SLE, DLA, and Shapes

presented by

Leo P. Kadanoff

University of Chicago

e-mail: LeoP@UChicago.edu

coworkers

Ilya Gruzberg, Bernard Nienhuis, Isabelle Claus, Wouter Kager, Paul Wiegmann, Panos Oikonomou, Marko Kleine Berkenbusch, Paul Wiegmann, Panos Oikonomou, Ilia Rushkin, Seung Yeop Lee

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Abstract

Two dimensional shapes have long been a preoccupation of mathematicians and theoretical physicists. They form the root of such subjects as critical phenomena, fluid motion (via Hele Shaw motion), growth dynamics (e.g. DLA), and analytic function theory. Fractal and scale-invariant shapes are among the most interesting ones

Conformal maps (analytic functions) automatically generate shapes. Iterated maps (functions of functions) are powerful tools for generating interesting shapes. These can be naturally formed using ordinary differential equations.

Important efforts exist to use iterated functions to generate shapes, based in part on the work of Loewner in the 1920s and the earlier work of Julia and Fatou. More recent work includes Duplantier on quantum gravity methods, Schramm, Rohde, Werner, Lawler, and others on SLE, percolation and self-avoiding walks, Hastings and Levitov on DLA, followed by Cardy and others on the relation to conformal field theory and critical phenomena problems

I hope to describe some of this development here

Universality: Another Invariance

The near-critical phase transition problems have another invariance called Universality. The concept was sort of current among peope who did critical phenomena in the 1960s. Among others, I pointed out its importance and borrowed the word from Sasha Migdal and Sasha Polyakov in a dollar bar in the USSR. Briefly the concept states that there are whole classes of different phase transion problems which have identally the same singularities and critical behavior. Not similar, idential. All you have to do is re-express one problem by translating its variables into smooth functions of the variables of the pother problem.

For example, the Ising model and real liquid gas phase transtions have the same critical behavior. POroblems whoich have identical critical behavior are same to belong to the same universality class.

A denumerable infinity of universality classes are known in two dimensions.

We next turn to the connection between DLA and Loewner evolution.

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A personal Note

I am a theoretical physicist, not a mathematician. I have done research related to many of the topics in this Fall's Thematic Program on Renormalization and Universality in Mathematics and Mathematical Physics. Specifically, I did lots of work which helped establish concepts related to scaling, universality, and renormalization in the context of the physics of phase transitions,

I shall be talking about SLE, Phase Transitions, and renormalization, using the approach of a physicist. For me, this approach means that I should argue as deeply as I can about different models, and then try to extrapolate the knowledge gained from these models to larger sets of problems. I won't state my premises with absolute clarity, and correspondingly I shall rarely prove anything.

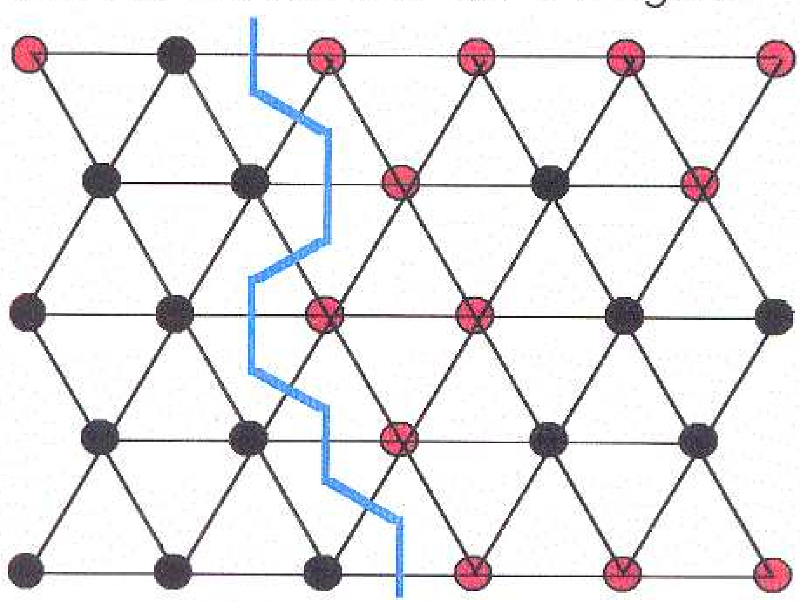
Often, but not always, this approach enables physicists to go more quickly than mathematics. Often, but not always, this approach leaves the outcome somewhat imprecise and fuzzy, necessitating further work to clean up and clear up what has been done.

