LOCAL SETS

of the

GAUSSIAN FREE FIELD

PART ONE

Scott Sheffield

based on work with Schramm; Schramm and Wilson; and Werner

The *standard Gaussian* on n-dimensional Hilbert space

has density function $e^{-(v,v)/2}$ (times an appropriate constant). We can write a sample from this distribution as

$$\sum_{i=1}^{n} \alpha_i v_i$$

where the v_i are an orthonormal basis for $\mathbb{R}^n$ under the given inner product, and the α_i are mean zero, unit variance Gaussians.

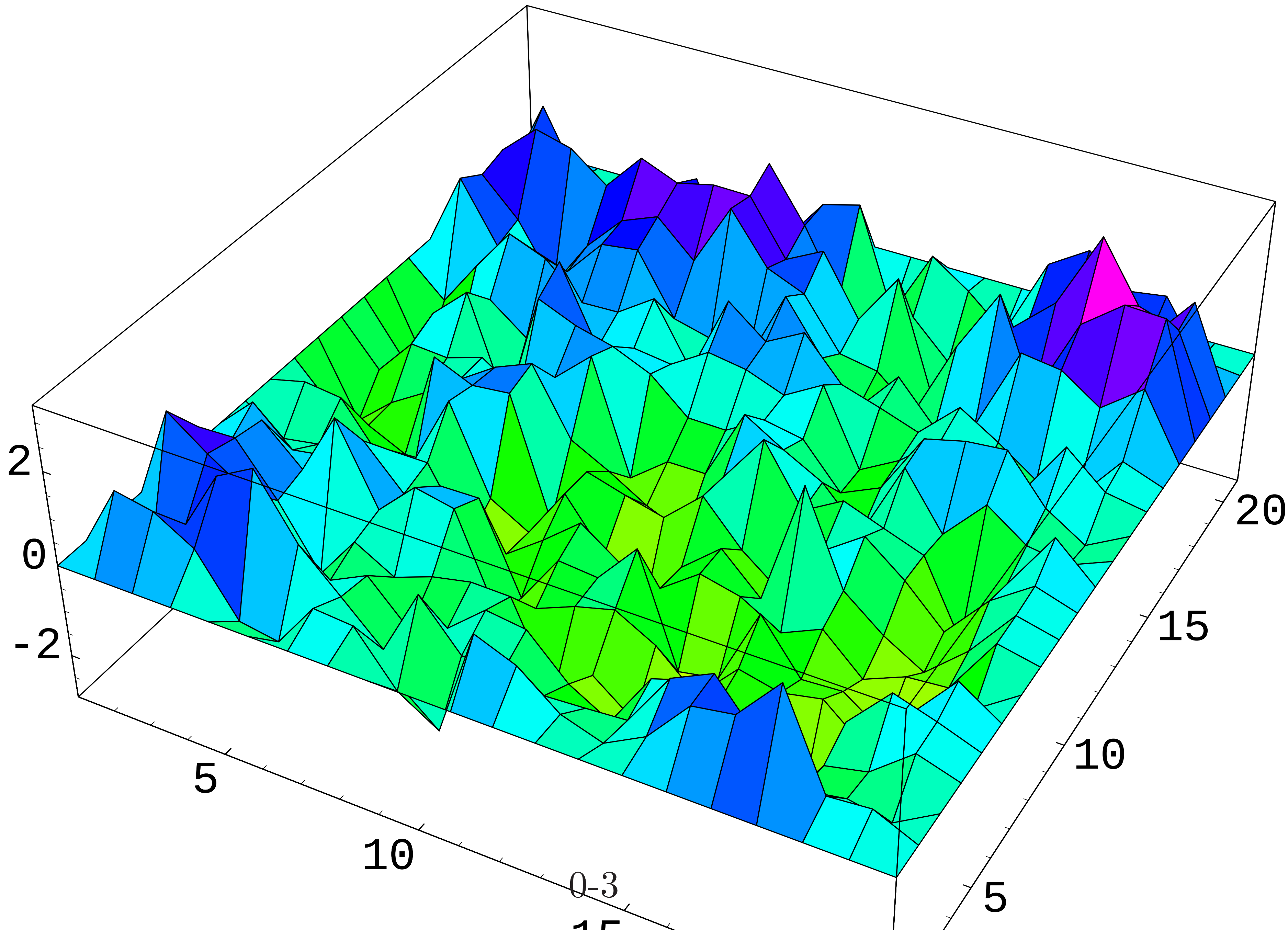
The discrete Gaussian free field

The **Dirichlet energy** of a real function f on the vertices of a planar graph Λ is $H(f) = (f, f)_\nabla$ where $(f, g)_\nabla$ is the **Dirichlet form**

$$(f, g)_\nabla = \sum_{x \sim y} (f(x) - f(y))\,(g(x) - g(y))\,.$$

Fix a function f_0 on boundary vertices of Λ. The set of functions f that agree with f_0 is isomorphic to $\mathbb{R}^n$, where n is the number of interior vertices. The **discrete Gaussian free field** is a random element of this space with probability density proportional to $e^{-H(f)/2}$.

Discrete GFF on 20×20 grid, zero boundary



Some DGFF properties:

Zero boundary conditions: The Dirichlet form $(f, f)_\nabla$ is an inner product on the space of functions with zero boundary, and the DGFF is a standard Gaussian on this space.

