Loop-erased random walk and Fomin's identity

Michael J. Kozdron
University of Regina

`http://www.math.uregina.ca/∼kozdron/`

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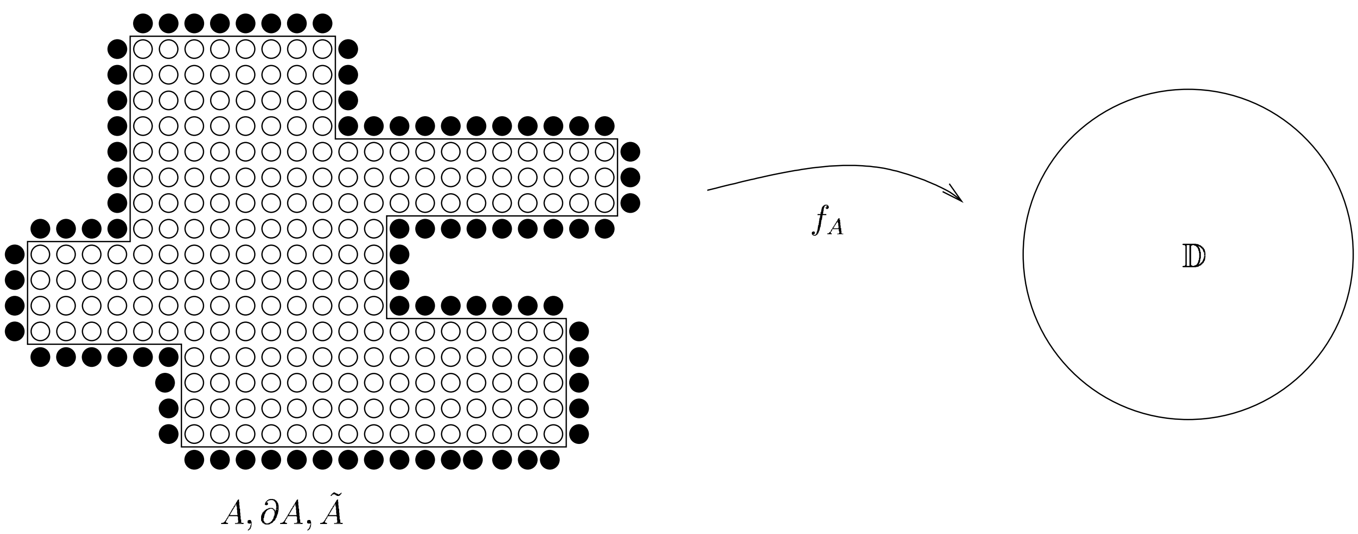
Our Discrete Domains

$A \subset \mathbb{Z}^2$ simply connected, $\mathcal{A}^n$: those A with $n \le \mathrm{inrad}(A) \le 2n$

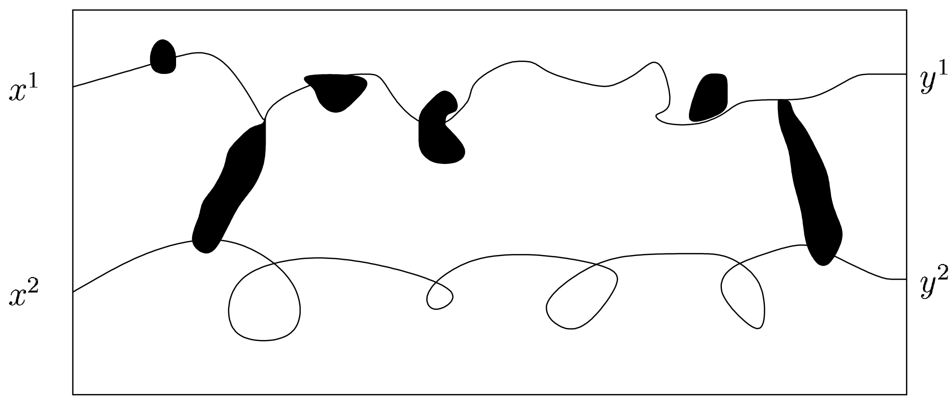
For $z \in A$, $x, y \in \partial A$, let

$$h_A(z,y) = \mathbb{P}^z\{S_\tau = y\}$$
$$h_{\partial A}(x,y) = \mathbb{P}^x\{S_\tau = y, S_1 \in A\}$$

Associate to each A the simply connected 'union of squares' domain $\tilde{A} \subset \mathbb{C}$



Fomin's Identity



$$\mathbb{P}\left\{S^1_{\tau^1_A} = y^1, \;\; S^2_{\tau^2_A} = y^2, \;\; S^2[0,\tau^2_A] \cap \mathcal{L}^1 = \emptyset\right\}$$

$$= h_{\partial A}(x^1, y^1) \cdot h_{\partial A}(x^2, y^2) - h_{\partial A}(x^1, y^2) \cdot h_{\partial A}(x^2, y^1)$$

Fomin's Identity

Suppose $A \in \mathcal{A}^n$, and $x^1, \ldots, x^k, y^k, \ldots, y^1$ are distinct points in ∂A, ordered counterclockwise.

