SLE & QUANTUM GRAVITY

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Service de Physique Théorique de Saclay

PERCOLATION, SLE, AND RELATED TOPICS WORKSHOP

Thematic program

Renormalization & Universality

in Mathematics & Mathematical Physics

The Fields Institute, Toronto

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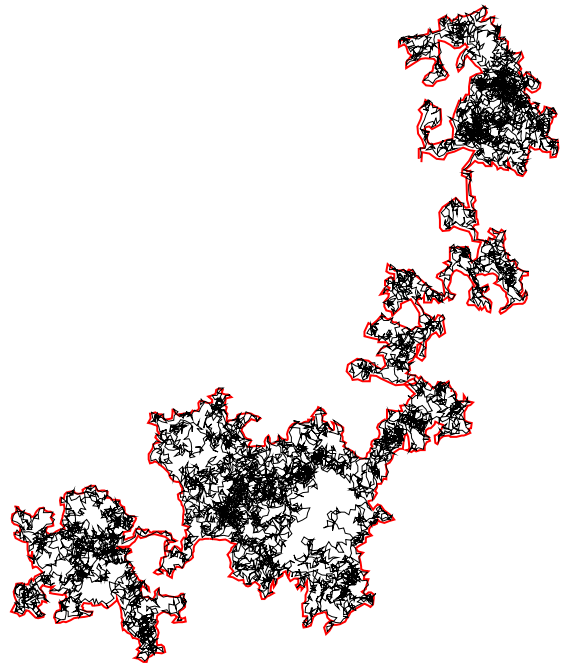
Brownian Path & Frontier

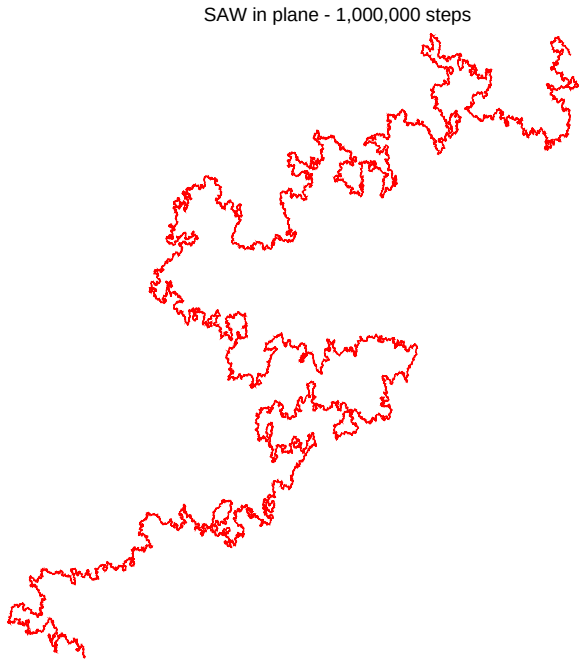


Paul Lévy: conformal invariance
Mandelbrot conjecture (1982):

Frontier Hausdorff dimension $D = \frac{4}{3}$, as a SAW

(Lawler, Schramm & Werner, 2001; L.-W.; B. D., 1998)

Self-Avoiding Walk

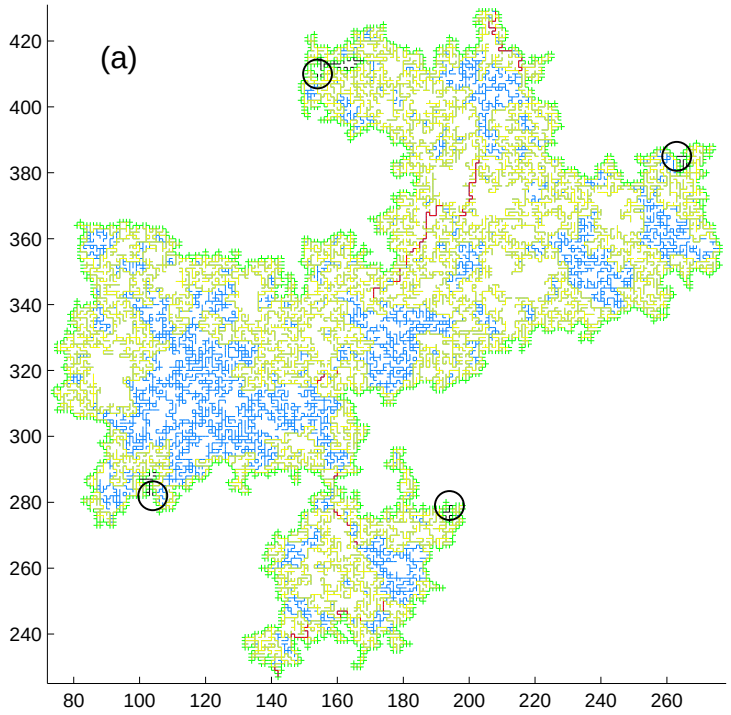


(Courtesy of T. Kennedy)

B. Nienhuis (1982): $D = \frac{4}{3}$; J. Cardy (1984): $\tilde{x}_1 = \frac{5}{8}$

D. & Saleur (1986): *Multiple SAWs*

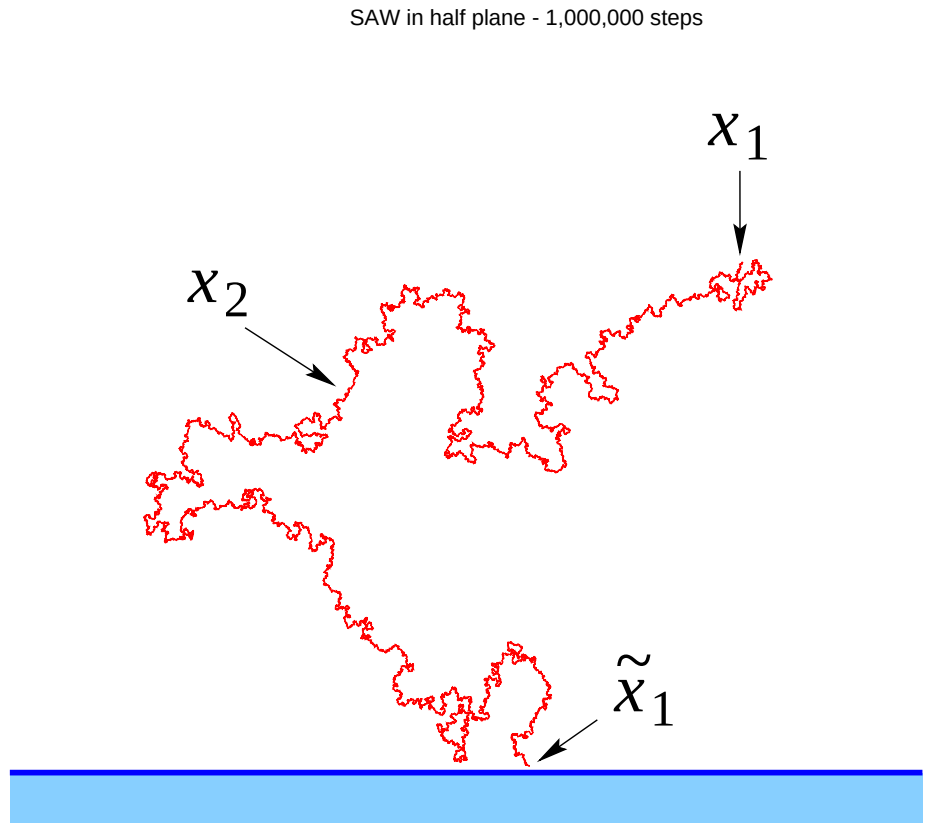
Percolation Hull & Frontier



(J. Asikainen et al., 2003)