Next: A List of Problems

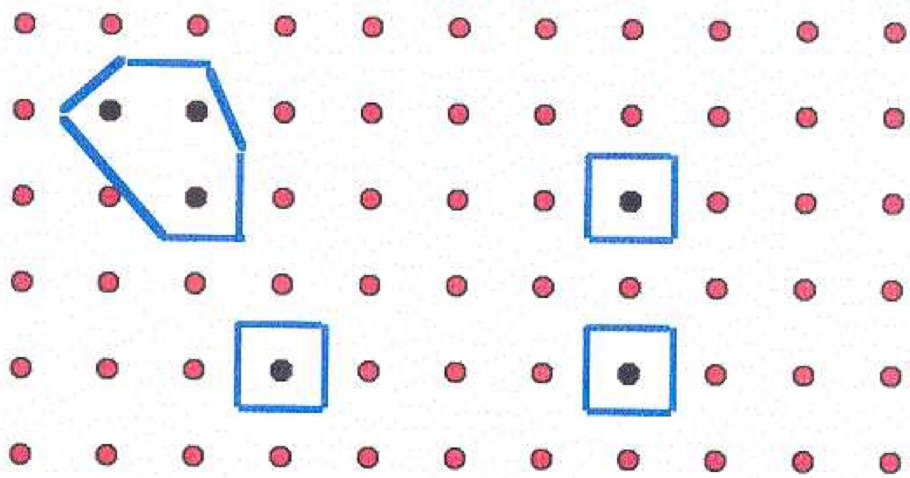
1.The Percolation Model.

Take a simple regular lattice. Each site on the lattice may be either occupied (black) or empty (red). Each lattice site is assigned the black color with a probability p. So the color assignment of each site is independent of all the others. We divide the lattice into different clusters by saying that a cluster is a set of sites of the same color, including all same-color nearest neighbors of the set members, and which cannot be divided into smaller sets.

A cluster boundary can be defined by a local process in which one walks through the centers of nearest neighbor bonds, always keeping black bonds on the left and the red ones on the right.



For small p--we mostly see small isolated clusters of occupied sites, small clusters of empty sites, and one very large empty cluster spanning the entire system.



If p is close to one, we same story with the colors reversed.

If p is close to one half, lots of large clusters of both colors are formed

see pics for talk 1!

Percolation: Is there left to right crossing?

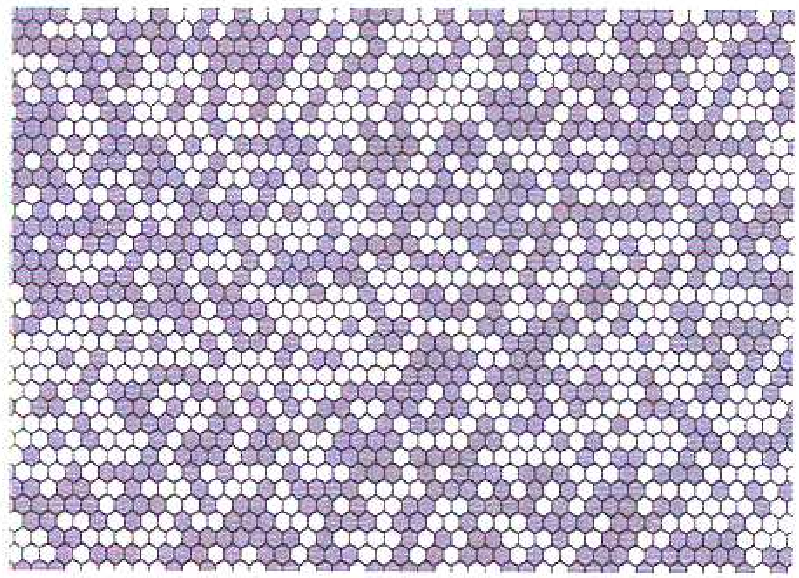
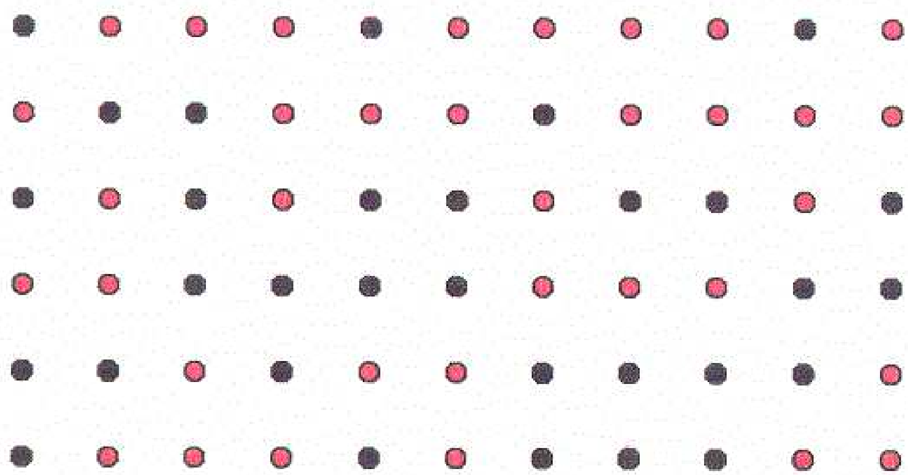