Other boundary conditions: DGFF with boundary conditions f_0 is an affine translation of DGFF with zero boundary; i.e., the same as DGFF with zero boundary conditions *plus* the (discrete) harmonic interpolation of f_0 to Λ.

Markov property: **Given** the values of f on the boundary of a subgraph Λ' of Λ, the values of f on the remainder of Λ' have the law of a DGFF on Λ', with boundary condition given by the observed values of f on $\partial\Lambda'$.

The continuum Gaussian free field

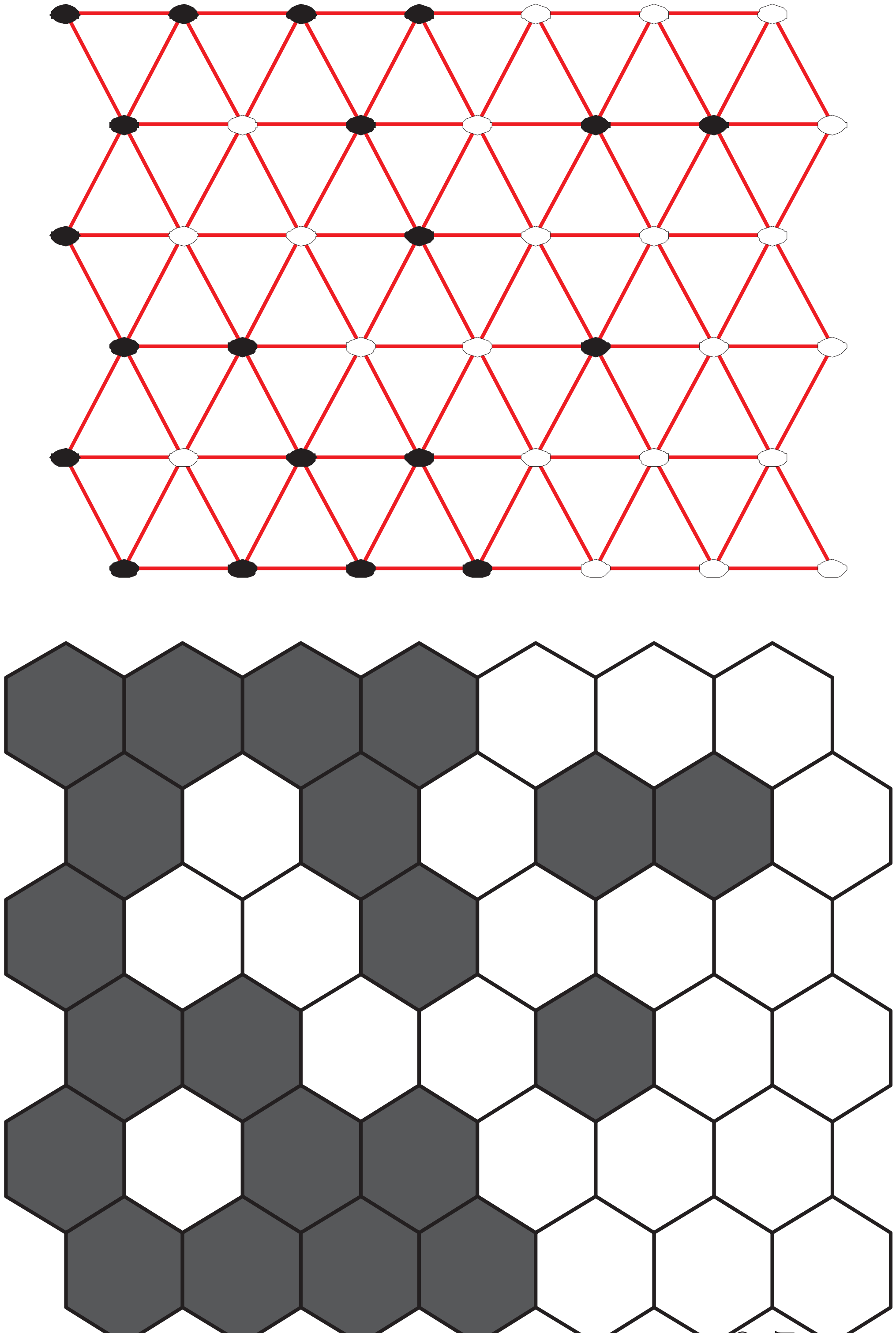
is a “standard Gaussian” on an *infinite* dimensional Hilbert space. Given a planar domain D, let $H(D)$ be the Hilbert space closure of the set of smooth, compactly supported functions on D under the conformally invariant *Dirichlet inner product*

$$(f_1, f_2)_\nabla = \int_D (\nabla f_1 \cdot \nabla f_2) dx dy.$$

One way to view GFF: A formal sum $h = \sum \alpha_i f_i$, where the f_i are an orthonormal basis for H and the α_i are i.i.d. Gaussians. The sum does not converge point-wise, but h can be defined as a *random distribution*—the pairings (h, ϕ) are well defined whenever ϕ is sufficiently smooth. The projection of the GFF onto the space of functions piecewise linear on triangle lattice triangles gives the DGFF (times the lattice-dependent constant $3^{1/4}$).

