Finiteness results for quadratic differentials

(joint with John Smillie)

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Definitions

S – a compact orientable surface, genus $g \geq 2$.

A *quadratic differential* (a.k.a. half-translation structure) q on S consists of

- $\Sigma_q = \{x_1, \ldots, x_r\} \subset S$;
- $\vec{k}(q) = \{k_1, \ldots, k_r\} \subset \mathbb{N}$;
- charts $S \smallsetminus \Sigma \to \mathbb{C}$, transition functions of form $z \mapsto \pm z + c$. At x_i have k_i-pronged singularity.

An *affine automorphism* is a homeomorphism $h : S \to S$ affine in each chart of q. Its *linear part* $Dh \in G = \mathrm{PSL}_2(\mathbb{R})$ independent of the chart. The Fuchsian group

$$\Gamma_q = \{Dh : h \text{ affine automorphism of } q\} \subset G$$

is the *Veech group* of q.

Example – polygonal billiards

$\mathcal{P} \subset \mathbb{R}^2$ a polygon with rational angles. *Billiard flow* – geodesic flow w.r.t. to flat structure, reflecting at sides and undefined at vertices.

Universal surface construction.
Δ – dihedral group generated by reflections in sides of $\mathcal{P}$.
$S = \bigcup_{g \in \Delta} g\mathcal{P}$, sides identified according to corresponding group elements. Billiard paths follow parallel line field on S.

Stratum of quadratic differentials

Fixing Σ, $\vec{k}$, a *stratum of quadratic differentials* is

$$\mathcal{M} = \{q : \Sigma_q = \Sigma,\ \vec{k}(q) = \vec{k},\ \text{area}(q) = 1\}$$

(considered up to self-homeos of S acting by precomposition on charts).

It parameterizes 'constructions of S out of pieces of graph paper.'

$\mathcal{M}$ is a noncompact orbifold, with an ergodic G-action preserving a finite smooth invariant measure. Union of strata for S is *quadratic differential space*, also a noncompact orbifold.

Veech group $\Gamma_q = \{g \in G : gq = q\}$.

Typically Γ_q is trivial, and Gq is dense. However exceptional q are of great interest.

Questions

1. Which Fuchsian groups Γ arise as Γ_q?

2. Classification of Veech groups.

3. Dynamical properties (of billiards or G-action) associated with group-theoretic properties.

Restrictions on Γ_q

- Non-uniform (G-action pinches a saddle).

- Arithmetic restriction (Gutkin-Judge, Kenyon-Smillie): can be conjugated into $G(k)$ for k a number field of degree at most g.

Existence results

- parabolic elements correspond to cylinder decompositions. Cyclic parabolic groups exist.

- hyperbolic elements (a.k.a. pseudo Anosovs). Have Markov partitions. Constructions by Thurston (bouillabaisse) and Arnoux-Yoccoz.

- Arithmetic lattices (Gutkin).

- Non-arithmetic lattices (Veech, . . .).

- Infinitely generated Fuchsian groups of the first kind (Hubert-Schmidt, McMullen).

- Group without parabolics (Hubert-Lanneau).

- All triangle groups (Möller-Bouw).

Classification results

- All lattices coming from billiards in acute rational triangles (Kenyon-Smillie, Puchta). Used a computer search, ordering triangles by gcd of angles.

- All lattices in genus 2 (Calta, McMullen).

- All non-elementary Veech groups in $\mathcal{H}(2)$ are lattices (Calta, McMullen).

- All non-elementary Veech groups in genus 2 are of the first kind (McMullen).

Dynamical properties for lattice surfaces

$\mathcal{L}_\theta(q)$ – saddle connections in direction θ.

- *Veech dichotomy*: If Γ_q is a lattice then

  I. if $\mathcal{L}_\theta(q) \neq \varnothing$ then flow in direction θ is completely periodic, $S \smallsetminus \mathcal{L}_\theta(q)$ a union of cylinders of commensurable moduli.