Fig. 10.1. Is there a left to right crossing of white hexagons?



In the limiting case of a very large system, $p<1/2$ has mostly particle clusters which have a size of order the distance between lattice sites while $p>1/2$ has unoccupied sites forming this kind of clusters. in between, $p=1/2$ has very large clusters of both types. This kind of qualitative change in large-scale behavior is called a phase transition.

A phase transition only makes sense in an infinite system. Therefore phase transitions always exhibit large-scale correlated behavior. Our job is to get at the geometry of this behavior.

Other Phase Transitions

Physicists, chemists, and mathematicians have worked with a large variety of phase transition problems. Among these is the change of ice into liquid water or the change of liquid water into vapor. In each case as one changes a parameter contolling the system (often temperature) there is an abrupt change in the long-distance correlations in the system. For example consider the Ising model on a finite lattice. Once again one puts a two-valued variable on all the sites of a lattice. Here we say the variable at site r is a spin, σ_r , with values plus or minus one. The different configurations are given a statitical weight which is

$$W = \prod e^{K\sigma_r\sigma_s}$$

a product over all nearest neighbor sites of a factor which is larger for equal spins than opposite ones.

Averages are sums over all spins using the weighting, W. As K is varied there is, once again, a qualitative transition. For very large distances **|r-s|** the product $<\sigma_r \sigma_s>$ goes exponentially to zero for small values of the coupling K, goes to a constant for K bigger than a critical value, and decays as a power **|r-s|** for $K=K_C$, the critical coupling.

The standard way of studying both percolation and the other phase transition problems before the SLE work was to study the behavior of pointlike functions, for example, $<\sigma_r \sigma_s>$ or perhaps the probability that two points separated by a distance |r-s| would fall into the same cluster.

At criticality in the Ising model $<\sigma_r \sigma_s>=C/|r-s|^{1/4}$. This result is universal in that additions of other kinds of interactions to the weighting function will not change the correlation at criticality.

SLE gives another perspective on the problem

Relation to SLE

SLE_6 will produce an ensemble of shapes-- so will the critical percolation problem. SLE's wonderful new result is that in the scaling limit, these two appraoches will give the same ensemble. The scaling limit considers very large objects which encompass very many lattice sites. In this limit, one observes only features of the object that are large in comparison to a lattice distance.

Shapes can be produced by many other interesting problems. For example, the two-dimensional Ising model is a set of spins on a lattice. Each spin takes on two possible values, say red or black. At the critical point, (given by $\sinh 2K_c = 1$), the weighting produces an esemble of clusters, which includes very large clusters. Although it has not been proven, most of the people who are experts in the field believe that, in the scaling limit, the ensemble of cluster shapes is the same as those generated by SLE_3.

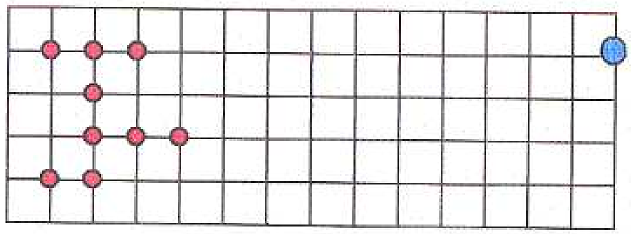
Thus, SLE gives a new way of approaching old physics problem and gaining new kinds of information, specifically information about shapes.

DLA- A problem of Dynamics

In 1981 Witten & Sander developed a dynamical model called DLA. This model was intended to construct scale-invariant, (fractal) objects. It did so, but despite a huge amount of work, we developed little understanding of the universality, scaling, or conformal properties of that model.