For $i = 1, \ldots, k$, let $\mathcal{L}^i$ denote loop erasure of path $[S^i_0 = x^i, S^i_1, \ldots, S^i_{\tau^i_A}]$.

Let $\mathcal{C} = \mathcal{C}(x^1, \ldots, x^k, y^k, \ldots, y^1; A)$ be event that both

$$S^i_{\tau^i_A} = y^i, \quad i = 1, \ldots, k, \tag{\dagger}$$

and

$$S^i[0, \tau^i_A] \cap (\mathcal{L}^1 \cup \cdots \cup \mathcal{L}^{i-1}) = \emptyset, \quad i = 2, \ldots, k. \tag{\ddagger}$$

Theorem (Fomin).

$$\mathbb{P}(\mathcal{C}) = \det \begin{bmatrix} h_{\partial A}(x^1, y^1) & \cdots & h_{\partial A}(x^1, y^k) \\ \vdots & \ddots & \vdots \\ h_{\partial A}(x^k, y^1) & \cdots & h_{\partial A}(x^k, y^k) \end{bmatrix}$$

Fomin's Conjecture

This theorem is a special case of an identity that Fomin established for general discrete stationary Markov processes.

> "In order for the statement of Theorem 7.5 to make sense, the Markov process under consideration does not have to be discrete. . . . The proofs can be obtained by passing to a limit in the discrete approximation. The same limiting procedure can be used to justify the well-definedness of the quantities involved; notice that in order to define a continuous analogue of Theorem 7.5, we do not need the notion of loop-erased Brownian motion. Instead, we discretize the model, compute the probability, and then pass to the limit. One can further extend these results to densities of the corresponding hitting distributions. Technical details are omitted."

$D, D' \subset \mathbb{C}$ simply connected domains; $f : D \to D'$ conformal transformation

$z \in D$; ∂D locally analytic at x, y; $\partial D'$ locally analytic at $f(x)$, $f(y)$

Poisson Kernel

Poisson kernel, density of harmonic measure wrt Lebesgue measure, exists

Fact. $H_D(z, y) = |f'(y)| \, H_{D'}(f(z), f(y))$

Excursion Poisson Kernel

Definition. $H_{\partial D}(x, y) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \, H_D(x + \varepsilon \mathbf{n}_x, y)$

Fact. $H_{\partial D}(x, y) = |f'(x)| \, |f'(y)| \, H_{\partial D'}(f(x), f(y))$

Main Estimate

Theorem (Lawler-K). *If* $A \in \mathcal{A}^n$ *with associated 'union of squares' domain* $\tilde{A} \subset \mathbb{C}$ *and* $f_A : \tilde{A} \to \mathbb{D}$, $f_A(0) = 0$, $f'_A(0) > 0$, $\theta_A(x) = \arg(f_A(x))$, *then*

$$h_{\partial A}(x,y) = \frac{(\pi/2)\, h_A(0,x)\, h_A(0,y)}{1 - \cos(\theta_A(x) - \theta_A(y))} \left[1 + O\left(\frac{\log n}{n^{1/16}\, |\theta_A(x) - \theta_A(y)|} \right) \right],$$

provided that $|\theta_A(x) - \theta_A(y)| \geq n^{-1/16} \log^2 n$.

