Lyapunov exponents of Teichmüller flows

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Moduli spaces

$\mathcal{A}_g$ = moduli space of Abelian differentials (holomorphic complex 1-forms) on Riemman surfaces of genus $g \geq 2$

- a complex orbifold of dimension $\dim_{\mathbb{C}} \mathcal{A}_g = 4g - 3$
- a fiber bundle over the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus $g \geq 2$

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$\mathcal{A}_g(m_1, \ldots, m_\sigma)$ = stratum of Abelian differentials having σ zeroes, with multiplicities $m_1, \ldots, m_\sigma$

$$\sum_{i=1}^{\sigma} m_i = 2g - 2$$

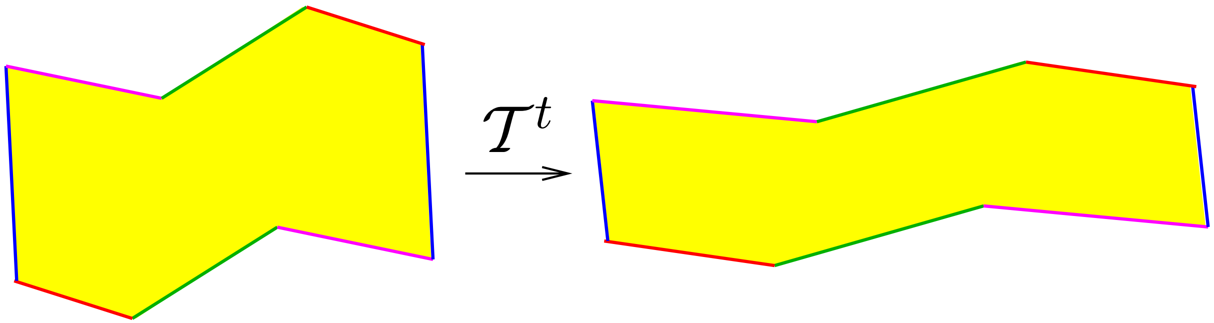
- complex orbifold of $\dim_{\mathbb{C}} \mathcal{A}_g(m_1, \ldots, m_\sigma) = 2g + \sigma - 1$

Teichmüller flow

The Teichmüller flow is the natural action $\mathcal{T}^t$ on the fiber bundle $\mathcal{A}_g$ by the diagonal subgroup of $\mathrm{SL}(2,\mathbb{R})$:

$$\mathcal{T}^t(\omega)_z = e^t\big(\Re\omega_z\big) + ie^{-t}\big(\Im\omega_z\big)$$

Geometrically:



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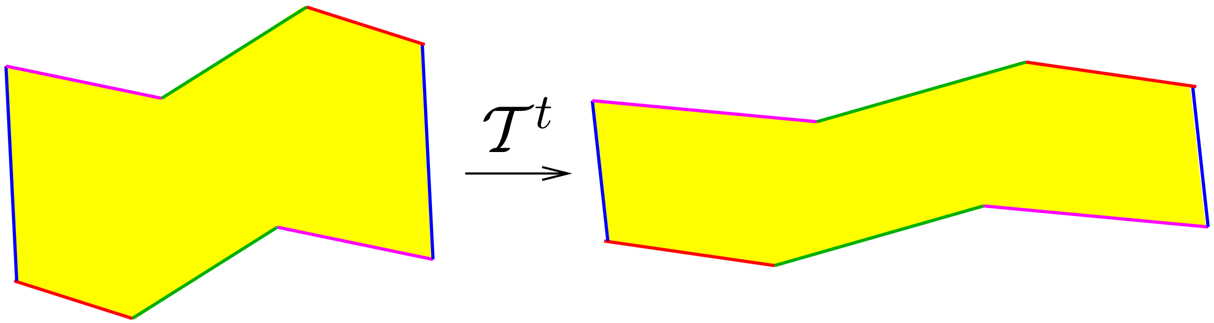

This flow preserves the area of the translation surface defined by the Abelian differential. In what follows we normalize the area.

Ergodicity

Masur, Veech:

Each stratum of $\mathcal{A}_g$ carries a canonical volume measure. This measure is finite and invariant under the Teichmüller flow.

The Teichmüller flow $\mathcal{T}^t$, restricted to any hypersurface of constant area, is ergodic on any connected component of every stratum.