Cluster; Hull: $D_{\mathrm{Hull}} = \frac{7}{4}$ *(D. & Saleur, 1987; Smirnov, 2001; Beffara, 2002); External Perimeter:* $D_{\mathrm{EP}} = \frac{4}{3}$ *(Aizenman, D. & Aharony, 1999; LSW; Beffara)*

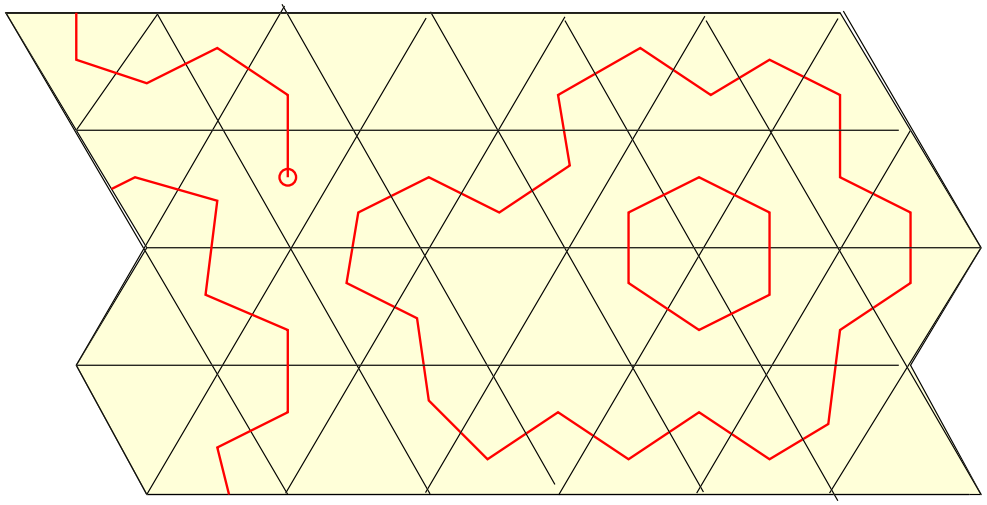
SLE_κ *(Schramm, 1999)*



(G. Lawler, O. Schramm. & W. Werner; S. Rohde & O. S.;
S. Smirnov; M. Bauer & D. Bernard; J. Cardy; W. Kager & B. Nienhuis;
V. Beffara; N.-G. Kang, J. Dubédat; S. Sheffield; F. Camia & C. Newman)

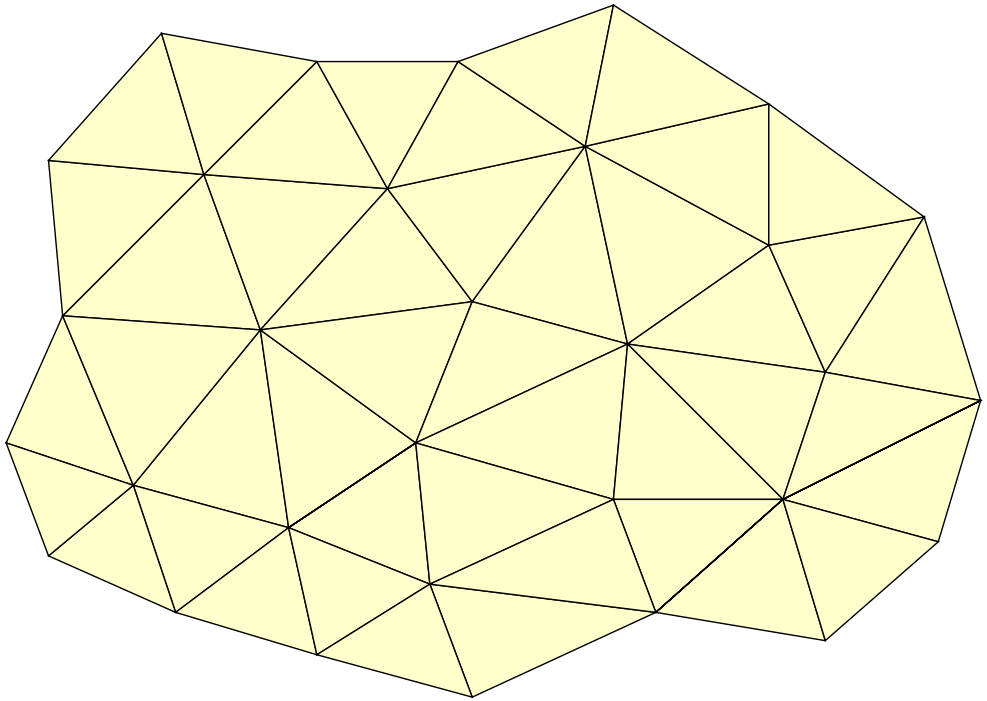
2D QUANTUM GRAVITY

Statistical Mechanics on a Regular Lattice



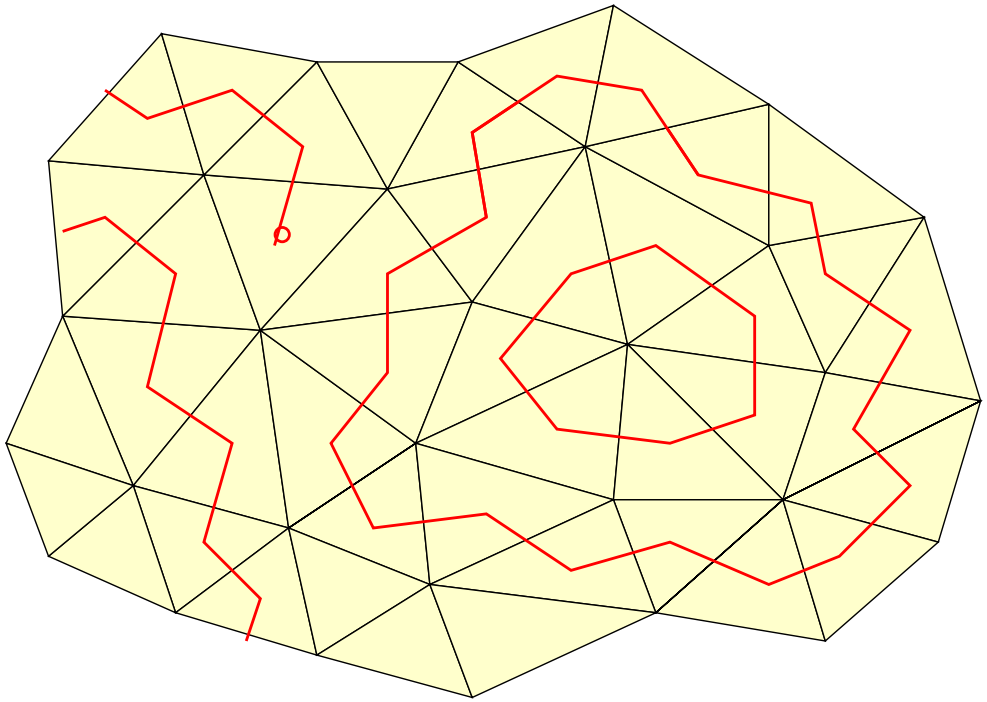
Random lines on the (dual of) a regular triangular lattice.