An Example and an Exact Formula

Example. If $u = e^{i\theta}, v = e^{i\theta'}$, $\theta \neq \theta'$, then

$$H_{\partial\mathbb{D}}(u,v) = \frac{1}{\pi}\,\frac{1}{|v-u|^2} = \frac{1}{2\pi}\,\frac{1}{1-\cos(\theta'-\theta)}.$$

Fact. For $x, y \in \partial D$

$$\boxed{H_{\partial D}(x,y) = \frac{2\pi\, H_D(0,x)\, H_D(0,y)}{1-\cos(\theta_D(x)-\theta_D(y))}}$$

Proof. $4\pi^2 H_{\mathbb{D}}(0, e^{i\theta_D(x)}) H_{\mathbb{D}}(0, e^{i\theta_D(y)}) = 1$

$$H_{\partial\mathbb{D}}(e^{i\theta_D(x)}, e^{i\theta_D(y)}) = \frac{2\pi H_{\mathbb{D}}(0, e^{i\theta_D(x)}) H_{\partial\mathbb{D}}(0, e^{i\theta_D(y)})}{1-\cos(\theta_D(x)-\theta_D(y))}$$

$$H_{\partial D}(x,y) = |f'(x)|\,|f'(y)| H_{\partial\mathbb{D}}(e^{i\theta_D(x)}, e^{i\theta_D(y)})$$

$$H_D(0,x) H_D(0,y) = |f'(x)|\,|f'(y)| H_{\mathbb{D}}(0, e^{i\theta_D(x)}) H_{\mathbb{D}}(0, e^{i\theta_D(y)})$$

Hitting Matrix Determinant Identities

If

$$\mathbf{h}_{\partial A}(\mathbf{x},\mathbf{y}) = \begin{bmatrix} h_{\partial A}(x^1, y^1) & \cdots & h_{\partial A}(x^1, y^k) \\ \vdots & \ddots & \vdots \\ h_{\partial A}(x^k, y^1) & \cdots & h_{\partial A}(x^k, y^k) \end{bmatrix}$$

then

$$\det \left[\frac{h_{\partial A}(x^j, y^l)}{h_{\partial A}(x^j, y^j)} \right]_{1 \le j,l \le k} = \frac{\det \mathbf{h}_{\partial A}(\mathbf{x},\mathbf{y})}{\prod_{j=1}^{k} h_{\partial A}(x^j, y^j)}$$

represents the conditional probability that (‡) holds given (†) holds.

The "Brownian Motion" Analogue

Let D be a smooth Jordan domain, and $x^1, \ldots, x^k, y^k, \ldots, y^1$ distinct points on ∂D ordered counterclockwise.

$$\Lambda_D(x^1, \ldots, x^k, y^k, \ldots, y^1) = \det \left[\frac{H_{\partial D}(x^j, y^l)}{H_{\partial D}(x^j, y^j)} \right]_{1 \le j,l \le k} = \frac{\det \mathbf{H}_{\partial D}(\mathbf{x}, \mathbf{y})}{\prod_{j=1}^{k} H_{\partial D}(x^j, y^j)}$$

Conformal covariance of the excursion Poisson kernel gives

$$\begin{aligned} \Lambda_D(x^1, \ldots, x^k, y^k, \ldots, y^1) &= \Lambda_{\mathbb{D}}(f(x^1), \ldots, f(x^k), f(y^k), \ldots, f(y^1)) \\ &= \det \left[\frac{1 - \cos(\theta_D(x^j) - \theta_D(y^j))}{1 - \cos(\theta_D(x^j) - \theta_D(y^l))} \right]_{1 \le j,l \le k} \end{aligned}$$

where f is a conformal transformation of D onto $\mathbb{D}$ and $\theta_D(z) = \arg(f(z))$.

An Important Corollary

Corollary. *Suppose* $A \in \mathcal{A}^n$ *and* $x^1, \ldots, x^k, y^k, \ldots, y^1$ *are distinct points in* ∂A *ordered counterclockwise. Let*

$$m = \min\{\, |\theta_A(x^1) - \theta_A(y^1)|,\ |\theta_A(x^k) - \theta_A(y^k)| \,\}.$$

If $m \geq n^{-1/16} \log^2 n$, *then*

$$\det \left[\frac{h_{\partial A}(x^j, y^l)}{h_{\partial A}(x^j, y^j)} \right]_{1 \leq j,l \leq k}$$

$$= \det \left[\frac{1 - \cos(\theta_A(x^j) - \theta_A(y^j))}{1 - \cos(\theta_A(x^j) - \theta_A(y^l))} \right]_{1 \leq j,l \leq k} + O_k\left(\frac{\log n}{n^{1/16}\, m^{2k+1}} \right).$$

An Important Corollary

Proof. The Main Estimate gives

$$\det\left[\frac{h_{\partial A}(x^j, y^l)}{h_{\partial A}(x^j, y^j)}\right]_{1\le j,l\le k}$$
$$= \det\left[\frac{1-\cos(\theta_A(x^j)-\theta_A(y^j))}{1-\cos(\theta_A(x^j)-\theta_A(y^l))}\left[1+O\left(\frac{\log n}{m\, n^{1/16}}\right)\right]\right]_{1\le j,l\le k}.$$