Laplacian of the Gaussian free field

If $\rho = -\Delta h$ describes an electric charge density, then h is its **Coulomb gas electrostatic potential function** (grounded at the boundary of D), and $(h, h)_\nabla$ is its total potential energy (i.e., the *energy of assembly* of the distribution). The Laplacian of a Gaussian free field is thus a random distribution that we may interpret as a random continuum charge distribution (a type of continuum charge Coulomb gas).



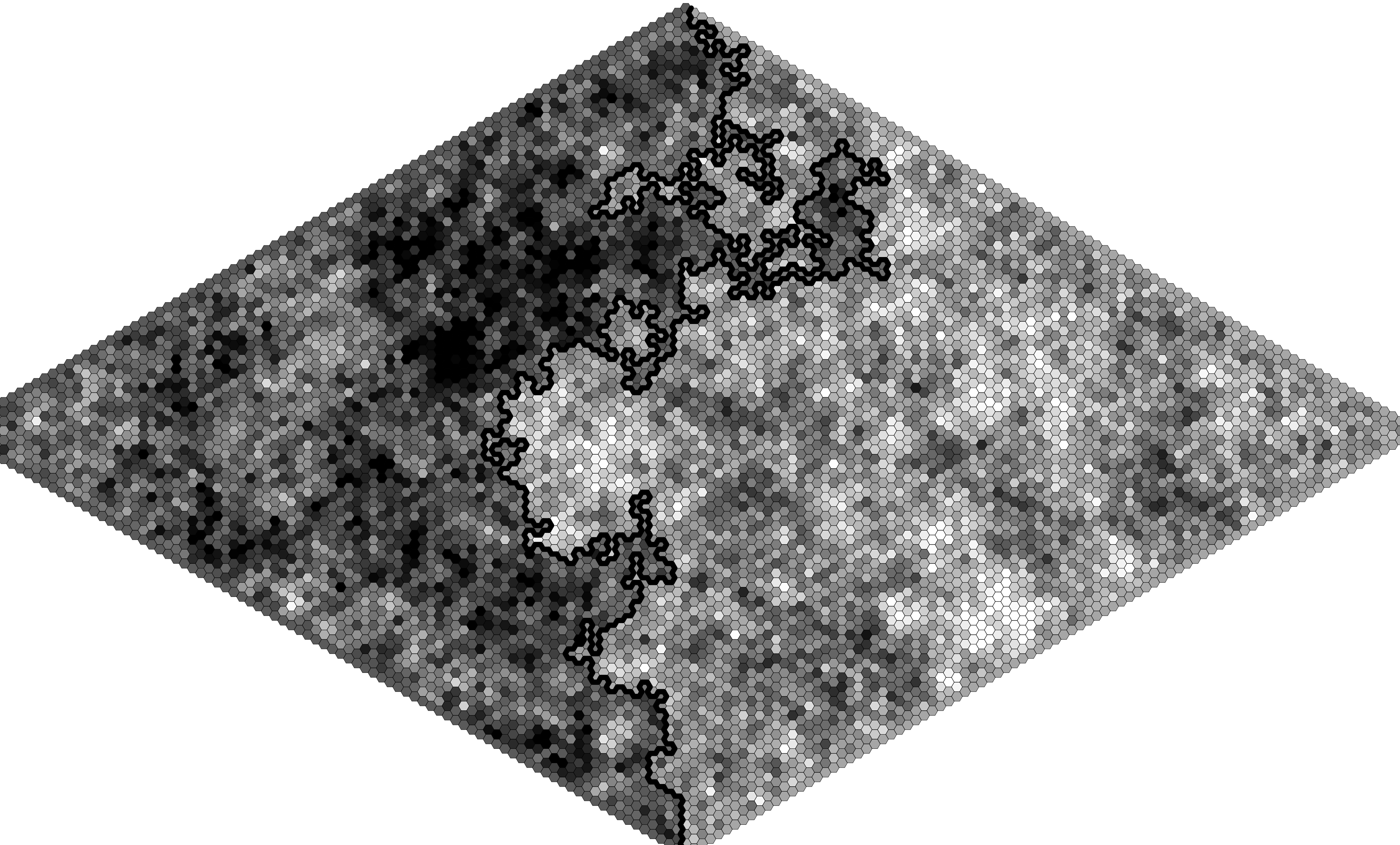
0-7

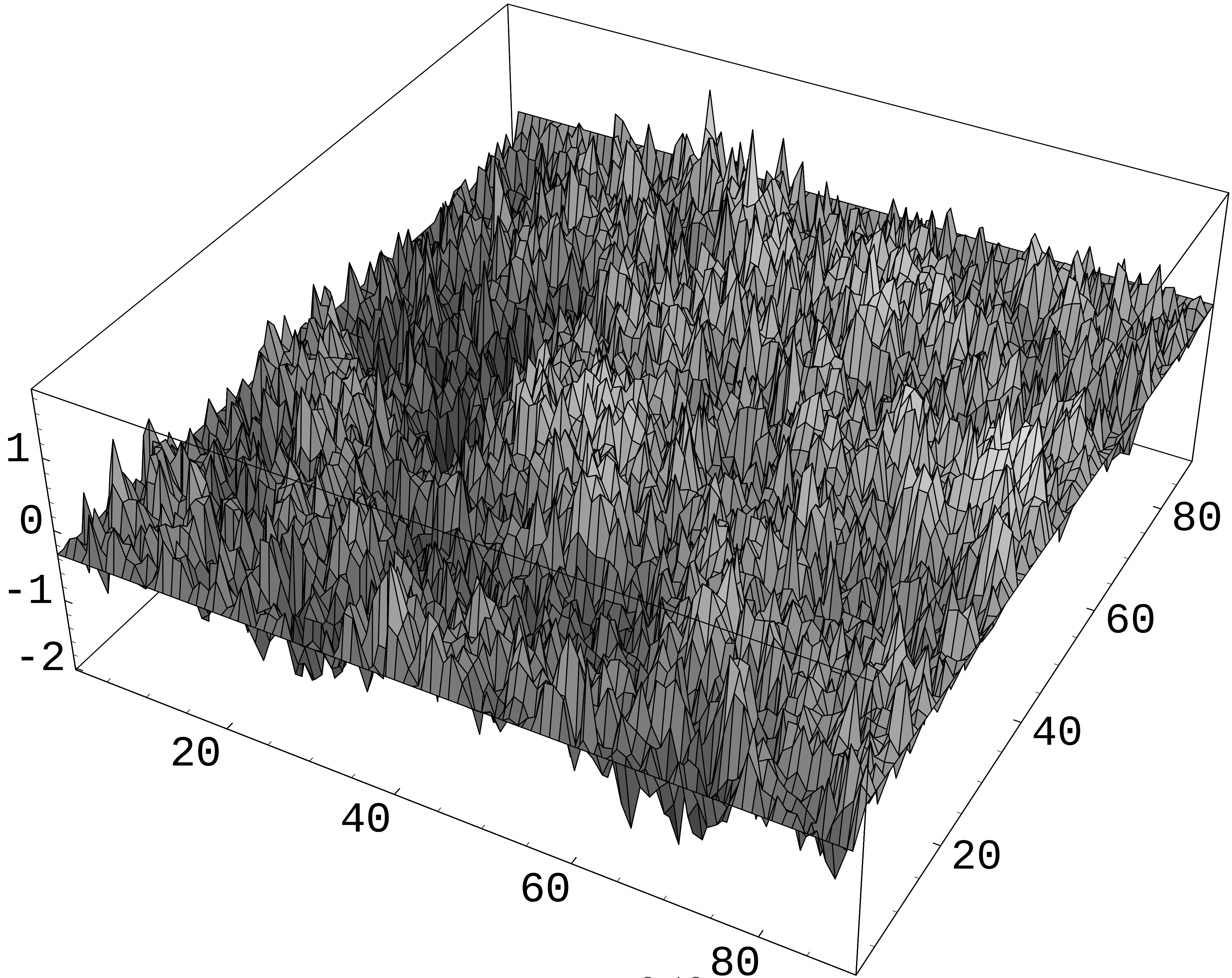
Scaling limit of zero-height contour line

Theorem (Schramm, S): If initial boundary heights are λ on one boundary arc and $-\lambda$ on the complementary arc, where λ is the constant $\sqrt{\frac{\pi}{8}}$, then the scaling limit of the zero-height interface (as the mesh size tends to zero) is SLE_4.