Randomly Triangulated Lattice



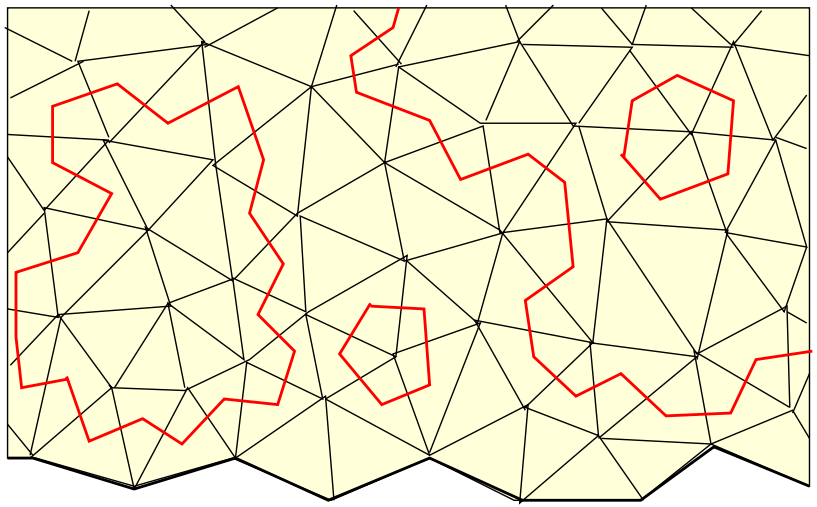
A random planar triangular lattice.

Statistical Mechanics on a Random Lattice



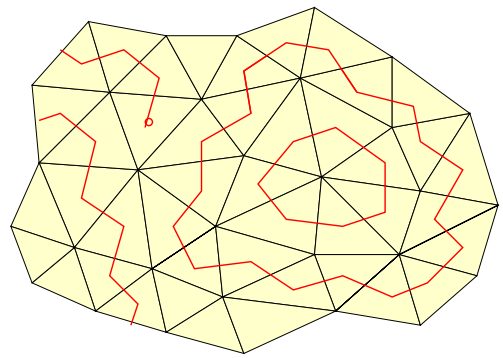
Statistical model on a random planar triangular lattice.

Boundary Effects



Dirichlet boundary conditions on a random disk.

Partition Function on a Random Lattice



Statistical model $\mathcal{M}$ on random lattice G.

$$Z(\beta) = \sum_{\text{planar } G} e^{-\beta|G|}\, \mathcal{Z}_G$$

$\mathcal{Z}_G$: partition function of the statistical model $\mathcal{M}$ on G.

DOUBLE CRITICAL POINT of $\mathcal{M}$ & G

$$Z(\beta) \sim (\beta - \beta_c)^{2-\gamma(c)}$$

The string susceptibility exponent γ depends on $\mathcal{M}$ through c

Double Critical Behavior

$\gamma_{\rm str}(c) \equiv \gamma$ is related to the "central charge" c of the CFT describing the statistical model by

$$c = 1 - 6\gamma^2/(1-\gamma)\,,\ \ \gamma \leqslant 0$$

$\mathrm{SLE}_\kappa,\ 0 \leqslant \kappa \leqslant +\infty$

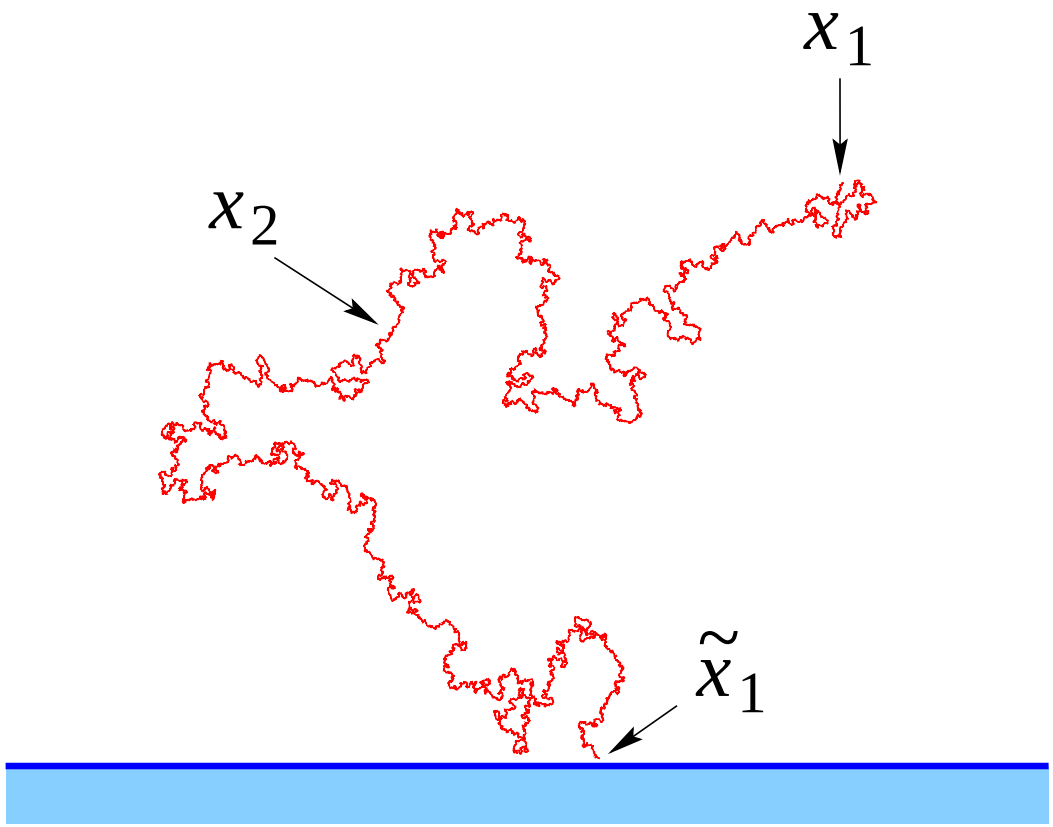
$$c = \frac{1}{4}(6-\kappa)\left(6 - \frac{16}{\kappa}\right)$$

$$\gamma = 1 - \frac{4}{\kappa},\ \kappa \leqslant 4\,, \quad \gamma = 1 - \frac{\kappa}{4},\ 4 \leqslant \kappa$$

