Existence of Hyperbolic Bernoulli Flows

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Main Theorem

Given a compact smooth Riemannian manifold M of $\dim M \geq 3$, there exists a C^∞ flow f^t s.t. for $t \neq 0$,

1. f^t preserves the Riemannian volume μ;
2. f^t has non-zero Lyapunov exponents (except for the exponent along the flow direction) at a.e. point $x \in M$;
3. f^t is a Bernoulli diffeomorphism.

1. Anosov flows:

- special flows over Anosov diffeomorphisms
- geodesic flows on negatively curved manifolds

2. A volume preserving non-Anosov flow on a 3-manifold with nonzero exponents — a locally slow-down along trajectories of an Anosov flow.

3. (Dolgopyat, P.) A volume preserving C^∞ Bernoulli diffeomorphism with nonzero exponents on any compact manifold of dim ≥ 2.

Let

$$A = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix},$$

be a hyperbolic automorphism of the two-torus T^2. It has four fixed points

$$q_1 = (0,0),\ q_2 = (\frac{1}{2}, 0),\ q_3 = (0, \frac{1}{2}),\ q_4 = (\frac{1}{2}, \frac{1}{2}).$$

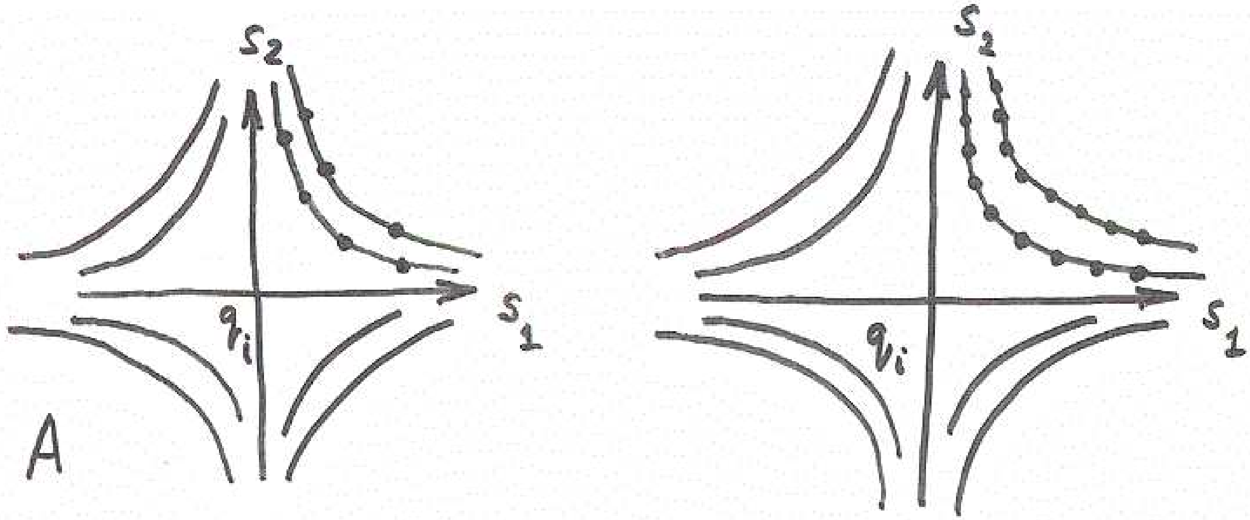
In a small neighborhood

$$D_r^i = \{(s_1, s_2) : s_1^2 + s_2^2 \leq r\}$$

of q_i, the map A is the time-1 map of the flow

$$\dot{s}_1 = -(\log \alpha) s_1, \quad \dot{s}_2 = (\log \alpha) s_2,$$

where $\alpha > 1$ is the larger eigenvalue of A and $\{s_1, s_2\}$ is the coordinate system in D_r^i generated by the eigenvectors of A.