Kontsevich-Zorich:

Each stratum has up to 3 connected components.

Lyapunov exponents

Fix a connected component of a stratum. The Lyapunov spectrum of the Teichmüller flow has the form

$$2 \geq 1+\nu_2 \geq \cdots \geq 1+\nu_g \geq 1 = \cdots = 1 \geq 1-\nu_g \geq \cdots \geq 1-\nu_2 \geq 0$$
$$\geq -1+\nu_2 \geq \cdots \geq -1+\nu_g \geq -1 = \cdots = -1 \geq -1-\nu_g \geq \cdots \geq -1-\nu_2 \geq -2$$

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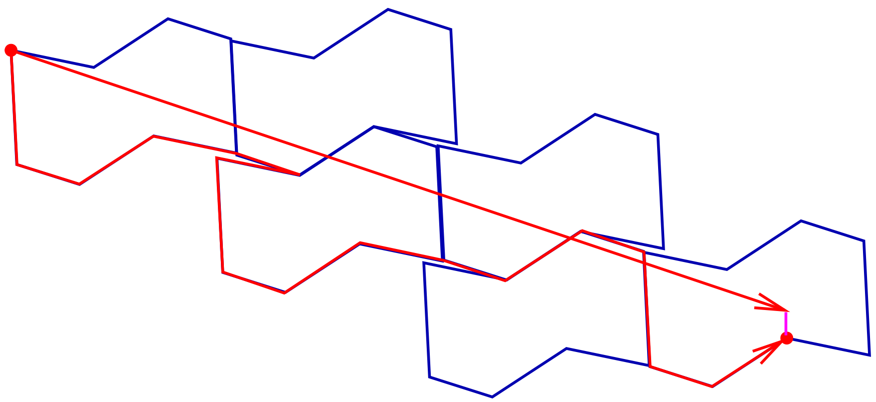
Theorem (Avila, Viana). $1 > \nu_2 > \cdots > \nu_g > 0$

Asymptotic cycles

Given any long geodesic segment γ in a given direction, "close" it to get an element $h(\gamma)$ of $H_1(S,\mathbb{Z})$:



Kerckhoff, Masur, Smillie: For almost any direction the geodesic flow is uniquely ergodic.

Then, $h(\gamma)/|\gamma|$ converges uniformly to some $c_1 \in H_1(S,\mathbb{R})$ when the length $|\gamma|$ goes to infinity, where the asymptotic cycle c_1 depends only the surface and the direction.

Asymptotic flag in homology

Corollary (Zorich). *There are subspaces* $L_1 \subset L_2 \subset \cdots \subset L_g$ *of* $H_1(S, \mathbb{R})$ *with* $\dim L_i = i$ *for every* i*, such that*

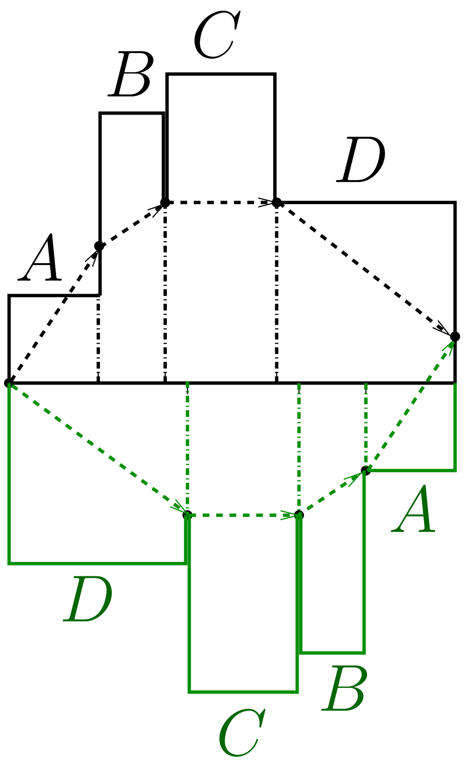
- *the deviation of* $h(\gamma)$ *from* L_i *has amplitude* $|\gamma|^{\nu_{i+1}}$ *for all* $i < g$*:*

$$\limsup_{|\gamma| \to \infty} \frac{\log \operatorname{dist}(h(\gamma), L_i)}{\log |\gamma|} = \nu_{i+1}$$

- *the deviation of* $h(\gamma)$ *from* L_g *is bounded* $\quad$ *(*$g =$ *genus).*