Symmetric under *duality:* $\kappa \to \kappa' = 16/\kappa$

Conformal Weights of a Random Path in $\mathbb{C}$ or $\mathbb{H}$

SAW in half plane - 1,000,000 steps



Critical Behavior

Partition functions in $\mathbb{C}$ or $\mathbb{H}$

$$\mathcal{Z} \propto \left(\frac{r}{R}\right)^{2x}, \quad \tilde{\mathcal{Z}} \propto \left(\frac{r}{R}\right)^{\tilde{x}}$$

Partition functions in QG

$$Z \propto \langle |G| \rangle^{-\Delta}, \quad \tilde{Z} \propto \langle |\partial G| \rangle^{-\tilde{\Delta}}$$

Average lattice area $\langle |G| \rangle$, boundary length $\langle |\partial G| \rangle$:

$$\langle |G| \rangle \sim (\beta - \beta_c)^{-1}$$

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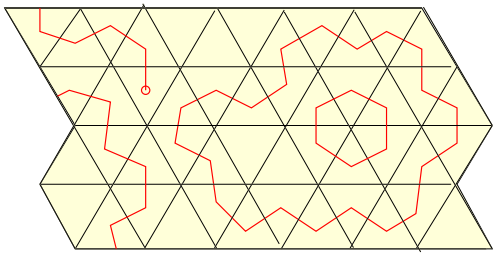
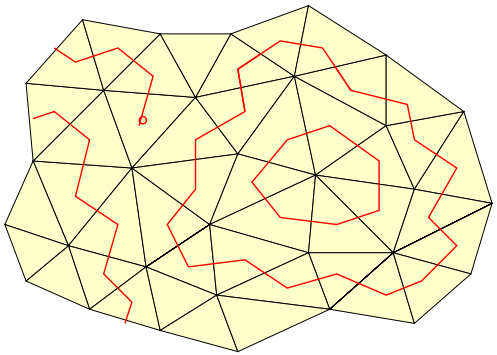
KPZ *Knizhnik, Polyakov, Zamolodchikov (88)*



A "conformal operator" O (e.g. creating a line extremity) has conformal weight $x = U(\Delta)$ in $\mathbb{C}$ (or $\tilde{x} = U(\tilde{\Delta})$ in $\mathbb{H}$)



where Δ (or $\tilde{\Delta}$) is the corresponding conformal weight in quantum gravity (or boundary Q. G.)

KPZ: The fundamental quadratic relation exists between the conformal dimensions Δ on a random planar surface and those x in $\mathbb{C}$ or $\mathbb{H}$

$$x = U(\Delta) = \Delta \frac{\Delta - \gamma}{1 - \gamma}$$

Inverse KPZ map

$$\Delta = U^{-1}(x) = \frac{1}{2}\left(\sqrt{4(1-\gamma)x + \gamma^2} + \gamma\right)$$

SLE & KPZ

Conformal dimensions Δ in (boundary) QG and x in $\mathbb{C}(\mathbb{H})$

$$x = U(\Delta) = \frac{1}{4}\Delta(\kappa\Delta + 4 - \kappa)$$

Inverse KPZ map

$$\Delta = U^{-1}(x) = \frac{1}{2\kappa}\left(\sqrt{16\kappa x + (\kappa - 4)^2} + \kappa - 4\right)$$

(duality $\kappa \to 16/\kappa$)

Duality & KPZ

Dual conformal dimensions Δ, Δ' in QG

$$x = U_\kappa(\Delta) = \Delta \times \frac{1}{4}(\kappa\Delta + 4 - \kappa) = \Delta \times \Delta'$$

Inverse KPZ κ-map

$$\begin{aligned} U_\kappa^{-1}(x) &= \frac{1}{2\kappa}\left(\sqrt{16\kappa x + (\kappa-4)^2} + \kappa - 4\right) \\ &= \Delta \quad (\kappa \leqslant 4) \quad \text{or} \quad \Delta' \quad (\kappa \geqslant 4) \\ x &= U_\kappa^{-1}(x) \times U_{16/\kappa}^{-1}(x) \end{aligned}$$

SLE Duality

$$D_{\mathrm{EP}}(\kappa) = D_{\mathrm{H}}(\kappa), \ \kappa \leqslant 4$$

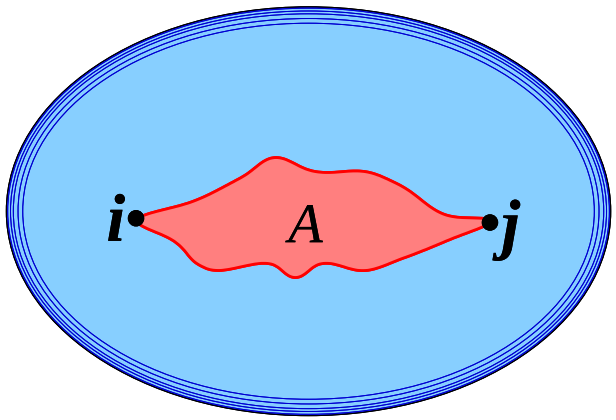
$$D_{\mathrm{EP}}(\kappa) = D_{\mathrm{H}}(\kappa' = 16/\kappa), \quad \kappa \geqslant 4$$

$$\frac{1}{4} = [D_{\mathrm{EP}}(\kappa) - 1]\,[D_{\mathrm{H}}(\kappa) - 1]$$

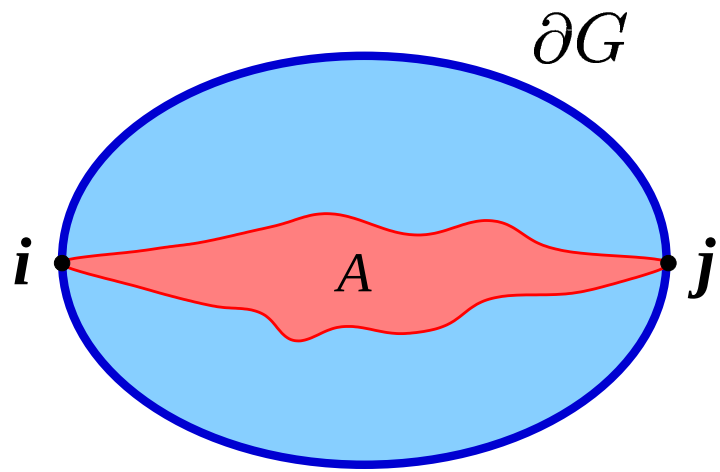
Duality: the external perimeter of $\mathrm{SLE}_{\kappa \geqslant 4}$ "is" the simple path of $\mathrm{SLE}_{[(16/\kappa) \leqslant 4]}$