g_1 is the time-1 map of the flow

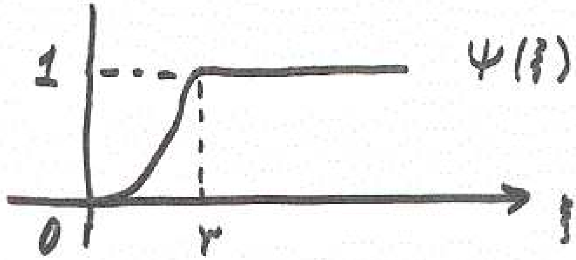
$$\dot{s}_1 = -(\log\alpha)s_1\psi(s_1^2 + s_2^2),$$

$$\dot{s}_2 = (\log\alpha)s_2\psi(s_1^2 + s_2^2)$$

in D_r^i, and $g_1 = A$ otherwise. Here ψ is a C^∞ function except at 0 and s.t. $\psi(0) = 0$, $\psi(\xi) \geq 0$, for $\xi \geq 0$, $\psi(\xi) = 1$ for $\xi \geq r$ and

$$\int_0^r \sqrt{\frac{1}{\psi(\xi)}}\, d\xi < \infty.$$

g_1 is conjugate to A via a conjugacy ϕ_0 (it slows down the motion near q_i).



g_1 preserves a measure $d\nu = \kappa_0^{-1} \kappa \, dm$, where $\kappa_0 = \int_{T^2} \kappa \, dm$ is a "normalizing factor", m is area and the density κ is a C^∞ function,

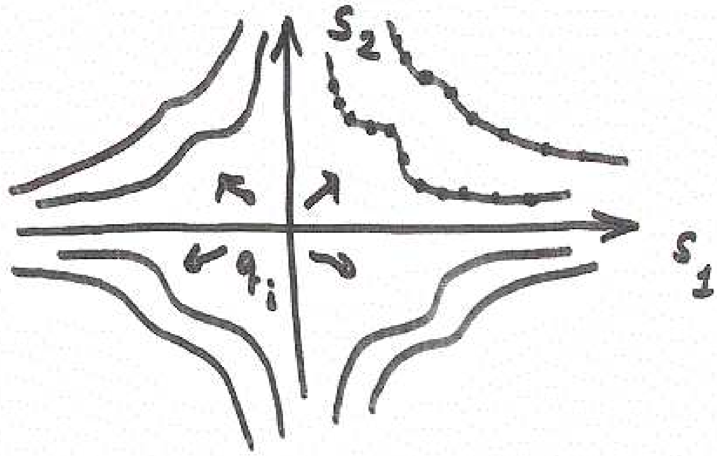
$$\kappa(s_1, s_2) = (\psi({s_1}^2 + {s_2}^2))^{-1}, \, (s_1, s_2) \in D_r^i$$

and $\kappa(s_1, s_2) = 1$ otherwise. Note that κ is infinite at q_i.

Define the map ϕ_1 by the formula

$$\phi_1(s_1, s_2) = \frac{1}{\sqrt{\kappa_0 a}} \left(\int_0^a \frac{du}{\psi(u)} \right)^{1/2} (s_1, s_2)$$

($a = {s_1}^2 + {s_2}^2$) near each q_i and extend it to T^2 s.t. ϕ_1 is C^∞ and satisfies $(\phi_1)_* \nu = m$. Hence, $g_2 = \phi_1 \circ g_1 \circ \phi_1^{-1}$ is a C^∞ area preserving map.

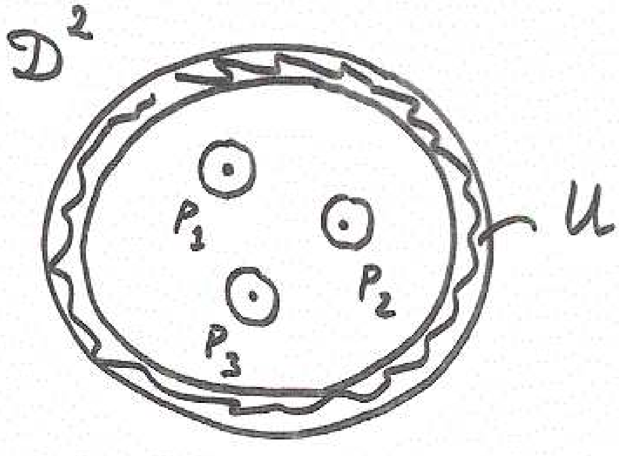


Let $\phi_2: T^2 \to S^2$ be a double branched covering satisfying $\phi_2 \circ J = \phi_2$, $(\phi_1)_* m = m$, and ϕ_2 is C^∞ everywhere except for q_i, where it branches and near q_i,

$$\phi_2(s_1, s_2) = \frac{1}{\sqrt{{s_1}^2 + {s_2}^2}}(s_1^2 - s_2^2,\ 2s_1 s_2).$$

The map $g_3 = \phi_2 \circ g_2 \circ \phi_2^{-1}$ is a C^∞ diffeo of the sphere S^2.