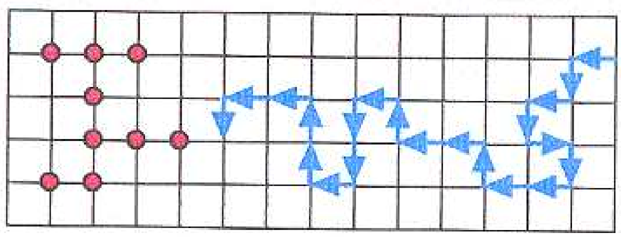
DLA describes how tiny bits of soot may come together and form one large, fractal aggregate. It was one of the first models to be initially expressed as a computer algorithm. Since the walking process is a time-independent diffusion, the probability that the walker appear at x,y , p(x,y) obeys the Laplace equation in the region outside the aggregate with the boundary condition that p vanish on the aggregate and that the probability of adding a walker is proportional to the normal gradient of p at the boundary.

The DLA Algorithm



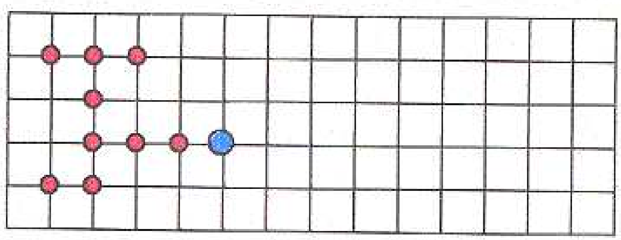
a

Start with
walker
at infinity



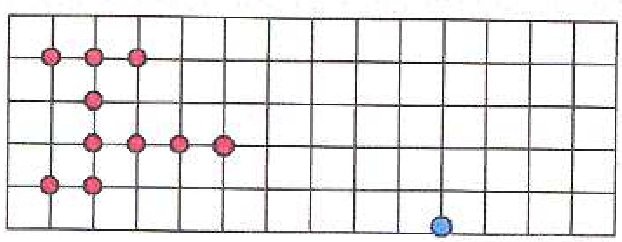
b

It does a
random walk
until it reaches
aggregate



c

It stops at
nearest
neighbor
site



d

A walker is
introduced at
infinity once
more

Mathematical Formulation

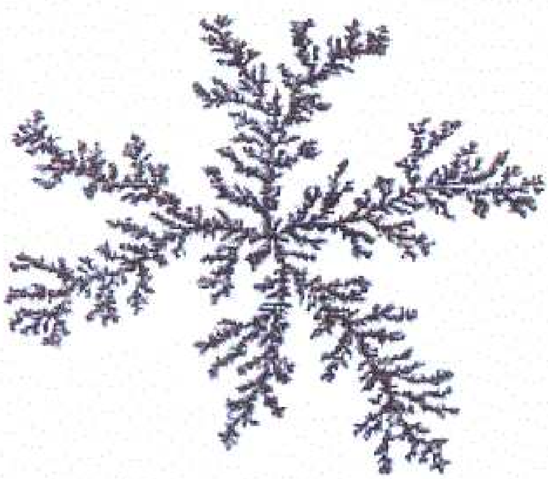
The random walk is described by a discretizd diffusion equation for p(x,y), the probability that we can find a walker at the position (x,y). The time-independent diffusion means that:

1. p obeys Laplace's equation in the region outside the aggregate (defines PDE)

2. There is a source of unit strength at infinity (defines boundary condition.)

3. p vanishes on aggregate (p is uniquely defined)

The probability of landing on the sites nearest neighbor to the aggregate site (x,y) is the particle's probability being added at (x,y). This is then the probability of growth at that site. A very interesting object is produced via an iterative process:



Invariance Properties of these problems

All of these problems, percolation, phase transitions, DLA have the same interesting invariance properties near the critical point. I list them here:

1. translation invariance $\mathbf{r} \to \mathbf{r} + \mathbf{a}$

2. rotation invariance $\mathbf{r} \to M\,\mathbf{r}$

$$M = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$$

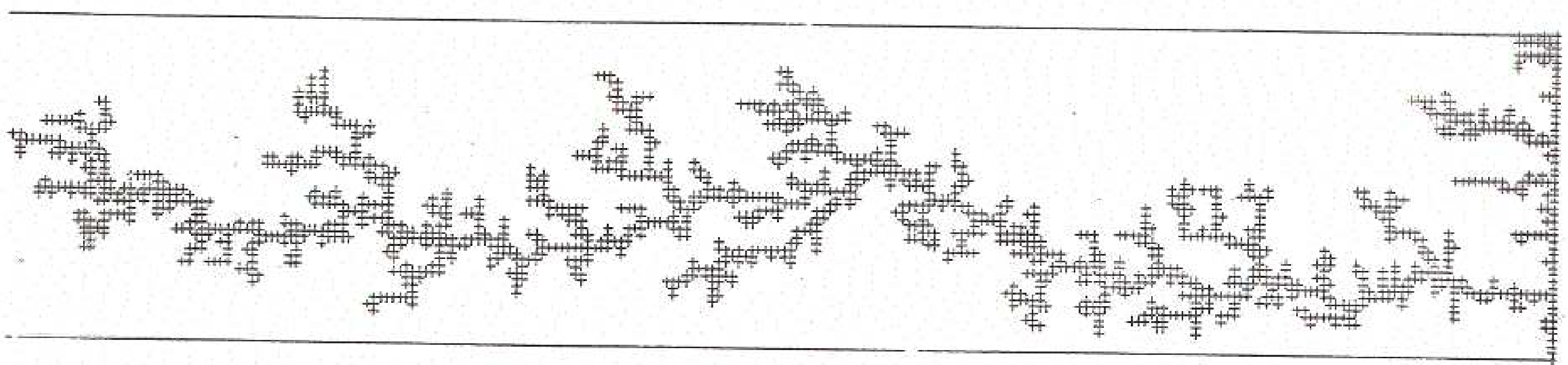
3. parity invariance $x \to x$, $y \to -y$

4. scale invariance $\mathbf{r} \to \mathbf{r}/\lambda$

5. Conformal invariance: all of above plus a boost operation: $\mathbf{r} \to \mathbf{r}/r^2$

The last implies (alas only in two dimensions) that the whole theory is invariant under $z=x+iy \to z+a$, $ze^{i\theta}$, $x-iy$, z/λ, $1/z$ and hence under conformal maps.

N=1200 M=20 I=H+3M K=H+6M



Let's Start Discussing the Maths of Loewner evolution.

From Dynamical Systems to Maps

Loewner evolution can be approached through the evolution equation, $dw/dt = 2/(w-\xi(t))$, and its solution, $w=g(t,z)$, where we are using the initial data $w=z$ at $t=0$. In writing the solution, we have translated the problem from finding one trajectory in the complex plane to finding many different trajectories, parametrized by the initial data, z.

It is best to approach this more generally. Let g obey the differential equation

$dw/dt = V(w,t)$

with initial data $w=z$ at time t_0. The solution can then be written down as $w = g_t(t_0,z)$.

g_t can be thought of as a map from the space of initial conditions, z, to later values, w. Iterated function theory enters naturally in every dynamical systems theory problem because if you calculate the solution in two pieces (from t_0 to t_1) **and** (from t_1 to t) you first perform one mapping operation and then another:

$w = g_t(t_1, g_{t_1}(t_0, z))$

In this talk, z, w, and g will all be complex variables and the g(z) will be an analytic function.