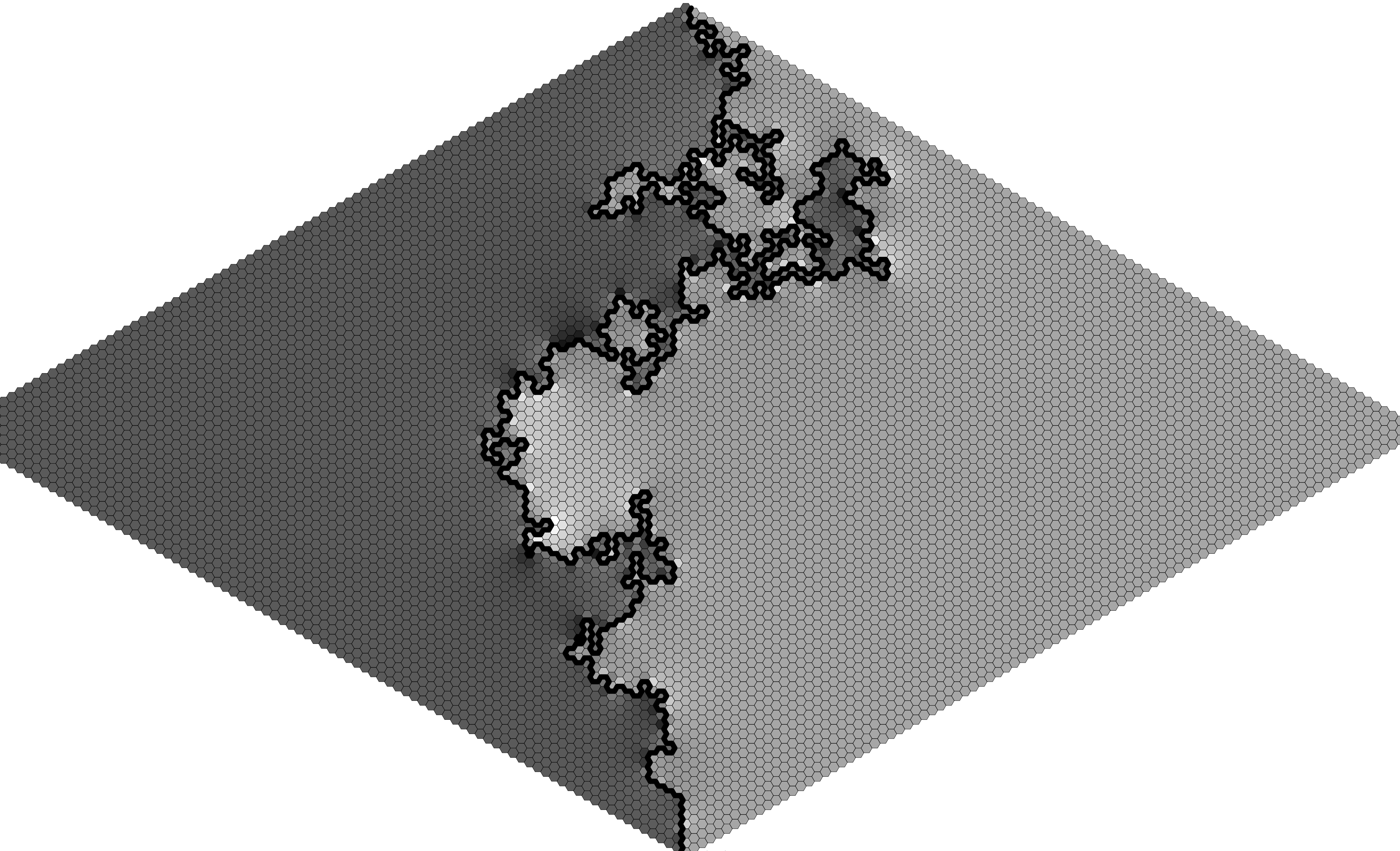
If the initial boundary heights are instead are instead $-(1+a)\lambda$ and $(1+b)\lambda$, then as the mesh gets finer, the laws of the random paths described above converge to the law of $\mathrm{SLE}_{4,a,b}$.

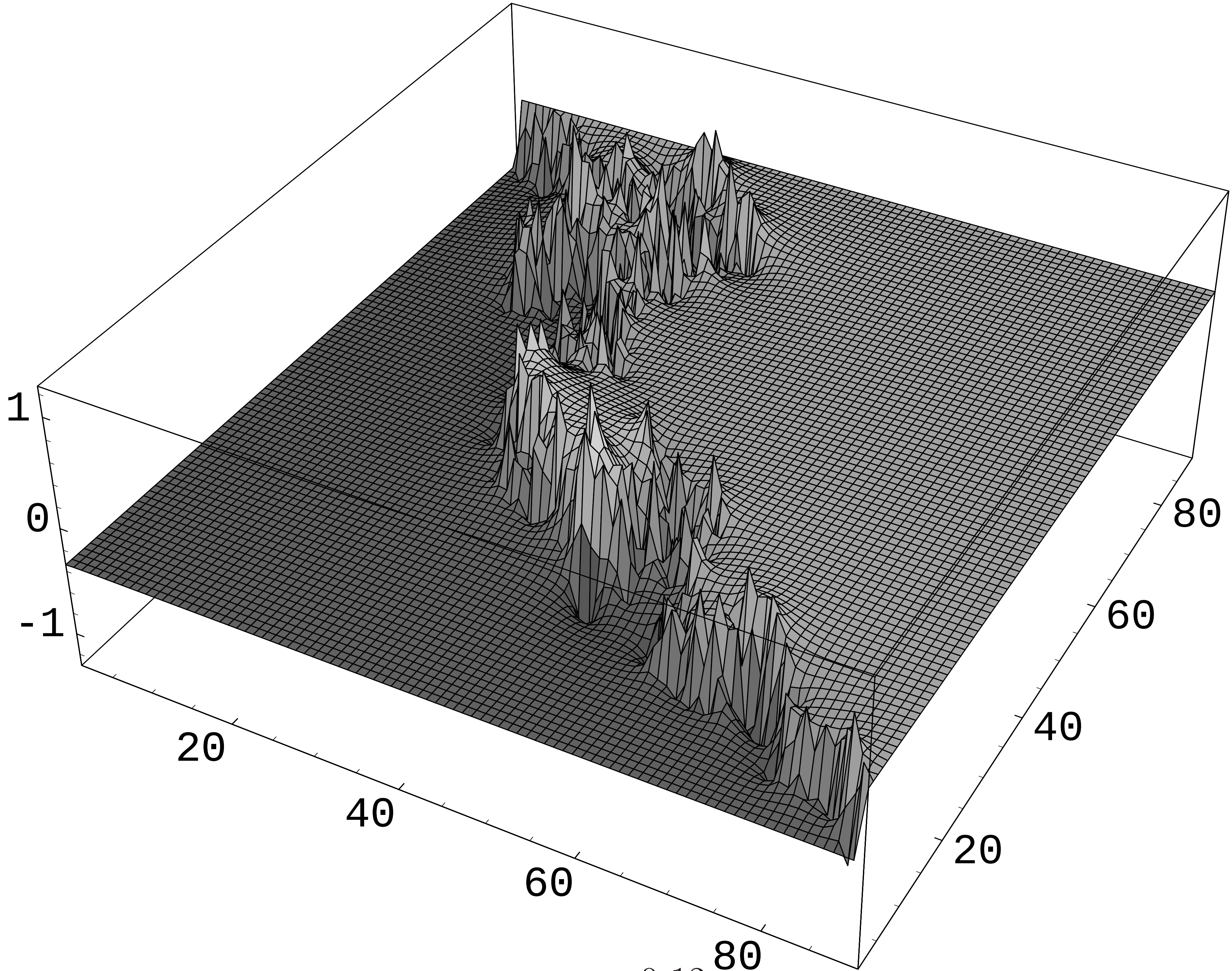
DGFF with $\pm\lambda$ boundary conditions