  II. if $\mathcal{L}_\theta(q) = \varnothing$ then flow in direction θ is uniquely ergodic.

- Γ_q is a lattice $\Longleftrightarrow$ Gq is closed (Smillie, Minsky-W).

Triangle areas

A triangle for q is a triangle with vertices in Σ_q, whose interior is isometrically embedded in S. If Δ is a triangle for q and $g \in G$ then $g\Delta$ is a triangle for gq.

Triangle area spectrum:

$\mathcal{T}(q) = \{\text{area}(\Delta) : \Delta \text{ a triangle for } q\}$.

Generically $\# \mathcal{T}(q) = \infty$, $\inf \mathcal{T}(q) = 0$.

No small triangles (Vorobets):

$\mathrm{NST}(\alpha) = \{q : \inf \mathcal{T}(q) \geq \alpha\}$
Note: closed, G-invariant.

$\widetilde{\mathrm{NST}}(\alpha) = \mathrm{NST}(\alpha)/G$.

Thm. $\#\widetilde{\mathrm{NST}}(\alpha) < \infty$.

Characterizes lattice property

Thm (Vorobets, S-W). TFAE:

- Γ_q is a lattice.
- $\# \mathcal{T}(q) < \infty$.
- $q \in \mathrm{NST}(\alpha)$ for some $\alpha > 0$.
- *Uniform Veech dichotomy:* there is $r > 0$ such that if $\mathcal{L}_\theta(q) \neq \varnothing$ then $S \smallsetminus \mathcal{L}_\theta(q)$ is a union of cylinders with commensurable moduli and for $\sigma_1, \sigma_2 \in \mathcal{L}_\theta(q)$, $\frac{|\sigma_1|}{|\sigma_2|} \leq r$.

Thm. There are q satisfying the Veech dichotomy for which Γ_q is not a lattice.

Remark: replacing triangles with pentagons does not characterize lattices.

Parabolic elements, cusp areas

Suppose $p \in \Gamma_q$ parabolic and primitive, Γ_q non-elementary. Denote:

- $k(p)$ – # cylinders in decomposition.
- area(p) – hyperbolic area of associated cusp, i.e. of B/Γ_q where $B \subset \mathbb{H}$ is maximal horoball with $B/\langle p \rangle \to B/\Gamma_q$ injective.

No large cusps: $\mathrm{NLC}(k, S) =$

$\{(q, p)$ as above : $k(p) = k$, area$(p) \leq S\}$,

$\widetilde{\mathrm{NLC}}(k, S) = \mathrm{NLC}(k, S)/G$.

Thm. $\# \widetilde{\mathrm{NLC}}(k, S) < \infty$.

Hyperbolic elements, Markov partitions

Suppose $h \in \Gamma_q$ is hyperbolic and primitive. Denote:

- $\lambda(h)$ – larger eigenvalue of h.
- $\ell(q, h)$ – minimal number of parallelograms in an associated Markov partition.

No simple Markov partitions: $\mathrm{NSMP}(\ell, T) =$

$\{(q, h) \text{ as above} : \lambda(h) \le T,\, \ell(q, h) = \ell\}$,

$\widetilde{\mathrm{NSMP}}(\ell, T) = \mathrm{NSMP}(\ell, T)/G$.

Thm. $\#\,\widetilde{\mathrm{NSMP}}(\ell, T) < \infty$.

Applications

1. Finiteness in a stratum. Fix $\mathcal{M}$, T.

- $\#\{Gq : q \in \mathcal{M},\ \mathrm{covol}(G/\Gamma_q) < T\} < \infty$.

  Proved independently by McMullen, without explicit bounds.

- $\#\{Gq : q \in \mathcal{M},\ \exists h \in \Gamma_q \text{ s.t. } \lambda(h) < T\} < \infty$.

  Proved by Veech with explicit bounds.

2. Restrictions on Γ_q.

- Cannot contain infinitely many cusps of same area or non-conjugate hyperbolics with same $\lambda(h)$.

- Example: a normal infinite index subgroup of a lattice in G cannot occur as Γ_q.