Let ϕ_3 be a C^∞ map that blows up the point q_4 into a circle and makes $g = \phi_3 \circ g_3 \circ \phi_3^{-1}$ to be the desired map of the disk.



Properties of the Map g

(1) g is C^∞, preserves area, has non-zero Lyapunov exponents a.e. and is a Bernoulli map.

(2) g is uniformly hyperbolic outside a small neighborhood of the *singularity* set $Q = \partial D^2 \cup \{p_1, p_2, p_3\}$, i.e., there exists $\lambda > 1$, s.t.

$$\|dg|E_g^s(x)\| \le \frac{1}{\lambda}, \quad \|dg^{-1}|E_g^u(x)\| \le \frac{1}{\lambda}.$$

(3) g possesses two one-dimensional continuous foliations which are extensions of the stable and unstable global foliations $W_g^s(x)$ and $W_g^u(x)$; we will use the same notations for these foliations.

(4) On the boundary of the disk g is the identity map and has all its derivatives zero; moreover, there are neighborhoods $U \subset U_1$ of ∂D^2 and a vector field $\mathbf{V}$ in U_1 which generates an area-preserving flow $g^t\colon U \to D^2$, $-2 < t < 2$ for which $g|U = g^1$.

(5) g is diffeotopic to the identity map – there exists a C^∞ map $G : D^2 \times [0,1] \to D^2$ s.t.

1. $G(\cdot, 0) = id$ and $G(\cdot, 1) = g$;

2. $G(x, t) = g^t(x)$ for $x \in U$ and $t \in [0,1]$;

3. the map $G(\cdot, t) : D^2 \to D^2$ is area-preserving;

4. $d^k G(x, 1) = d^k G(g(x), 0)$ for any $k \geq 0$.

Proof. Extend the vector field $\mathbf{V}$ to a smooth vector field $\widehat{\mathbf{V}}$ on the whole D^2, and let $\widehat{g}^t$ be the corresponding flow. Note that $g|U = \widehat{g}^1|U$.

Lemma (Smale) Let $\mathcal{A}$ be the space of C^∞ diffeo of the unit square (disk) which are the identity in a neighborhood of the boundary. Endow $\mathcal{A}$ with the C^r topology, $1 < r \leq \infty$. Then $\mathcal{A}$ is contractible to a point.

Applying this result to $g \circ \widehat{g}^{-1}$, which is the identity on U, we obtain a homotopy

$$\widetilde{G} : D^2 \times [0,1] \to D^2$$

such that

$$\widetilde{G}(\cdot, 0) = id|D^2 \text{ and } \widetilde{G}(\cdot, 1) = g \circ \widehat{g}^{-1}.$$

Moreover, $\widetilde{G}$ is C^∞ in (x,t), i.e., $\widetilde{G}$ is a diffeotopy in $\mathcal{A}$. Therefore, for $t \in [0,1]$, there is a neighborhood U_t of ∂D^2 s.t. $\widetilde{G}(\cdot, t)|U_t = id|U_t$. The set

$$U = int \bigcap_{t \in [0,1]} U_t$$

is not empty and is a neighborhood of ∂D^2. If $\widetilde{g}^t = \widetilde{G}(\cdot, t)$ then

$$G_1(\cdot, t) = \widetilde{g}^t \circ \widehat{g}^t$$

satisfies Statements 1 and 2. We shall further modify the diffeotopy $G_1(x,t)$, so it satisfies Statements 3 and 4.

We need:

– Anosov

Lemma (Mo[redacted]er–Katok) Let $\{O_0^t\}$ and $\{O_1^t\}$ be two families of volume forms on D^2 that are C^∞ in (x,t). Assume that $O_0^t|U = O_1^t|U$ for any t and $O_0^t = O_1^t$ for $t \in [0,\epsilon) \cup (1-\epsilon, 1]$. Then there exists a map $\bar{G} : D^2 \times [0,1] \to D^2$ s.t.