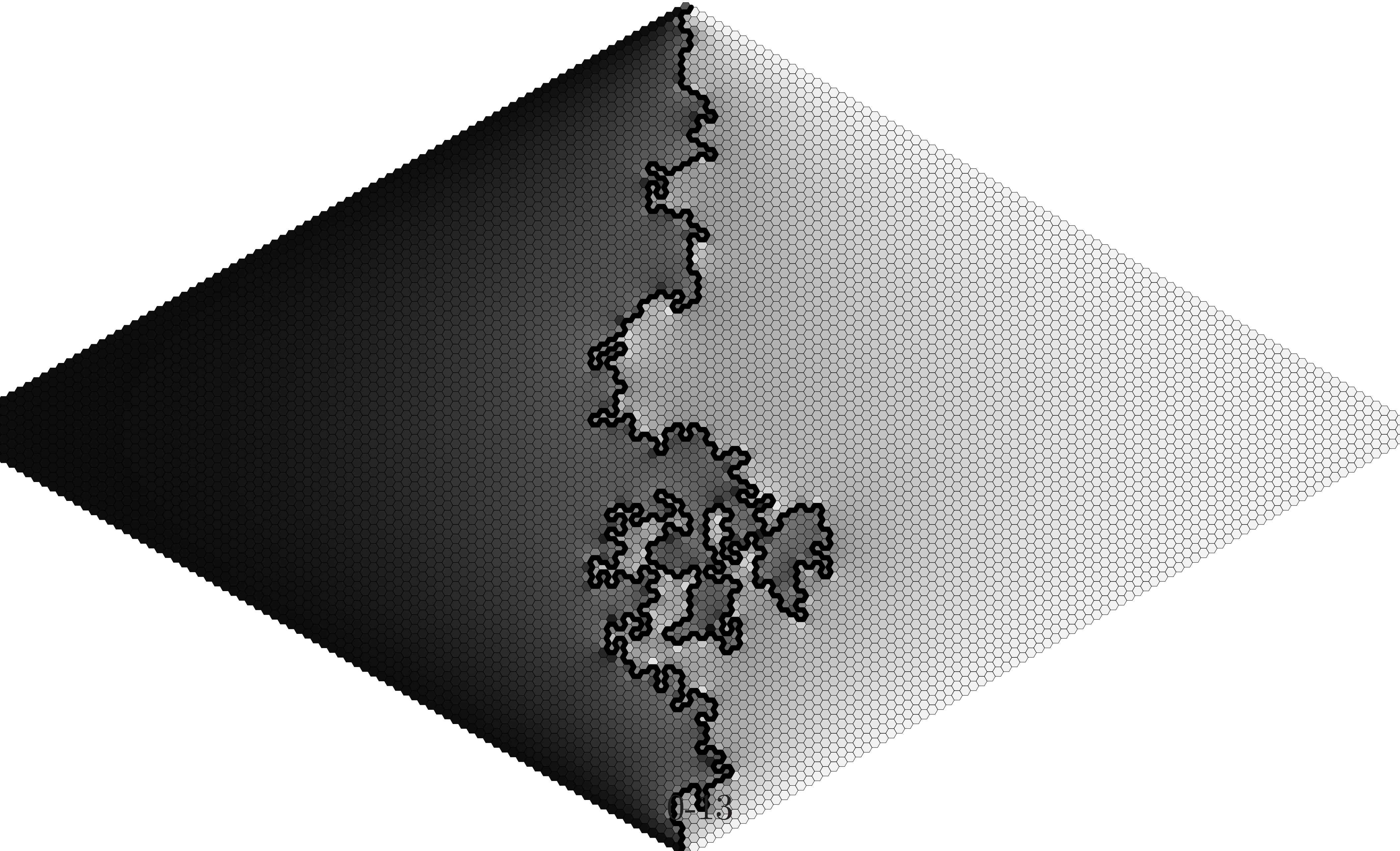

1
0
-1
-2
20
40
60
80
20
40
60
80

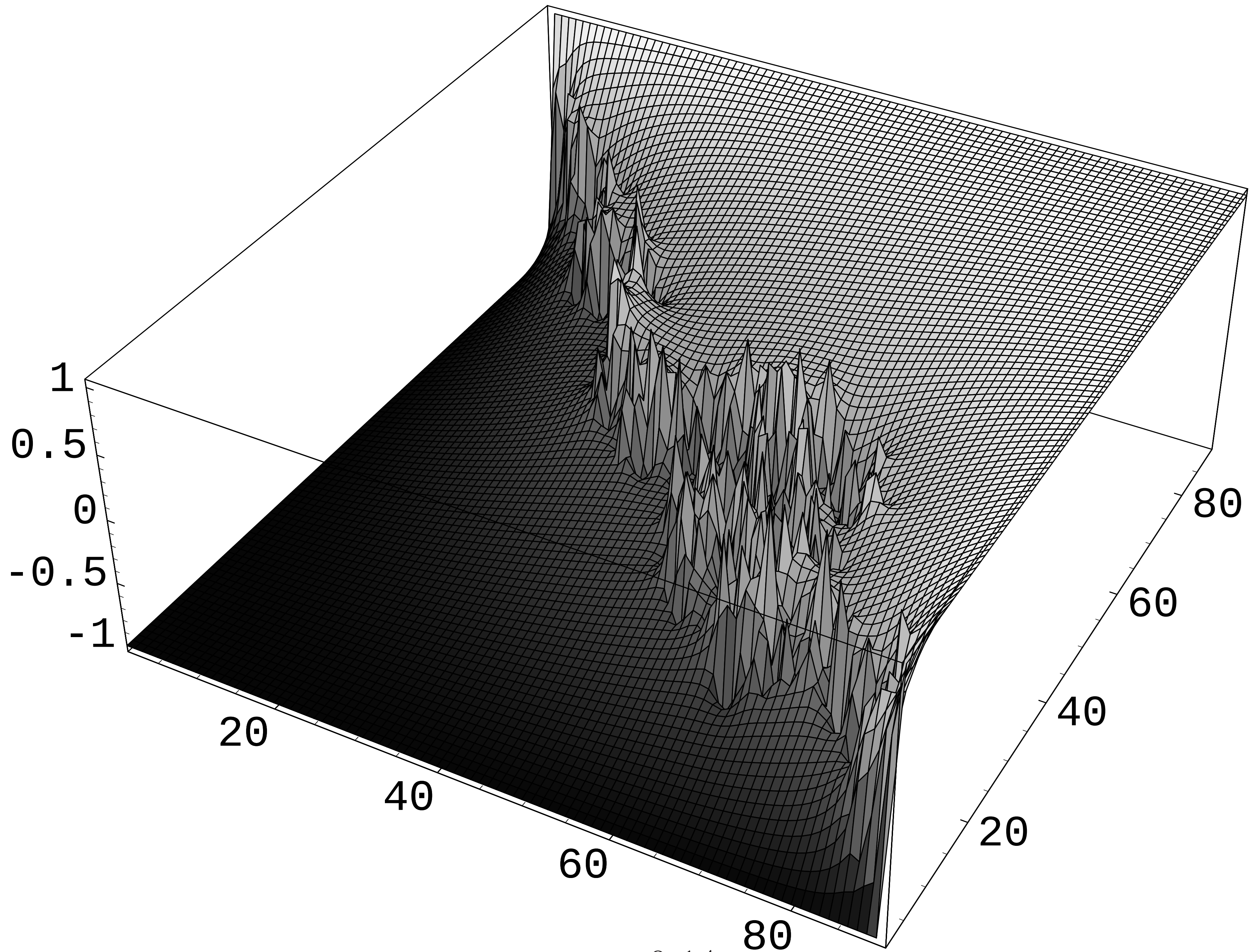
Expectations given values along interface




1
0
-1
20
40
60
80
20
40
60
80

Expectations given interface, $\pm 3\lambda$ boundary conditions




1
0.5
0
-0.5
-1
20
40
60
80
20
40
60
80

“As mesh gets finer”

Let TG be triangular lattice, D a domain whose boundary is simple curve comprised of edges and vertices of TG. The discrete (zero-boundary) GFF is a projection of the continuum GFF onto the subspace $H_{\mathrm{T}G}(D)$ of $H(D)$ comprised of continuous functions that are linear on each triangle. Let ϕ_D be conformal map from D to $\mathbb{H}$. Write $r_D = \mathrm{inr}_{\phi_D^{-1}(i)}(D)$ where $\mathrm{inr}_x(D)$ denotes the radius of D viewed from x. As $r_D \to \infty$, the subspaces $\{f \circ \phi_D^{-1} : f \in \mathbb{H}_{\mathrm{T}G}(D)\}$ become **asymptotically dense** in $H(\mathbb{H})$, i.e.,

LEMMA: For each $f \in H(\mathbb{H})$, the values $||P_D(f) - f||_\nabla$ tend to zero as $r_D \to \infty$, where P_D is projection onto $\{f \circ \phi_D^{-1} : f \in \mathbb{H}_{\mathrm{T}G}(D)\}$. In fact, if $f \in H_s(D)$, then $||P_D(f) - f||_\nabla = O(\frac{1}{r_D})$.