Life in QG is Easy

I • Bulk-Boundary Conformal Weights Relation



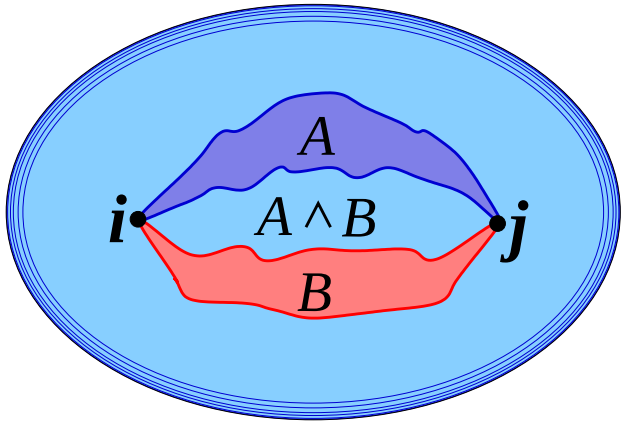
Δ_A

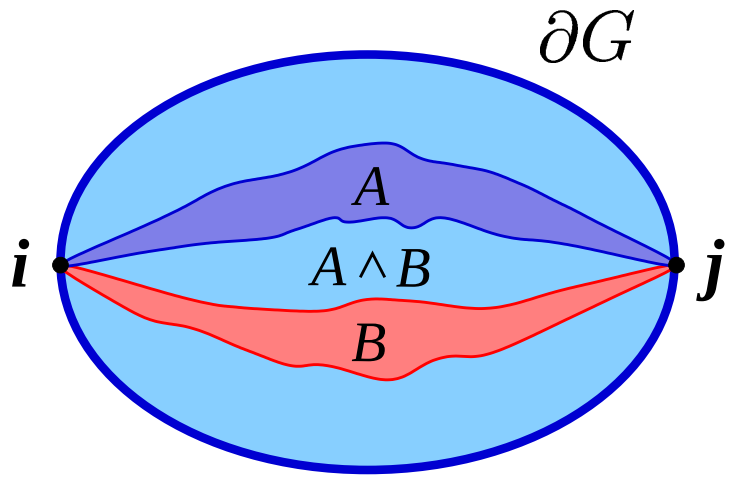


$\tilde{\Delta}_A$

$$2\Delta_A - \gamma = \tilde{\Delta}_A$$

II • QG Boundary Additivity & Mutual Avoidance

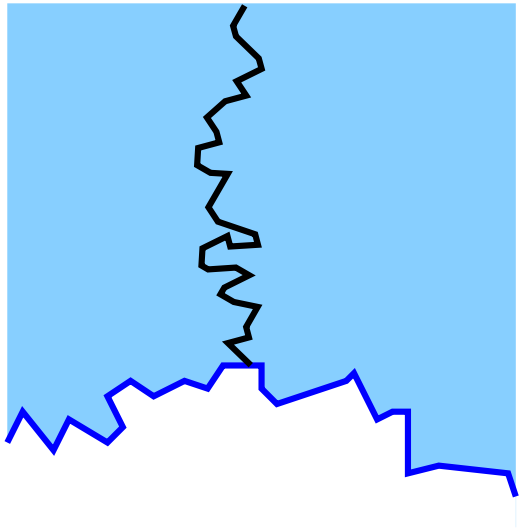




$A \wedge B$: random sets A & B avoid each other

$$2\Delta_{A\wedge B} - \gamma = \tilde{\Delta}_{A\wedge B} = \tilde{\Delta}_A + \tilde{\Delta}_B$$