Zippered rectangles

Almost every translation surface may be represented in the form of zippered rectangles (minimal number of rectangles is $d = 2g - \sigma - 1$):



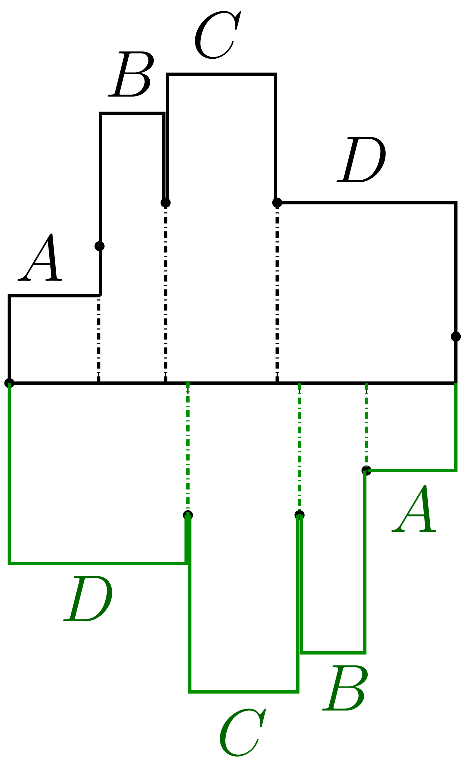
Coordinates in the stratum

This defines local coordinates (π, λ, τ, h) in the stratum:

- $\pi = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$ describes the combinatorics of the associated interval exchange map
- $\lambda = (\lambda_A, \lambda_B, \lambda_C, \lambda_D)$ are the horizontal coordinates of the saddle-connections (= widths of the rectangles)
- $\tau = (\tau_A, \tau_B, \tau_C, \tau_D)$ are the vertical components of the saddle-connections
- The heights $h = (h_A, \lambda_B, \lambda_C, h_D)$ of the rectangles are linear functions of τ.

Poincaré return map

We consider the return map to the cross section $\sum_\alpha \lambda_\alpha = 1$.
First step:



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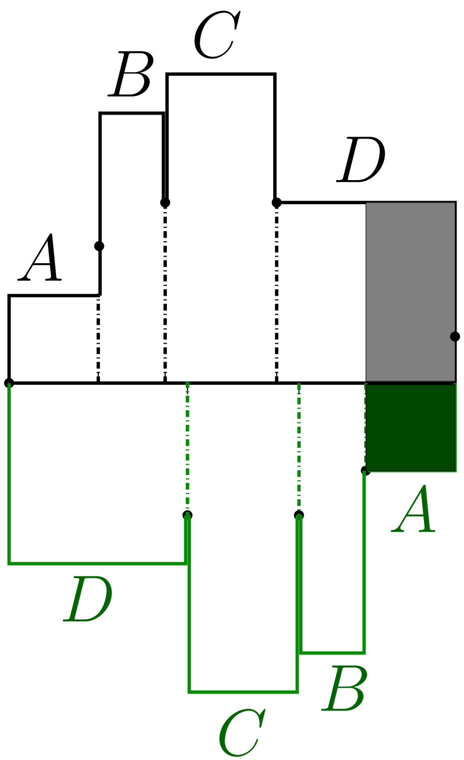

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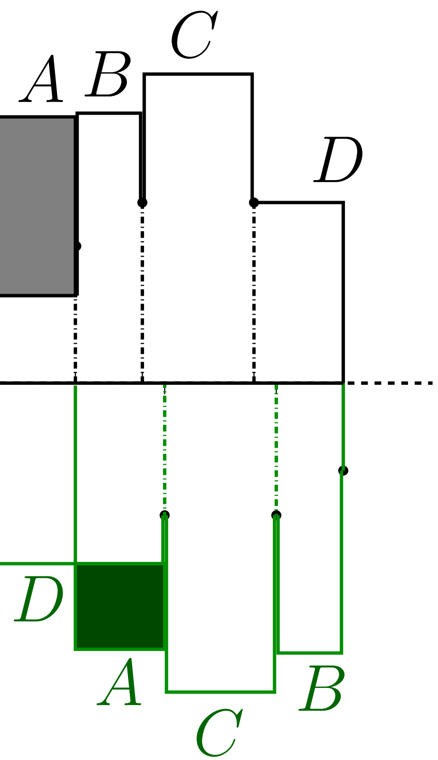