Height gap lemma

Take any boundary conditions for a DGFF bounded above by some universal constant M, non-negative on a right boundary arc and non-positive on the left. Let γ denote the discrete interface as above and let T be some discrete stopping time for γ and let γ^T denote γ stopped at T. Let v be some vertex in D. Let F_T denote the function that is $+\lambda$ on right side $V_+(\gamma^T)$ of γ^T, $-\lambda$ on left side $V_-(\gamma^T)$ of γ^T, equal to boundary values of h on ∂D, and discrete-harmonic at all other vertices in $\overline{D}$. Let h_T be the discrete harmonic interpolation of the values of h on $V_-(\gamma^T) \cup V_+(\gamma_T)$ and on all TG-vertices in ∂D.
LEMMA: Assume setting as above. Then

$$h_T - F_T(v) \to 0$$

in probability as D and v are taken so that $\mathrm{dist}(v, \partial D) \to \infty$, while M is held fixed. The same holds as $r \to \infty$ when v is a random vertex (with law independent of h) supported on the set of points of distance at least r from ∂D.

Property of SLE_4

Observe: SLE_4 is the only random path γ with the following property: *Given* $\gamma([0,t])$, the probability that γ passes z on right equals the probability that Brownian motion started at z first hits $\mathbb{R} \cap \gamma[0,t]$ on the left side of $\gamma(t)$. Similar characterizations apply to the $\mathrm{SLE}_{4,a,b}$.

This is the idea behind proof that discrete paths converge in law to $\mathrm{SLE}_{4,a,b}$. To formally define level lines of the continuum field—and show that the discrete paths converge in probability to these—we will need some more abstract machinery.

Almost independence

Say two coupled variables X and Y are **almost independent** if their joint law is absolutely continuous with respect to the product of the marginal laws. Equivalently, for almost all X, the conditional law of Y given X is absolutely continuous with respect to the unconditioned law.

LEMMA: If D is the unit disc with subdomains A and B, then the law of the GFF restricted to A and restricted to B are almost independent whenever the distance between A and B is positive.

Discrete deterministic local sets

A vertex-subset valued function A defined on the set of possible instances h of the GFF (i.e., set of real-valued functions on the vertices of G) is called **local** if $A(h_1) = A(h_2)$ whenever h_1 and h_2 agree on A. Such an A is called a **deterministic local set** (i.e., given h, it is a deterministic function of A).

Discrete non-deterministic local sets

A coupling (h, A) of a subset A of the vertices with a DGFF h is is called **local** if for every deterministic set A_0, the conditional probability $P(A \subset A_0 | h)$ is a measurable function of the values of h in A_0.

In case of the DGFF, this is equivalent to saying that $P(A \subset A_0 | h)$ is a measurable function of the projection of h onto the space of functions that are harmonic on the complement of A_0. Equivalently, it is independent of the projection of h onto the orthogonal space of functions supported on A_0.

Equivalent definition of local

LEMMA: A random subset A of the vertices of D, coupled with an instance h of the discrete Gaussian free field on G with boundary conditions h_∂, is **local** if and only if for every deterministic subset A_0 of the vertices of G and function ϕ on the vertices of G that vanishes outside of A_0, the event $A \subset A_0$ is independent of the random variable $(h, \phi)_\nabla$.

Space of closed subsets of $\overline{\mathbb{H}}$

Let Γ be the space all closed subsets of $\overline{\mathbb{H}} \cup \{\infty\}$ (with respect to the d_* metric). Then Γ is a compact metric space when it is endowed with the **Hausdorff** metric induced by d_*, i.e., the distance between sets $S_1, S_2 \in \Gamma$ is

$$\max\{\sup_{x \in S_1} d_*(x, S_2), \sup_{y \in S_2} d_*(y, S_1)\}.$$

Let $\mathcal{G}$ be the Borel σ-algebra on Γ induced by this metric.

Continuum local sets

Following the discrete definitions, we say a random closed set A (with law given by a measure on $(\Gamma, \mathcal{G})$), coupled with the GFF h, is **local** if for every deterministic open $B \subset D$ and function $\phi \in H(B)$ (which vanishes in $D \backslash B$), the event $B \cap A \neq \emptyset$ is independent of the random variable $(h, \phi)_\nabla$.

Equivalently, for every deterministic closed $A_0 \subset D$, the conditional probability $P(A \subset A_0 | h)$ is a measurable function of the projection of h onto the space of functions that are harmonic off of A_0—i.e., it does not depend on the projection of h onto the orthogonal space of functions supported on A_0.

Denote by η_A the expectation of h in the complement of A conditioned on the heights on (an infinitesimal neighborhood of) A. This η_A is harmonic off of A.

Unions of local sets

Given two local sets A_1 and A_2 (coupled with GFF) we define a coupling of the triple (A_1, A_2, h) in a way that preserves the marginal laws of (h, A_1) and (h, A_2) and such that *conditioned* on h, the conditional laws of A_1 and A_2 are almost surely independent of one another.

LEMMA: If A_1 and A_2 are boundary-connected local sets coupled with h, then their union $A_1 \cup A_2$ (with the coupling described above) is also local. Moreover, $\eta_{A_1 \cup A_2}$ almost surely tends to η_{A_1} on paths in $D \backslash (A_1 \cup A_2)$ approaching points in $A_1 \backslash A_2$.

Limits of discrete local sets are local

LEMMA: Let D_n be a sequence of $\mathrm{T}G$-domains with maps $\phi_n : D_n \to \mathbb{H}$ such that $r_D \to \infty$ as $n \to \infty$, and let A_n be a sequence of discrete local subsets of $D_n \cap \mathrm{T}G$. Then there is a subsequence along which $(h, \phi_n A_n)$ converges weakly to a limiting coupling (h, A). In any such limit, A is local.