But, if $|\delta_{j,l}| \le \varepsilon$, multilinearity of the determinant and the estimate $\det[b_{j,l}] \le k^{k/2}\,[\sup|b_{j,l}|]^k$ shows that

$$\big|\det[b_{j,l}(1+\delta_{j,l})] - \det[b_{j,l}]\big| \le [(1+\varepsilon)^k - 1]k^{k/2}\,[\sup|b_{j,l}|]^k.$$

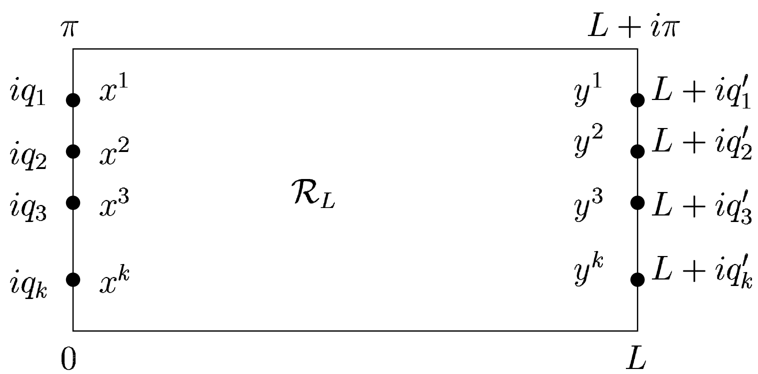
Kenyon's Crossing Exponent for LERW

Using the corollary, we can approximate the determinant for random walks, and hence the probability of the crossing event $\mathcal{C}$, in terms of the corresponding quantity for Brownian motion, at least for simply connected domains.

We consider the asymptotics of $\Lambda_D(x^1, \ldots, x^k, y^k, \ldots, y^1)$ when $x^1, \ldots, x^k$ get close and $y^1, \ldots, y^k$ get close.

Since Λ_D is a conformal invariant, we take $D = \mathcal{R}_L$, where

$$\mathcal{R}_L = \{\, z :\ 0 < \mathcal{Re}(z) < L,\ 0 < \mathcal{Im}(z) < \pi \,\},$$



Kenyon's Crossing Exponent for LERW

Example. Using separation of variables,

$$H_{\partial\mathcal{R}_L}(iq, L+iq') = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n\,\sin(nq)\,\sin(nq')}{\sinh(nL)}$$

Theorem (Lawler-K). *As* $L \to \infty$,

$$\Lambda_{\mathcal{R}_L}(iq_1, \ldots, iq_k, L+iq'_k, \ldots, L+iq'_1)$$

$$= k!\,\frac{\det[\sin(lq_j)]_{1\le j,l\le k}\ \det[\sin(lq'_j)]_{1\le j,l\le k}}{\prod_{j=1}^{k}\sin(q_j)\,\sin(q'_j)}\,e^{-k(k-1)L/2} + O_k(e^{-k(k+1)L/2})$$

This crossing exponent $k(k-1)/2$ was first proved by Kenyon for loop-erased walk.

Kenyon's Crossing Exponent for LERW

Proof. Let $\mathbf{q} = (q_1, \ldots, q_k)$, $\mathbf{q}' = (q_1', \ldots, q_k')$,

$$\vec{u}_j = \begin{bmatrix} \sin(jq_1) \\ \sin(jq_2) \\ \vdots \\ \sin(jq_k) \end{bmatrix}, \qquad \vec{v}_j = \begin{bmatrix} \sin(jq_1') \\ \sin(jq_2') \\ \vdots \\ \sin(jq_k') \end{bmatrix}.$$

Using the example, $(\pi/2)^k \det \mathbf{H}_{\partial \mathcal{R}_L}(i\mathbf{q}, L + i\mathbf{q}')$ can be written as

$$\det \left[\sum_{j=1}^{\infty} \frac{j \sin(jq_1)}{\sinh(jL)} \vec{v}_j, \sum_{j=1}^{\infty} \frac{j \sin(jq_2)}{\sinh(jL)} \vec{v}_j, \ldots, \sum_{j=1}^{\infty} \frac{j \sin(jq_k)}{\sinh(jL)} \vec{v}_j \right]_{1 \le j \le k}.$$

By multilinearity of the determinant, we can write the determinant above as

$$\sum_{j_1, \ldots, j_k} \frac{(j_1 \cdots j_k) \ \sin(j_1 q_1) \cdots \sin(j_k q_k)}{\sinh(j_1 L) \cdots \sinh(j_k L)} \det[\vec{v}_{j_1}, \ldots, \vec{v}_{j_k}].$$

The determinants in the last sum equal zero if the indices are not distinct.