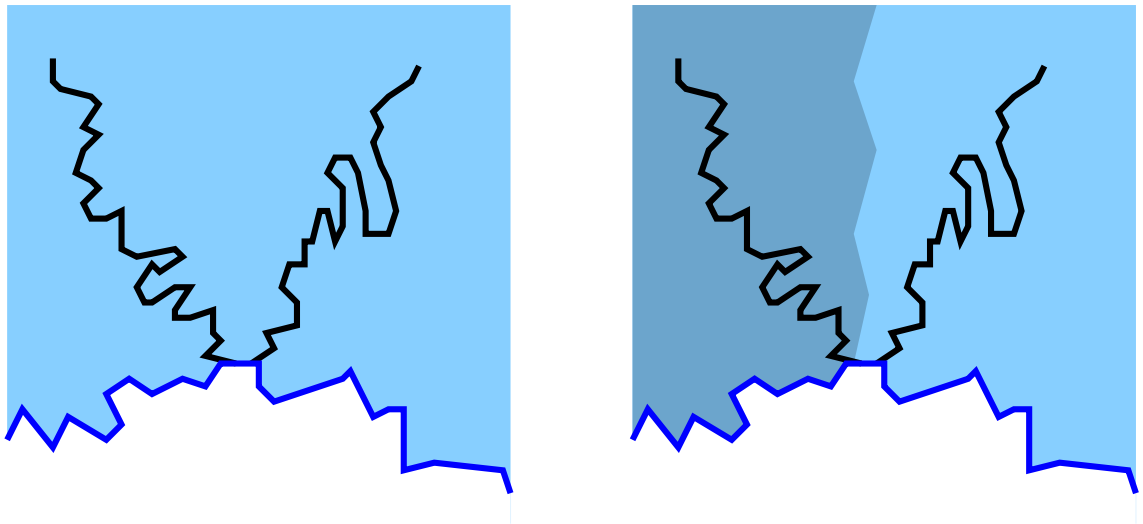
SLE & Boundary QG



$$\tilde{\Delta}_1 = U^{-1}(\tilde{x}_1)$$
$$= \frac{1-\gamma}{2}$$

$$\tilde{\Delta}_1 = U_\kappa^{-1}(\tilde{x}_1) = \frac{2}{\kappa}$$

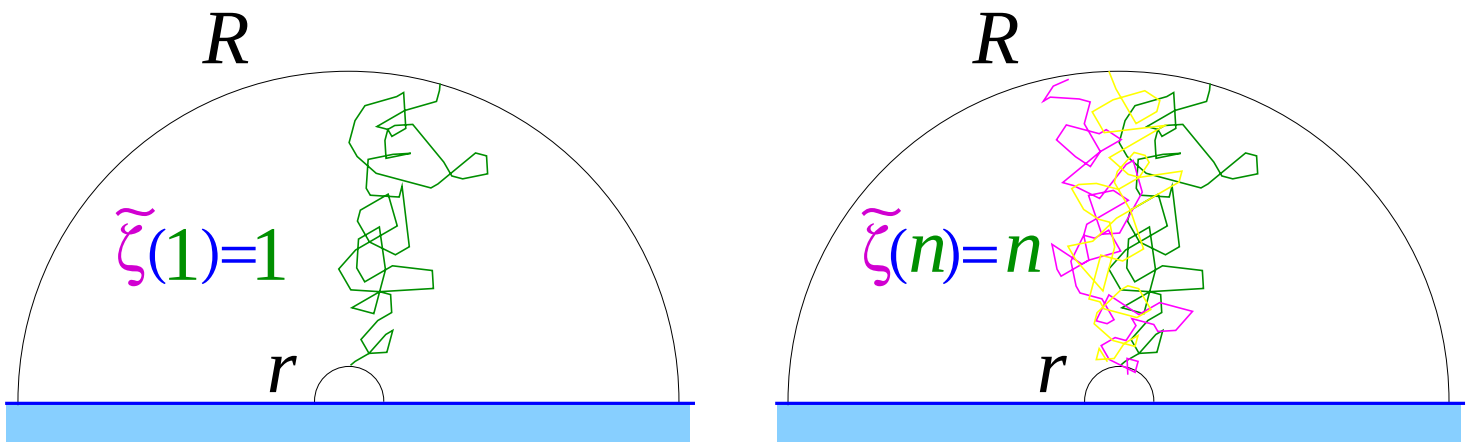
Boundary Quantum Gravity is Additive



$$U^{-1}(\tilde{x}_2) \quad = \quad 2U^{-1}(\tilde{x}_1)$$

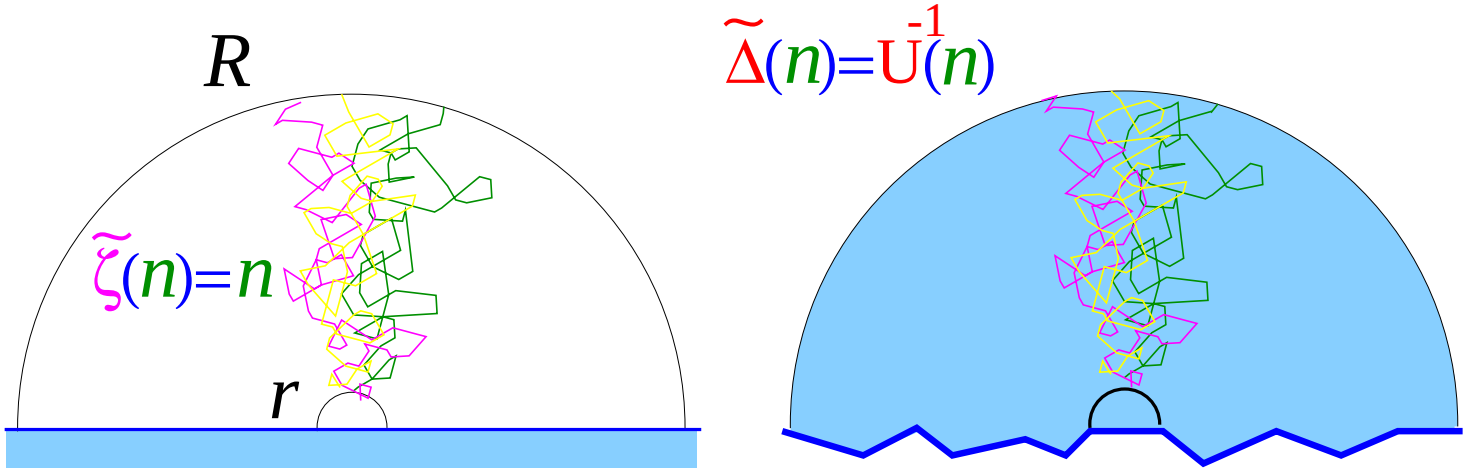
$$\tilde{\Delta}_2 = 2\tilde{\Delta}_1 = 2U_\kappa^{-1}(\tilde{x}_1) = \frac{4}{\kappa}$$

Brownian Packet Conformal Weight in $\mathbb{H}$



Dirichlet boundary conformal weights in $\mathbb{H}$ of a single Brownian path (left), and of a packet of n independent Brownian paths (right).

Brownian Packet Conformal Weight in QG



Left: Dirichlet boundary conformal weight n in $\mathbb{H}$ of a packet of n independent Brownian paths; right: its conformal weight $\tilde{\Delta}$ in boundary QG.

Brownian Packet in QG

Boundary conformal weight in $\mathbb{H}$ of a packet of n independent Brownian paths:

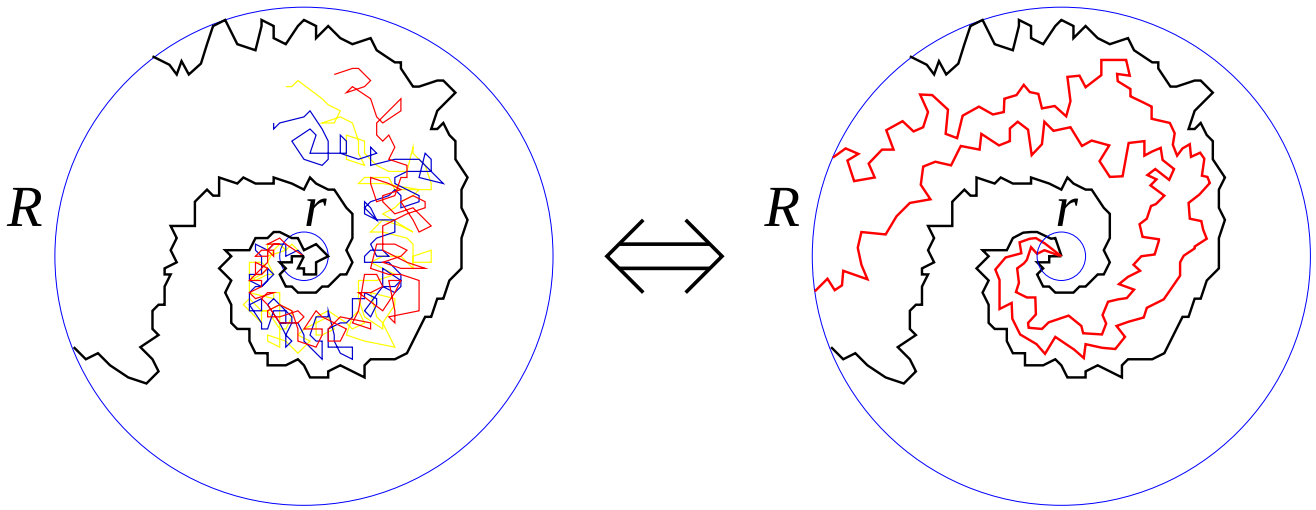
$$\tilde{\zeta}(n) = n$$

In QG, by inverting KPZ

$$\tilde{\Delta}(n) = U_\kappa^{-1}(n) = \frac{1}{2\kappa}\left(\sqrt{16\kappa n + (\kappa-4)^2} + \kappa - 4\right)$$

The Brownian paths, independent in a fixed metric, are strongly coupled by the metric fluctuations in quantum gravity.