Ideas in proofs: I. Perron-Frobenius.

Thurston (bouillabaisse), Veech:
Two non-commuting parabolics give intersection matrix. Knowing Dehn twists around cylinders gives

$$\mu_1 \mu_2 \vec{a} = D_1 B D_2 B^{\mathrm{t}} \vec{a},$$

where μ_i shear for each parabolic, D_i diagonal recording twists, B intersection matrix, $\vec{h}$ heights of rectangles.
Matrix $D_1 B D_2 B^{\mathrm{t}}$ is positive.

'Bottom of cusp': cylinder decomposition intersects itself.

Markov partition: intersection matrix of rectangles and their images.

Cor. Uniqueness of PV eigenvector $\Longrightarrow$ Combinatorics determines affine equivalence class.

II. Isolation results.

If one has a bound on possible combinatorics, nearby surfaces must belong to the same G-orbit.

Prop. Let $q_n \to q$, and suppose one of the following holds:

- $q, q_n \in \mathsf{NST}(\alpha)$;
- Γ_q and each Γ_{q_n} are non-elementary and contain parabolics p, p_n with $p_n \to p$ and corresponding cusp areas bounded above;
- There are $h \in \Gamma_q, h_n \in \Gamma_{q_n}$ hyperbolic with $h_n \to h$;

Then $q_n \in Gq$ for large enough n.

III. Closed implies finite volume.

Thm (Minsky -W.) Any G-invariant ergodic locally finite measure on $\mathcal{M}$ is finite.

Quantitative non-divergence estimates for horocycles:

For each $q \in \mathcal{M}$ and $\varepsilon > 0$ there is $K \subset \mathcal{M}$ compact such that

$$\liminf_{T \to \infty} \frac{1}{T} |\{s \in [0, T] : h_s q \in K\}| \geq 1 - \varepsilon.$$

Here $\{h_s\}$ is Teichmüller horocycle flow.

Modelled on similar arguments of Dani and Margulis for homogeneous spaces.

If Gq is closed, orbit map $G/\Gamma_q \to Gq$ is a homeomorphism so obtain a locally finite measure on $Gq \subset \mathcal{M}$. Measure is finite to Γ_q a lattice.

IV. Non-lattices with Veech dichotomy.

Hubert-Schmidt: There are non-lattice surfaces q such that if $\mathcal{L}_\theta(q) \neq \varnothing$ then flow in direction θ completely periodic.

Obtained by covers over lattice surfaces, branched over 'aperiodic connection points.'

Note: $\mathcal{L}_\theta(q) = \varnothing \implies$ direction θ minimal. So need to upgrade to uniquely ergodic.

Masur's lemma: If q has minimal but not uniquely ergodic vertical direction, then $X_t \to \infty$ in moduli space of Riemann surfaces, where X_t is Riemann surface underlying $g_t q$, $\{g_t\}$ is Teichmüller horocycle flow.

Observation: If q' obtained from q by branching over a single point and $X'_t \to \infty$ then also $X_t \to \infty$.

Remarks: 1. smallest example is in genus 5.
2. essentially use topology on moduli space.

Additional directions

1. Explicit bounds. What is growth of numbers $\#\,\widetilde{\mathsf{NST}}(\alpha)$, $\#\,\widetilde{\mathsf{NLC}}(k,S),\ldots$?

What is the growth in a given stratum?

In all cases our arguments give explicit bounds.

2. Effective searches. Fix α (resp. (k,S), or (ℓ,T)). One can find in finite time a list containing all $q \in \widetilde{\mathsf{NST}}(\alpha)$ (resp. $q \in \widetilde{\mathsf{NLC}}(k,S)$ or $q \in \mathsf{NSMP}(\ell,T)$). Results are given explicitly as polygons in plane with gluing pattern.

3. Bottom of spectrum. Which q have largest α, smallest cusp area, covolume, etc.

4. Other configurations. Lower bounds on areas for other polygons gives closed G-invariant sets. Classify e.g. $\mathsf{NSQ}(\alpha)$.