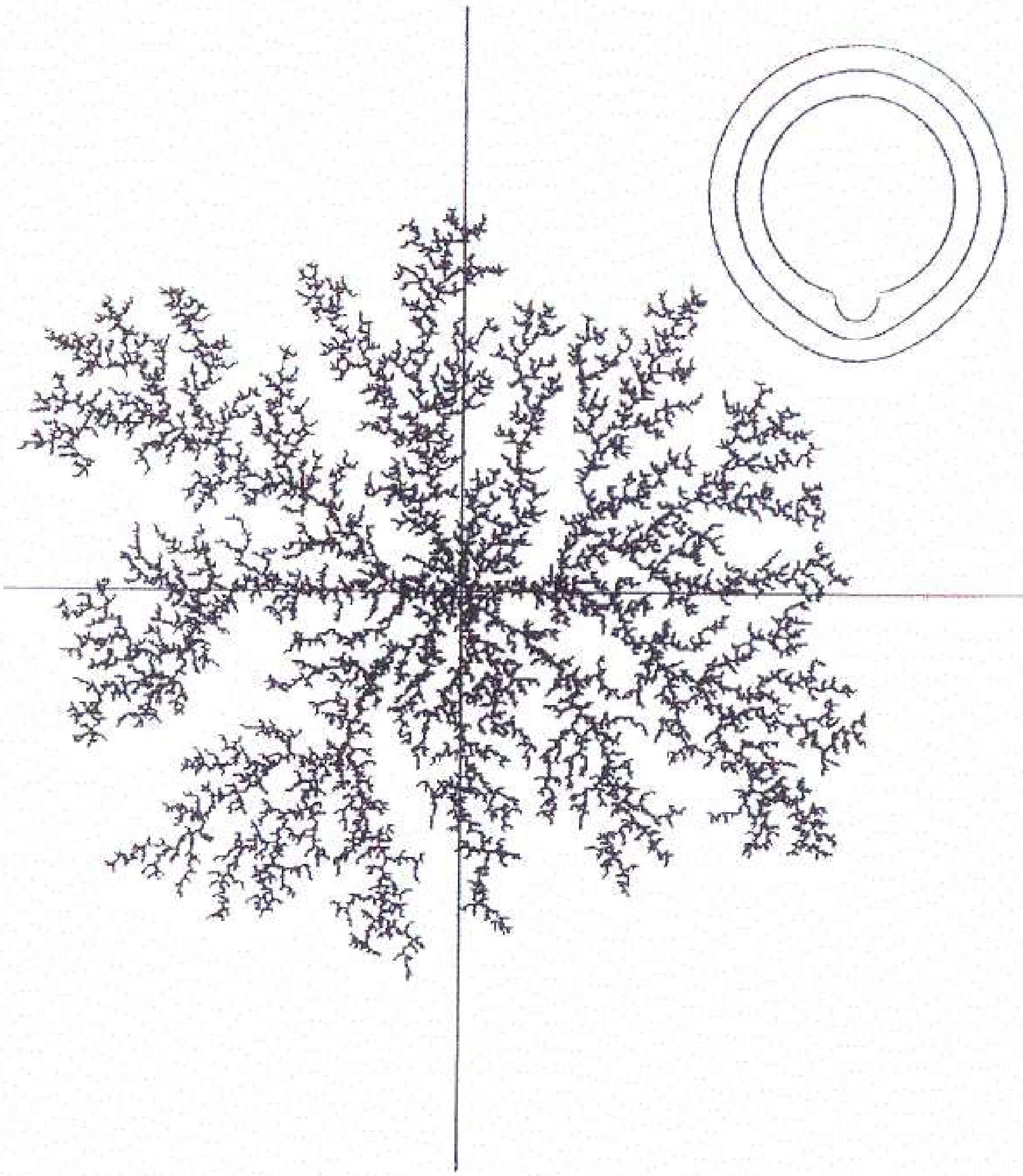
Thus iterated function theory is a very natural tool for discussing Loewner's evolution equation.

There are two natural forms of the tool: In chordal Loewner Evolution (LE) $g_t(t_0,z)$ is a conformal map from the upper half plane to the unit circle while in radial LE g conformally maps the region outside a simply connected finite part of the plane to the exterior of the unit circle.

The most important property of such mappings is their composition rule. For completeness I first defined conformal maps and then describe their composition rules

Make a DLA Cluster

From a very large number of small bumps placed on the circle one can make a DLA cluster as was done by Hastings and Levitov