1. $\bar{G}(x,t)$ is C^∞ in (x,t) and $\bar{G}(\cdot,0) = \bar{G}(\cdot,1) = id$;

3. for any $t \in [0,1]$ the map $\bar{G}(\cdot,t) : D^2 \to D^2$ is a diffeo with $\bar{G}(\cdot,t)^* O_1^t = O_0^t$;

4. $\bar{G}(x,t) = x$ for any $t \in [0,1]$ and x in some neighborhood $U' \subset U$ of ∂D^2.

Consider $O_0^t = dx_1 \wedge dx_2$ and $O_1^t = (d\tilde{g}^t d\hat{g}^t)^* O_0^t$. Let $\bar{g}^t = \bar{G}(\cdot,t)$. The map $G(x,t) = \bar{g}^t \circ \tilde{g}^t \circ \hat{g}^t$ satisfies Statements $1-3$. One can change $G(\cdot,t)$ in a small neighborhood of the sets $D^2 \times 0$ and $D^2 \times 1$ so that it will satisfy Statement 4.

Proof of the theorem: $\dim M \geq 5$

Consider the map

$$R = g \times A : D^2 \times T^{n-3} \to D^2 \times T^{n-3},$$

where g is Katok's diffeo of the disk D^2 and A a hyperbolic automorphism of the torus T^{n-3}. Consider the suspension flow $\varphi^t_{\mathbf{Z}}$ over R with the roof function $H = 1$ and the suspension manifold $K = D^2 \times T^{n-3} \times [0,1]/\sim$, where $\sim$ is the identification $(x, y, 1) = (g(x), A(y), 0)$. Here $\mathbf{Z}$ is the vector field of the suspension flow and in the coordinate system (x, y, t) we have $\mathbf{Z} = (0, 0, 1)$.

The strategy:

The manifold K has a boundary and due to its particular structure it can be embedded into any manifold of dimension≥ 5 (Brin–Katok). A vector field $\mathbf{X}$ on K can be carried over to M provided it is identity on the boundary along with all its derivatives. Starting with the vector field $\mathbf{Z}$ we will construct a desired vector filed $\mathbf{X}$.

Step [illegible]. Consider a C^∞ function $\alpha : D^2 \to [0, 1]$ s.t.

1. α and all its partial derivatives of any order are equal to zero on ∂D^2;

2. $\alpha(x) > 0$ outside ∂D^2 and $\alpha(x) = 1$ for $x \in D^2 \setminus U$;

3. $\alpha(x)^{-1}\mathbf{V}(x) \to 0$ as $x \to \partial D^2$.

Define the vector field $\mathbf{X}$ on N by

$$\mathbf{X}(G(x,t), y, t) = (\frac{\partial G}{\partial t}(x,t), 0, \alpha(G(x,t))).$$

Note that $\frac{\partial G}{\partial t}(x,t) = \mathbf{V}(G(x,t))$ for $x \in U$. Therefore, for $(x, y, t) \in N$ with $x \in U$,

$$\mathbf{X}(x, y, t) = (\mathbf{V}(x), 0, \alpha(x)).$$

$\varphi^t = \varphi^t_{\mathbf{X}}$ is the flow on N generated by the vector field $\mathbf{X}$ and it has all the desired properties.

$$(x, y, 1) \sim (g(x), Ay, 0)$$

$$X(G(x,1), y, 1) = X(G(g(x),0), Ay, 0)$$

$$X(G(x,1), y, 1) =$$

$$= \left(\frac{\partial G(x,t)}{\partial t}\Big|_{t=1}, 0, \alpha(G(x,1)) \right) =$$

$$= \left(\frac{\partial G(g(x),t)}{\partial t}\Big|_{t=0}, 0, \alpha(g(x)) \right) =$$

$$= \left(\frac{\partial G(g(x),t)}{\partial t}\Big|_{t=0}, 0, \alpha(G(g(x),0)) \right) =$$

$$= X(G(g(x),0), Ay, 0)$$

Step II. h^t is the suspension flow over A with the roof function $H = 1$ and L is the suspension manifold. h^t preserves volume.