Poincaré return map

In this first step $(\pi, \lambda, \tau, h) \mapsto (\pi', \lambda', \tau', h')$, where

- $\pi' = \begin{pmatrix} A & B & C & D \\ D & A & C & B \end{pmatrix}$ (*top* case)
- $\lambda'_\alpha = \lambda_\alpha$ except $\lambda'_D = \lambda_D - \lambda_A$
- same for τ
- $h'_\alpha = h_\alpha$ except $h'_A = h_A + h_D$

Write $h' = \Theta(h)$ where Θ is the linear operator thus defined. Notice that it has nonnegative coefficients.

Then $\lambda' = \Theta^{-1*}(\lambda)$ and analogously for τ.

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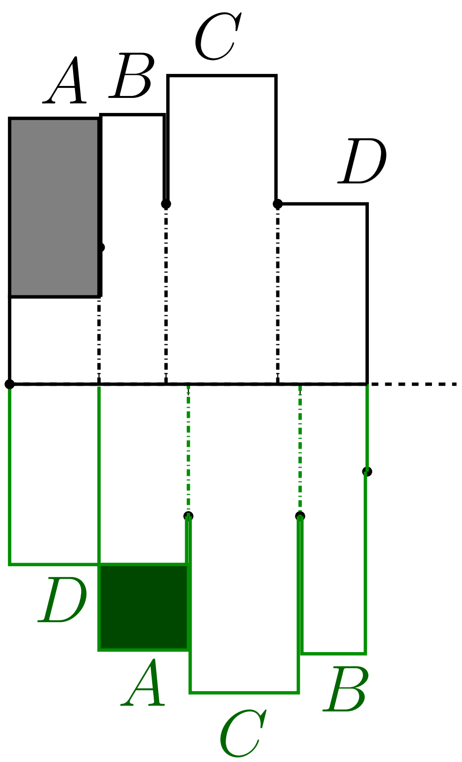

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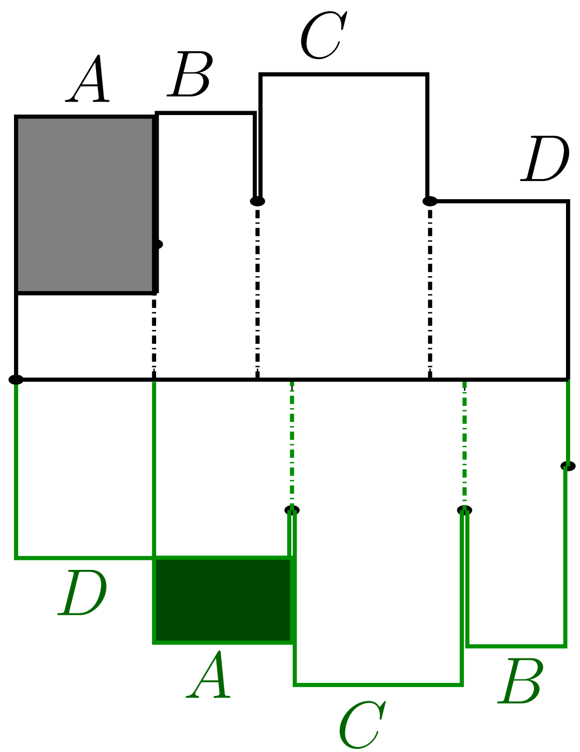

Now $(\pi', \lambda', \tau', h') \mapsto (\pi'', \lambda'', \tau'', h'')$, with $\pi'' = \pi'$ and $\lambda'' = c\lambda'$ and $\tau'' = c^{-1}\tau'$, where $c = c(\lambda')$ is the normalizing factor.

Rauzy-Veech cocycle

This Poincaré return map $\mathcal{R} : (\pi, \lambda, \tau) \mapsto (\pi'', \lambda'', \tau'')$ is called (invertible) Rauzy-Veech renormalization.

