our first really interesting local sets using SLE. From definition of SLE, we have $dg_t(z) = \frac{2}{g_t(z)-W(t)}$ and $dW_t = \sqrt{\kappa} dB_t$. Write $f_t(z) = g_t(z) - W_t$ and apply Ito's formula to compute time derivatives of $f_t(z)$, $\log f_t(z)$, $f_t'(z)$, and $\log f_t'(z)$:

$$\begin{aligned}
df_t(z) &= \frac{2}{f_t(z)} dt - \sqrt{\kappa} dB_t \\
d \log f_t(z) &= \frac{2}{f_t(z)^2} dt - \frac{\sqrt{\kappa}}{f_t(z)} dB_t - \frac{\kappa}{2 f_t(z)^2} dt \\
&= \frac{(4-\kappa)}{2 f_t(z)^2} dt - \frac{\sqrt{\kappa}}{f_t(z)} dB_t \\
df_t' &= \frac{-f_t'}{f_t(z)^2} dt \\
d \log f_t'(z) &= \frac{-2}{f_t(z)^2} dt
\end{aligned}$$

Important martingale of SLE

Observe:

$$d[\log f_t(z) + \frac{4-\kappa}{4} \log g_t'(z)] = -\sqrt{\kappa} f_t(z)^{-1} dB_t.$$

Thus, for any fixed value of z, the following linear combination of the angle and the winding number is a martingale:

$$h_t(z) = -\frac{2\lambda}{\pi} \arg(f_t(z)) - \chi \arg f_t'(z) + \lambda$$

where $\lambda := \lambda(\kappa) := \sqrt{\frac{\pi}{2\kappa}}$ and $\chi := \chi(\kappa) := (4-\kappa)\lambda$.

We chose λ and χ in such a way that makes $dh_t(z)$ (which is a multiple of $\mathrm{Im}(f_t(z)^{-1})dB_t$) independent of κ.

Harmonic measure of the tip

The function $-2\mathrm{Im}(f_t(z)^{-1})dB_t$ is significant. At time $t = 0$, the function $-2\mathrm{Im}(f_t(z)^{-1})$ is simply $-2\mathrm{Im}(z^{-1})$. This is a positive harmonic function whose level sets are circles in $\mathbb{H}$ that are tangent to $\mathbb{R}$ at the origin. And in fact, it is a derivative of the Green's function $G(x,y) = \log\left|\frac{x-y}{x-\overline{y}}\right|$ in the following sense:

$$[\frac{\partial}{\partial s}G(is,z)]_{s=0} = \frac{\partial}{\partial s}\left|\frac{z-is}{z+is}\right|_{s=0} = \mathrm{Re}\frac{-2iz}{|z^2|} = -2\mathrm{Im}(z^{-1}).$$

Intuitively, the value of $-2\mathrm{Im}(f_t(z)^{-1})$ represents the harmonic measure of the tip of γ_t as seen from the point z.

In this setup, h_0 is the harmonic function on $\mathbb{H}$ with boundary conditions λ on the negative real axis and $-\lambda$ on the positive real axis. Observe that when $\kappa = 4$, we have $\chi = 0$ and hence each h_t is the harmonic function on $\mathbb{H}\backslash\gamma_t$ with boundary conditions λ on the left side of the tip of γ_t and $-\lambda$ on the right side. In this case, $h_t(z)$ is simply a linear function of the angle $\arg f_t(z)$.

Log conformal radius parameterization

Write $C_t(z)$ for the conformal radius of z in $\mathbb{H}\backslash\gamma_t$. Observe that we can also write:

$$\log C_t(z) = -\mathrm{Re}[\log g_t'(z)] + \log \mathrm{Im} g_t(z)$$

Write $\theta = \arg f_t(z)$. Then
$d\mathrm{Im} f_t(z) = 2\mathrm{Im} f_t(z)^{-1} dt = -2|f_t(z)|^{-1}\sin(\theta) dt$ and
$d \log \mathrm{Im} g_t(z) = \frac{-2|f_t(z)|^{-1}\sin(\theta)}{-\sin(\theta)|f_t(z)|} dt = 2|f_t(z)|^{-2} dt$. Now we can compute:

$$d\log C_t(z) = -2[\mathrm{Re}(|f_t(z)|^{-2}dt) - |f_t(z)|^2]dt = -2[\cos(2\theta)-1]|f_t(z)|^{-2} =$$

$$-2[-2\sin^2\theta]|f_t(z)|^{-2} = -4[\sin(\theta)]^2|f_t(z)|^{-2} = -4[\mathrm{Im} f_t(z)^{-1}]^2 dt$$

Using the convention $dB_t dB_t = dt$, we have

$$(dh_t)^2 \quad = \quad -d\log C_t. \tag{1}$$

By standard Ito calculus, this implies that if time is parameterized by the negative log conformal radius $-\log C_t(z)$, then $h_t(z)$ is a standard Brownian motion.

What about multiple points?

Weighted averages of h_t over multiple points in $\mathbb{H}$ are also Brownian motions when properly parametrized. Our computation uses the *Green's function on* $\mathbb{H}$: $G(x,y) = \log \left|\frac{x-y}{x-\overline{y}}\right|$. (Here $\overline{y}$ is the complex conjugate of y.)