Also it is not difficult to show that

$$\sum_{j_1+\cdots+j_k \geq R} \frac{j_1 \cdots j_k}{\sinh(j_1 L) \cdots \sinh(j_k L)} \leq C(k, R)\, e^{-RL}.$$

$\therefore$ except for error of $O_k(e^{-(k^2+k+2)L/2})$, $(\pi/2)^k \det \mathbf{H}_{\partial \mathcal{R}_L}(i\mathbf{q}, L + i\mathbf{q}')$ equals

$$k! \sum_{\sigma} \frac{\sin(\sigma(1)q_1) \cdots \sin(\sigma(k)q_k)}{\sinh(L)\, \sinh(2L) \cdots \sinh(kL)} \det[\vec{v}_{\sigma(1)}, \ldots, \vec{v}_{\sigma(k)}], \qquad (*)$$

where the sum is over all permutations σ of $\{1, \ldots, k\}$.

But, $\det[\vec{v}_{\sigma(1)}, \ldots, \vec{v}_{\sigma(k)}] = (\mathrm{sgn}\ \sigma) \det[\vec{v}_1, \ldots, \vec{v}_k]$.

Hence $(*)$ equals

$$\frac{k!\ \det[\vec{u}_1, \ldots, \vec{u}_k]\ \det[\vec{v}_1, \ldots, \vec{v}_k]}{\sinh(L)\ \sinh(2L) \cdots \sinh(kL)},$$

which up to an error of $O_k(e^{-(k^2+k+2)L/2})$ equals

$$2^k\, k!\, e^{-k(k+1)L/2}\ \det[\vec{u}_1, \ldots, \vec{u}_k]\ \det[\vec{v}_1, \ldots, \vec{v}_k].$$

To finish the proof, note that we can also write

$$H_{\partial \mathcal{R}_L}(iq, L + iq') = \frac{4}{\pi}\, e^{-L}\ \sin q\ \sin q'\ [1 + O(e^{-L})],$$

so that

$$(\pi/2)^k\ \prod_{j=1}^{k} H_{\partial \mathcal{R}_L}(iq_j, L + iq_j') = 2^k\, e^{-kL}\ \left(\prod_{j=1}^{k} \sin q_j\ \sin q_j'\right)\ [1 + O_k(e^{-L})].$$

Green's Function Estimate

$G_A(x)$: the expected number of visits to x before leaving A of a simple random walk starting at 0

a: potential kernel for two-dimensional simple random walk

Fact. $\exists\, k_0$ such that $a(x) = \frac{2}{\pi} \log|x| + k_0 + O(|x|^{-2})$ as $x \to \infty$.

Theorem (Lawler-K). *If $A \in \mathcal{A}^n$ and $x \neq 0$, then*

$$G_A(x) = \frac{2}{\pi}\, g_A(x) + k_x + O(n^{-1/3} \log n).$$

where $g_A(x) = -\log|f_A(x)|$ is Green's function for Brownian motion in $\tilde{A}$ and

$$k_x = k_0 + \frac{2}{\pi} \log|x| - a(x).$$

Notes on the Proof of the Main Estimate

- The error term $O(n^{-1/16} \log n)$ is not optimal, and we probably could have improved it slightly. However, our methods are not strong enough to give the optimal error term. The importance of this result is that the error is bounded uniformly over all simply connected domains and that the error is in terms of a power of n. For domains with "smooth" boundaries, one can definitely improve the power of n.

- To help understand this estimate, one should consider $h_{\partial A}(x, y)$ as having a "local" and a "global" part. The local part, which is very dependent on the structure of A near x and y, is represented by the $h_A(0, x)\, h_A(0, y)$ term. The global part, which is $[1 - \cos(\theta_A(x) - \theta_A(y))]^{-1}$, is the conformal invariant and depends only on the image of the points under the conformal transformation of $\tilde{A}$ onto the unit disk.

Notes on the Proof of the Main Estimate

To prove the main result required the following tools:

- conformal invariance
- Koebe $1/4$ Theorem
- Beurling estimates for simple random walk and Brownian motion
- Komlós-Major-Tusnády strong approximation
- Brownian motion/simple random walk near the boundary is likely to exit nearby, and quickly
- Brownian motion and simple random walk can be strongly coupled
- establishing Green's function estimates so that we can establish Poisson kernel estimates