We consider the linear cocycle over the map $\mathcal{R}$ defined by

$$F_{\mathcal{R}} : (\pi, \lambda, \tau, h) \mapsto (\mathcal{R}(\pi, \lambda, \tau), \Theta(h))$$

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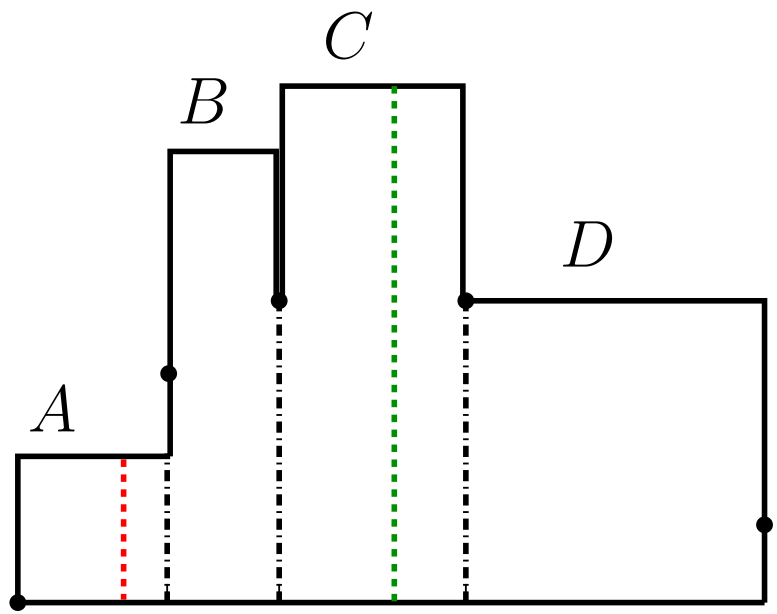
The strategy is to try

- to relate the Lyapunov exponents of the Teichmüller flow to the Lyapunov exponents of $F_{\mathcal{R}}$
- and to analyze the latter through general methods of linear cocycles

(there is a technical difficulty, as we shall see in a while).

Rauzy cocycle and long geodesics

The Rauzy-Veech cocycle gives a way to construct long geodesics:



Consider a geodesic segment that crosses each rectangle α vertically. "Close" it by joining the endpoints to some base point in the cross-section. This defines some $v_\alpha \in H_1(S, \mathbb{Z})$.

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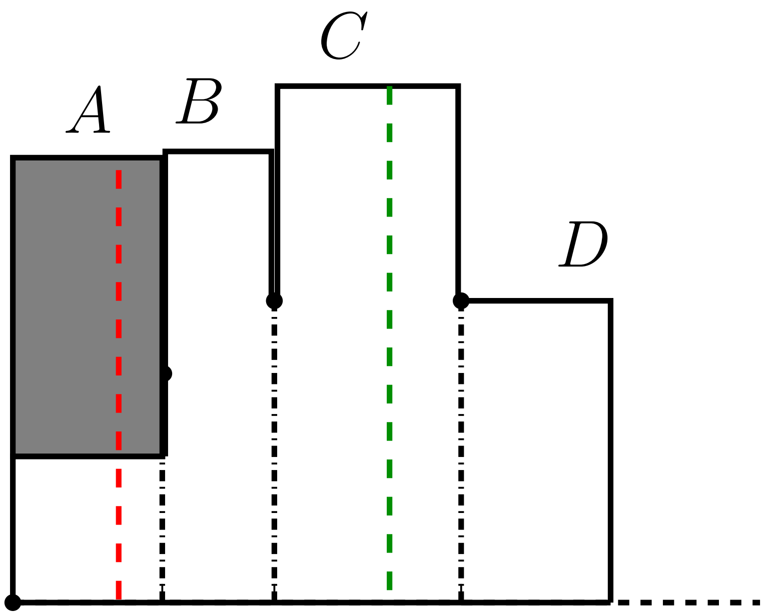

We get $\quad v'_A = v_A + v_D \quad v'_B = v_B \quad v'_C = v_C \quad v'_D = v_D$.

In other words, $\quad v' = \Theta(v)$.

Zorich cocycle

The Rauzy-Veech renormalization $\mathcal{R}$ has an invariant volume measure (for each choice of the stratum), related to the invariant volume of the Teichmuller flow. However, this measure is infinite.

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Zorich introduced an accelerated renormalization and an accelerated cocycle

$$\mathcal{Z}(\pi,\lambda,\tau) = \mathcal{R}^n(\pi,\lambda,\tau) \quad \text{and} \quad F_{\mathcal{Z}}(\pi,\lambda,\tau,h) = F_{\mathcal{R}}^n(\pi,\lambda,\tau,h),$$

where $n = n(\pi,\lambda)$ is smallest such that the Rauzy iteration changes from “top” to “bottom” or vice-versa.