Write $G_t(x,y) = G(f_t(x) - f_t(y))$ when x and y are both in the infinite component of $\mathbb{H}\backslash\gamma_t$. Otherwise, we let $G_t(x,y)$ be the limiting value of $G_s(x,y)$ as s approaches the first time at which one of x or y ceases to be in this infinite component. For fixed x and y, this limit exists almost surely when $4 < \kappa < 8$: it is equal to zero when x and y are in different connected components of $\mathbb{H}\backslash\gamma_t$, and when x and y lie in the same component, it is simply the Green's function of x and y on this bounded domain. Now we have:

$$\begin{aligned}
dG_t(x,y) &= d\,\mathrm{Re}\log[g_t(x)-g_t(y)] - d\,\mathrm{Re}\log[g_t(x)-\overline{g_t(y)}] \\
&= 2\mathrm{Re}\frac{(f_t(x))^{-1}-(f_t(y))^{-1}}{g_t(x)-g_t(y)}dt - \\
&\quad\; 2\mathrm{Re}\frac{(g_t(x)-W_t)^{-1}-(\overline{g_t(y)}-W_t)^{-1}}{g_t(x)-\overline{g_t(y)}}dt \\
&= -2\mathrm{Re} f_t(x)^{-1} f_t(z)^{-1} dt + 2\mathrm{Re} f_t(x)\overline{f_t(y)}^{-1} dt \\
&= -4\mathrm{Re}[i f_t(x)^{-1}\mathrm{Im} f_t(z)^{-1}] \\
&= -4\mathrm{Im} f_t(x)^{-1}\mathrm{Im} f_t(z)^{-1} dt.
\end{aligned}$$

Using the convention $dB_t dB_t = dt$ and the expression for $dh_t(x)$ in the previous section, this gives:

$$dG_t(x,y) = -(dh_t(x)dh_t(y)) \tag{2}$$

The above equations imply:

LEMMA: For any $x_1, \ldots, x_m \in \mathbb{H}$, $a_1, \ldots, a_m \in \mathbb{R}$, and h_t and G_t defined as above, we have:

$$\left(d \sum_{j=1}^{m} a_j h_t(x_j) \right)^2 = -d \left(\sum_{1 \le j,k \le m} a_j a_k \tilde{G}_t(x_j, x_k) \right)$$

where

$$\tilde{G}_t(x,y) = \begin{cases} G_t(x,y) & x \neq y \\ \log C_t(x) & x = y \end{cases}.$$

Note that if $\kappa \geq 8$ (so that γ is space-filling) and $x \in \mathbb{H}$, then the value $-\log C_t(x)$ tends (almost surely) to infinity as γ approaches and finally hits the point x. If $\kappa < 8$ and $x, y \in \mathbb{H}$, then $-\log C_t(x)$, $-G_t(x,y)$, and $h_t(x)$ each tend almost surely to a finite limit as t tends to infinity.

What about a continuous density function?

We extend to the case that ν is a measure with a density function $\rho \in \Delta H(D)$ (so that, in particular, ν has no point masses, and thus we have no $-\log C_t(x)$ terms). Write

$$E_t(\rho) := \int G_t(x,y)\rho(x)\rho(y)dxdy$$

for the energy of assembly of ρ in the domain or union of domains $\mathbb{H}\backslash\gamma_t$.

LEMMA: Fix $0 < \kappa < 8$. Then for any $\rho \in \Delta H(D)$, we have $(d(h_t,\rho))^2 = -dE_t(\rho)$. In other words, $d(h_t,\rho)$ is a Brownian motion when parametrized by minus the energy of assembly of ρ in the union of domains $\mathbb{H}\backslash\gamma_t$.

PROOF: By Fubini's theorem we have:

$$
\begin{aligned}
d(h_t, \rho) &= \left(\int \rho(x) dh_t(x) dx \right) \\
(d(h_t, \rho))^2 &= \left(d \int \rho(x) h_t(x) \right)^2 \\
&= -d \left(\int \rho(x) \rho(y) G_t(x, y) \right)
\end{aligned}
$$

Now, define $h_\infty(z) = \lim_{t\to\infty} h_t(z)$, $G_\infty(x, y) = \lim_{t\to\infty} h_t(x, y)$, and $E_\infty(\rho) = \lim_{t\to\infty} E_t(\rho)$. If $\kappa < 8$, then the reader may check that for fixed x $h_\infty(x)$ is almost surely finite and harmonic in $\mathbb{H}\backslash\gamma$. Similarly, since $G_t(x, y)$ and $E_\infty(\rho)$ are decreasing functions of t, these limits also exist almost surely.

THEOREM: Let $0 < \kappa < 8$, let h be equal to h_∞ plus an independent sum of zero-boundary GFF's, one in each of component of $\mathbb{H} \backslash \gamma$. Then h is a GFF in $\mathbb{H}$ with boundary conditions given by

$$\phi(x) = \begin{cases} \lambda & x \geq 0 \\ -\lambda & x < 0 \end{cases},$$

for $x \in \mathbb{R}$.

GFF convergence

Since each (h, ρ) is a sum of a Brownian motion stopped at time $E_0(\rho) - E_\infty(\rho)$ and a Gaussian of variance $E_\infty(\rho)$, it has the same law as a Gaussian of variance $E_0(\rho)$. The fact that the limiting field has the stated boundary conditions follows from the fact that each h_t has these boundary conditions.

If $\kappa \geq 8$, SLE$_\kappa$ is space-filling, and h_t is not a function a.e., be may still define (h_t, ρ) to be the solution to $d(h_t, \rho) = (-2\mathrm{Im}(f_t)^{-1}, \rho)dB_t$.

THEOREM: When $\kappa \geq 8$ the variables (h, ρ), for $\rho \in \Delta H(D)$, are the Gaussian free field on $\mathbb{H}$ with boundary conditions

$$\epsilon(x) = \begin{cases} \lambda & x \geq 0 \\ -\lambda & x < 0 \end{cases},$$

for all $x \in \mathbb{R}$.

Contour lines: local and deterministic?

THEOREM: In the couplings (h, γ) of the free field h and an SLE_κ, as described above, the random set $\gamma([0, \infty])$ is a local set. In fact, for any stopping time T, the set $\gamma([0, T])$ is local.