SLE Equivalence

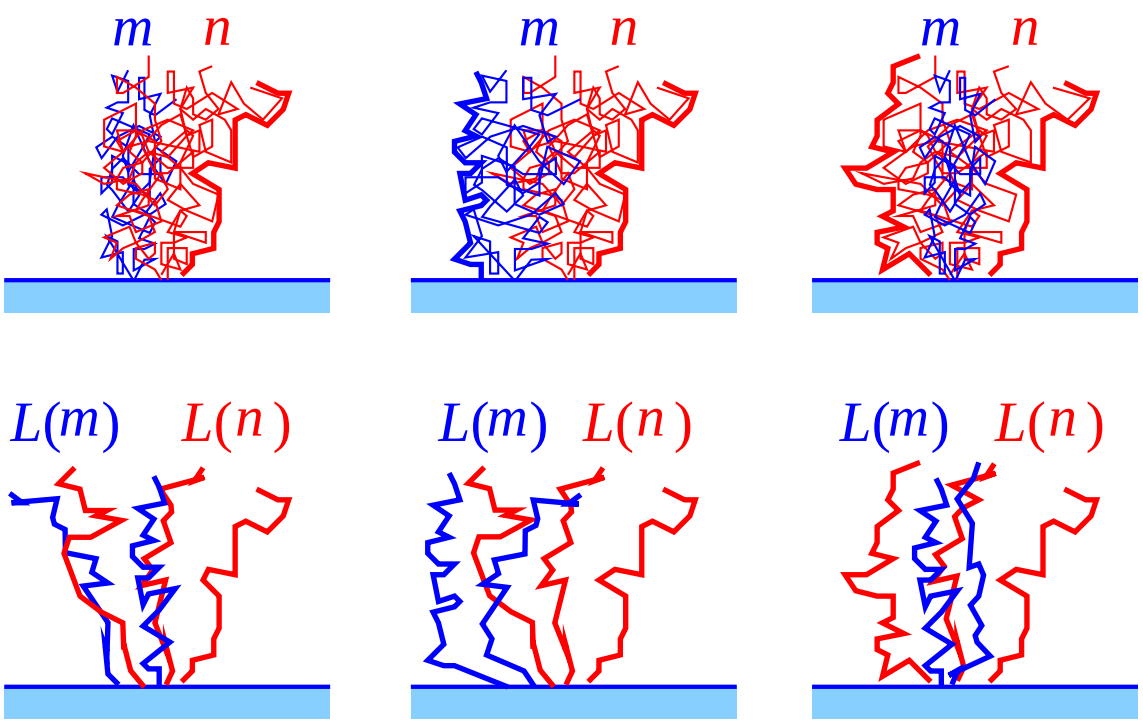


Total Number of Simple Paths: $\# = 2 + L$

n independent Brownian paths $\Longleftrightarrow$ L mutually-avoiding SLE paths:

$L = \frac{U_\kappa^{-1}(n)}{U_\kappa^{-1}(\tilde{x}_1)} = \frac{\kappa}{2} U_\kappa^{-1}(n)$, from ADDITIVITY OF BOUNDARY QG

Brownian Hiding Exponents and SLE(8/3)



$$U^{-1}_{\kappa=8/3}(n) = \frac{2}{\kappa}L(n) = \frac{3}{4}L(n)$$

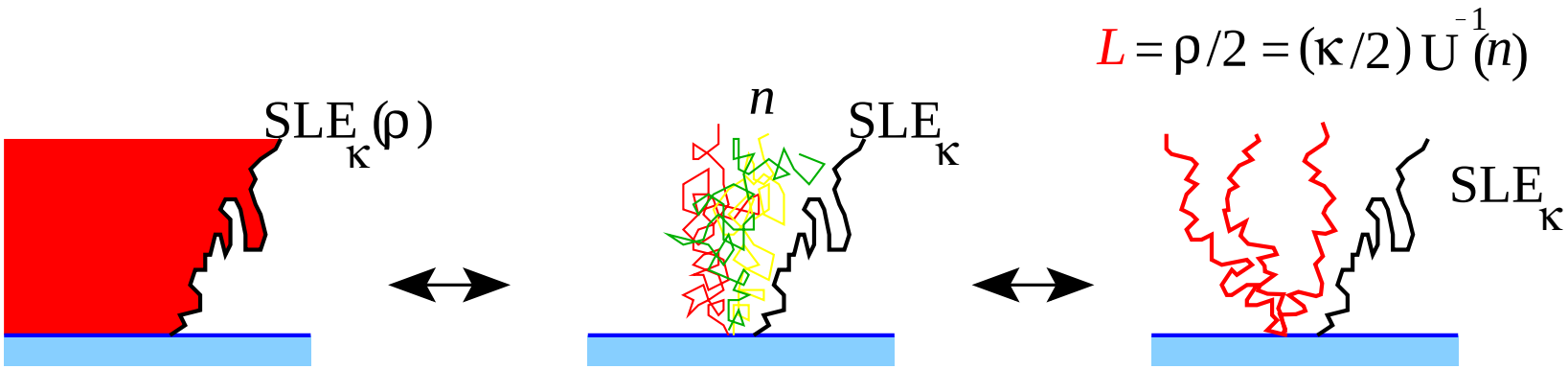
Hiding Exponents

Combining conformal dimensions $\tilde{\Delta}$ in boundary QG and $\tilde{x}$ in $\mathbb{H}$

$$\tilde{x}_{m,n} = U\left[\frac{3}{4} + U^{-1}\left[m + U\left(U^{-1}(n) - \frac{3}{4}\right)\right]\right]$$

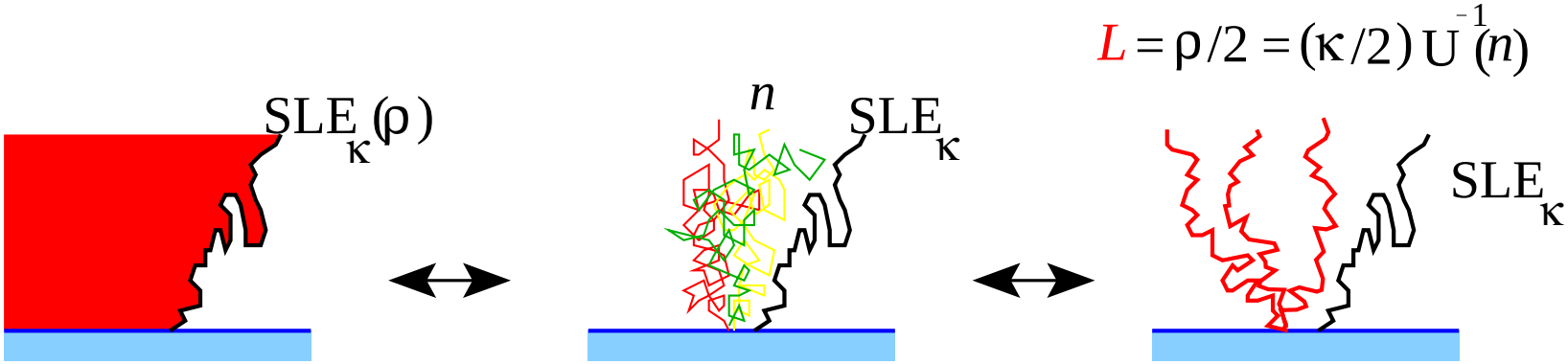
$$\tilde{x}_{m,n} = m + n + \frac{1}{4}\sqrt{24m + \left(\sqrt{1+24n} - 3\right)^2} - \frac{1}{4}\left(\sqrt{1+24n} - 3\right)$$

SLE(κ,ρ) & QG



Left: SLE(κ,ρ) in $\mathbb{H}$; middle: SLE(κ) and its counterpart of n independent Brownian paths; right: the counterpart as L equivalent SLE(κ)s from QG.