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where $n = n(\pi,\lambda)$ is smallest such that the Rauzy iteration changes from “top” to “bottom” or vice-versa.

This map $\mathcal{Z}$ has a natural invariant volume probability (for each choice of the stratum) and this probability is ergodic.

Zorich cocycle

The Lyapunov spectrum of the Zorich cocycle $F_{\mathcal{Z}}$ has the form

$$\theta_1 \geq \cdots \geq \theta_g \geq 0 = \cdots = 0 \geq -\theta_g \geq \cdots \geq -\theta_1$$

and is related to the spectrum of the Teichmüller flow by

$$\nu_i = \frac{\theta_i}{\theta_1} \quad i = 2, \ldots, g.$$

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One may find an explicit subbundle with dimension $2g$ that avoids the trivial exponents. Moreover, restricted to this subbundle the cocycle $F_{\mathcal{Z}}$ is symplectic.

Main result

Theorem (Avila, Viana). *The Lyapunov spectrum of the (restricted) Zorich cocycle is simple:* $\theta_1 > \theta_2 > \cdots > \theta_g > 0$.

The proof has two main steps:

(1) A sufficient condition for the Lyapunov spectrum of a very general linear cocycle to be simple.

(2) To prove that this criterium is satisfied by the Zorich cocycle, for every stratum.

Simplicity criterium

Let $f : M \to M$ be a finite or countable shift map,

$F : M \times \mathbb{R}^d \to M \times \mathbb{R}^d$ be a locally constant linear cocycle over f,

and μ be an f-invariant probability with continuous local product structure:

$$\mu = \rho(\mu^+ \times \mu^-)$$

where $\log \rho$ is continuous and locally bounded.

(there is an extension to non-locally constant cocycles)

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The linear cocycle F is called simple if it is

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twisting: there is some point z homoclinic to p such that, for all F_p^κ-invariant subspaces V_1, V_2 with $\dim V_1 + \dim V_2 \leq d$,

$$(F_z^{l\kappa})(V_1) \cap V_2 = \{0\}.$$

$$p = (\ldots, p_{-1}, p_0, p_1, \ldots, p_{l\kappa}, p_{l\kappa+1}, \cdots)$$

$$z = (\ldots, p_{-1}, p_0, z_1, \ldots, z_{l\kappa}, p_{l\kappa+1}, \cdots)$$

Simplicity criterium

Theorem 1. *If F is simple then all its Lyapunov exponents have multiplicity 1.*

This improves a criterium of Bonatti-Viana, based on Guivarch-Raugi and Goldsheid-Margulis.

Checking the Criterium

Theorem 2. *Every Zorich cocycle is simple.*

Checking the Criterium

Theorem 3. *Every Zorich cocycle is simple.*

The proof is by induction on the number of intervals. We consider combinatorial operations of reduction/extension:

$$\pi = \begin{pmatrix} a_1 & \cdots & a_{i-1} & c & a_{i+1} & \cdots & \cdots & \cdots & a_d \\ b_1 & \cdots & & \cdots\cdots & b_{j-1} & c & b_{j+1} & \cdots & b_d \end{pmatrix}$$

$$\updownarrow$$

$$\pi' = \begin{pmatrix} a_1 & \cdots & a_{i-1} & a_{i+1} & \cdots & \cdots & a_d \\ b_1 & \cdots & \cdots & b_{j-1} & b_{j+1} & \cdots & b_d \end{pmatrix}$$

In terms of the corresponding translation surfaces this may correspond to collapsing singularities or even changing the genus.

Checking the Criterium

Given π with d symbols, there exists π' with $d-1$ symbols such that π is an extension of π'. Then, either $g(\pi) = g(\pi')$ or $g(\pi) = g(\pi') + 1$.

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2. If $g(\pi) = g(\pi') + 1$ then $H_1(S(\pi'), \mathbb{R})$ may be seen as a symplectic reduction of $H_1(S(\pi), \mathbb{R})$

$=$ symplectic orthogonal of v_c inside $H_1(S(\pi), \mathbb{R})/v_c$

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In this way one can prove twisting for π from twisting for π'. Pinching also requires a careful combinatorial analysis.