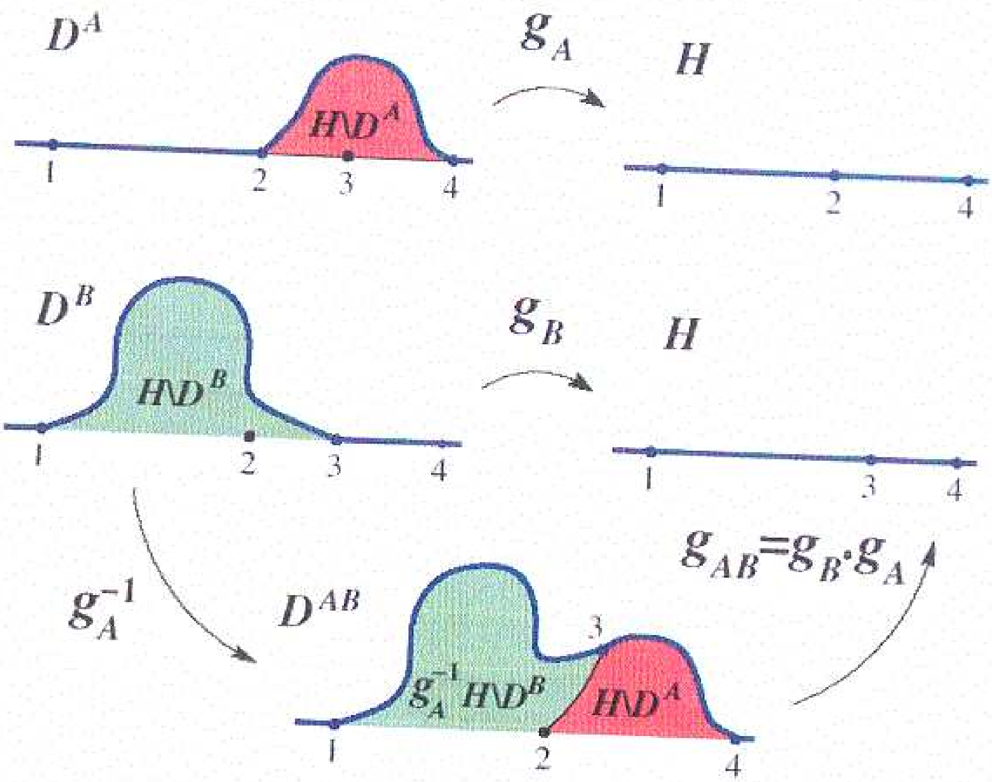


Definition: Conformal Map

A *conformal mapping*, is a function g(z) analytic within a simply connected region $\mathbb{D}$ of the complex z-plane, and which has the property that dg/dz is never zero in $\mathbb{D}$. It then provides a one-to-one mapping of the interior of $\mathbb{D}$ into the interior of another simply connected region $\mathbb{R}$ and likewise maps the curves which bounds these regions into one another. Riemann proved that for finite regions the mapping is unique. To go backwards from $\mathbb{R}$ to $\mathbb{D}$ you use the inverse function f(w) which obeys g(f(w))=z. This too is unique.

In the complex plane analysis and geometry are the same thing.

Composition Properties

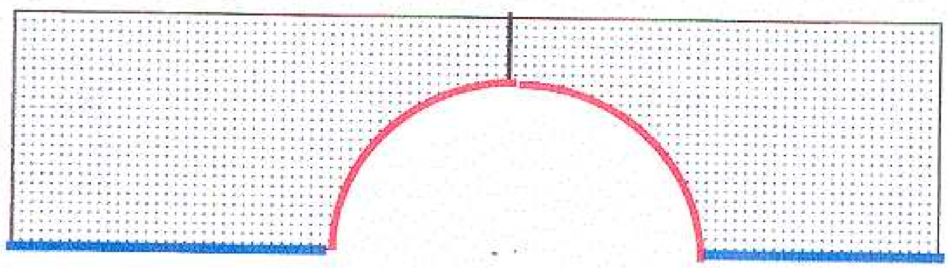


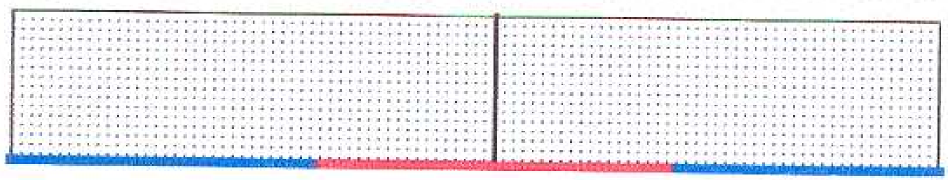
picture from Gruzberg and Kadanoff.

Notice how the boundary of the first mapping region has remained unchanged by the second.

Example

$w=g(z)=(z+z^{-1})$ takes $\mathbb{D}$ into $\mathbb{H}$, the upper half plane.