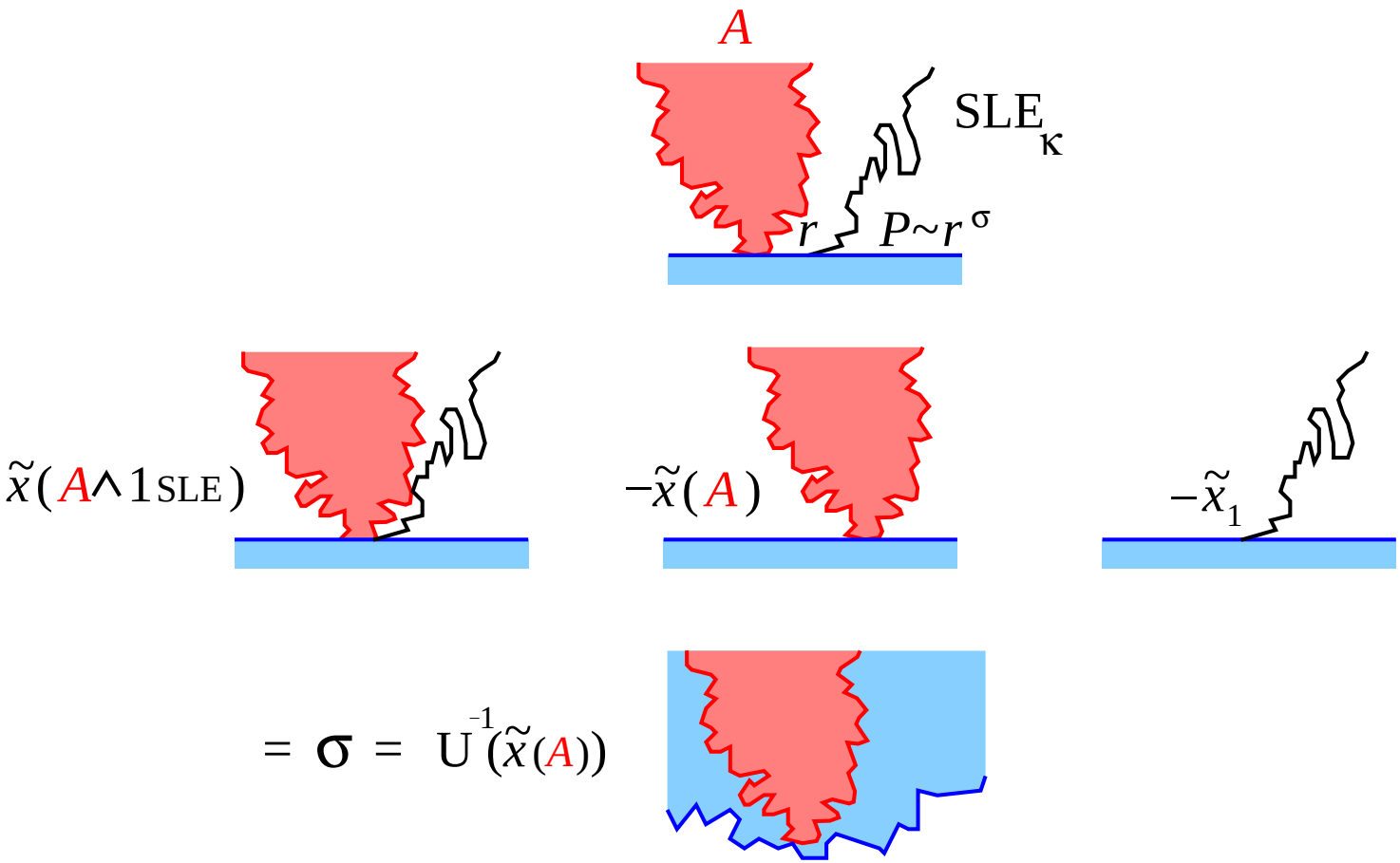
$\mathrm{SLE}(\kappa \geqslant 4, \rho = \kappa - 4)$



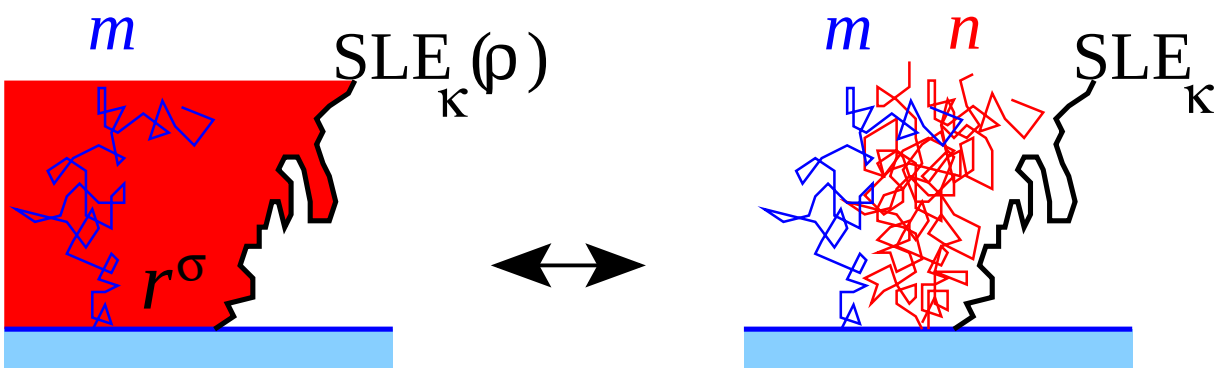
Left: $SLE(\kappa, \rho = \kappa - 4)$ *conditioned to avoid* $\mathbb{R}^-$*; middle:* $SLE(\kappa)$ *and its counterpart of* $n = 0$ *Brownian paths; right: the counterpart as* L *equivalent* $SLE(\kappa)s$ *from* QG

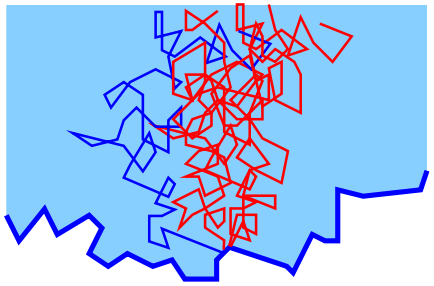
$$\rho = \kappa U_\kappa^{-1}(0), \quad U_\kappa^{-1}(0) = \theta(\kappa - 4)\left(1 - \frac{4}{\kappa}\right)$$

Contact Exponents and QG



Contact Exponents of SLE(κ, ρ)



$\sigma = U^{-1}(m+n)$  $- U^{-1}(n)$