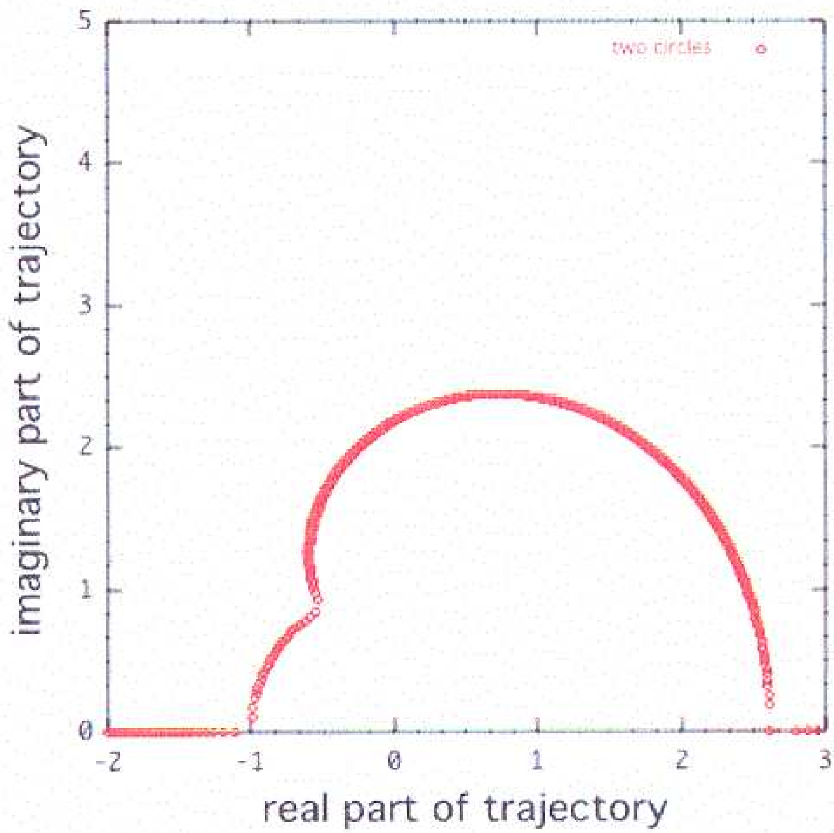


Solve for z. $z=(w+\sqrt{w^2-4})/2$ z=f(w) Then f takes $\mathbb{H}$ into $\mathbb{D}$.

Note that non-analyticity at w=2,-2, z=0 and vanishing derivatives z=1,-1 at all give interesting behavior.

Further

The map $w=z-a+r^2/(z-a)=g_{a,r}(z)$ describes a bump with radius r centered on the real axis at a. One can compose these bumps



Where should we Place the next Bump? We might

1. Let a random walker find the place, or equally

2. Solve the Laplace equation with a source at infinity and p=0 on the aggregate. The probability for landing is proportional to the normal gradient of the potential on the aggregate, or equally

3. Solve an electrostatics problem with charge one on the aggregate and a potential, φ, obeying the Laplace equation outside the aggregate. Again, the gradient gives the probability, or equally

4. Form the map g(w) taking the aggregate into the unit circle, with the extra condition that g --> w as w--> infinity. Write $\ln g = \varphi + i\psi$. Again the gradient gives the probability in the physical- (w-) plane.

(Note that 3 and 4 are identical since the φ in #4 is identical to the quantity defined in #3. It obeys the Laplace equation, is logarithmic at infinity, and vanishes on the aggregate.)

5. Equally, using g, one can say that the charges are uniformly distributed on the circle, i.e. in the math plane.

Therefore we can eliminate the walker entirely

We construct the DLA cluster for many very small bumps.

Just use g or g^{-1}=f to describe the shape of the interface.

Placed the next bump randomly on the circle. That is pick the value of a. Pick its radius, r, to make all the bumps on the physical plane be of the same size, i.e. make the bump size inversely proportional to the derivative of g.

Construct the map for the previous shape plus the new bump as $g' = g_a \circ g$.

This approach gives an iterative method for constructing the DLA cluster. It was used by Hastings and Levitov to construct the cluster previously shown.

DLA, summarized

One can form DLA aggregates by doing a bump map many many times. On picks a map of the form

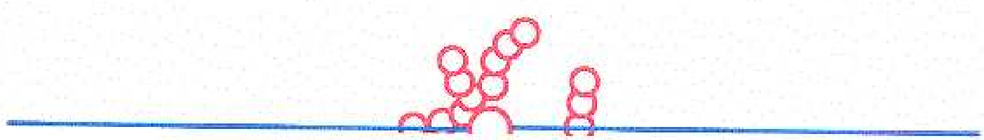
$$w = g_a(z) = (z - a + \frac{r^2}{z-a})/2,$$

which makes bumps with radius r and position a. (r and a must be real.) The one puts together a bunch of bumps by successively forming the maps one after the other:

$$g(z) = g_{a_5}(g_{a_4}(g_{a_3}(g_{a_2}(g_{a_1}(z)))))$$

One bump:

many bumps tend to clump



g(z) serves in the dual role of guiding the placement of the next bump and defining the shape of the aggregate.

Physical models:

DLA: bumps appear at random positions in w, all equal in size in the physical plane. (Hastings and Levitov) This model is almost certainly in the universality class of DLA..

Noise-reduced DLA: The landing probability is proportional to $|\nabla\varphi|^{\eta}$ where $\eta=1$ is DLA. $0<\eta<4$ is called the dielectric breakdown model and is probably a different universality class for each η. My student Chao Tang argued that large η produced a new universality class which he called noise-reduced DLA. I said it was not so. He was right

Other models: Vary bump size. New universality classes.

Other models yet: In DLA, if the radius of the bump is the small number ε, then a new bump is produced at a "displacement" $\Delta w = \pm\, \kappa^{0.5}\varepsilon$ from last bump. This random walk along the aggregate produces fractals very different from DLA. Structure of fractals differs depending on value of κ and is probably SLE_{κ}(Oded Schramm had the basic insights, further developed by Lawler and Warner, and then Hastings translated them into this language.)