Set $N = D^2 \times L$ and write

$$N = D^2 \times (T^{n-3} \times [0,1]/\sim)$$

where $\sim$ is the identification $(y, 1) = (A(y), 0)$, $y \in T^{n-3}$. Consider $F\colon K \to N$

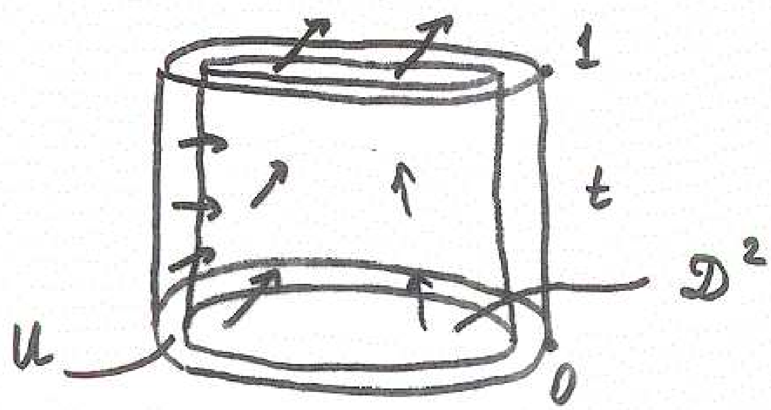
$$F(x, y, t) = (G(x, t),\ y,\ t),$$

where $G : D^2 \times [0,1] \to D^2$ is the diffeotopy constructed above. We have

$$F(x, y, 1) = (g(x), y, 1)$$

$$= (g(x), A(y), 0) = F(g(x), A(y), 0).$$

Therefore, F is well-defined; it preserves volume, is one-to-one and continuous. Hence, it is a homeo. One can show that F is a C^∞ diffeo.



Lemma 1. The vector filed $\mathbf{X}$ is divergence free and φ^t is volume-preserving.

Proof. $F\colon K \to N$, $F(x,y,t) = (G(x,t),\, y,\, t)$. Consider the vector field $\mathbf{Y} = dF\mathbb{Z}$ on N and let $\varphi^t_{\mathbf{Y}}$ be the corresponding flow. In the coordinate system (x,y,t), we have

$$\mathbf{Y}(G(x,t),y,t) = (\frac{\partial G}{\partial t}(x,t), 0, 1),\ (x,y,t) \in K.$$

The vector field $\mathbf{Y}$ is divergence free since it is the image of the divergence free vector field $\mathbb{Z}$ under the volume-preserving map F and the result follows.

Lemma 2. All but one Lyapunov exponents of the flow φ^t are non-zero a.e.

Proof. Consider the map $g^* : D^2 \to D^2$ s.t. $g^* = g$ on $D^2 \setminus U$ and g^* is the time-1 map of the flow $(g^*)^t$ generated by the vector field $\mathbf{V}^*(x) = \alpha^{-1}(x)\mathbf{V}(x)$, $x \in U$. The map g^* is a diffeo and preserves a measure μ^* which is absolutely continuous w.r.t. area with positive density; the latter is unbounded as x approaches ∂D^2.

We proceed as before replacing g by g^*. Define $G^* : D^2 \times [0,1] \to D^2$ by $G^*(x,t) = G(x,t)$ if $x \in D^2 \setminus U$, and $G^*(x,t) = (g^*)^t(x)$ otherwise. Let $\phi^t_{\mathbf{Z}^*}$ be the suspension flow over $g^* \times A$ with the suspension manifold $K^* = D^2 \times T^{n-3} \times [0,1]/\sim$, where $\sim$ is the identification $(x,y,1) = (g^*(x), A(y), 0)$ and $\mathbf{Z}^*$ is the vector field of the suspension flow. Define the map $F^* : K^* \to N$ by

$$F^*(x,y,t) = (G^*(x,t), y, t).$$

Define the vector field $\tilde{\mathbf{Z}}$ on K^* by

$$\tilde{\mathbf{Z}}(x,y,t) = (dF^*)^{-1}\mathbf{X}(F^*(x,y,t)).$$

We have $\phi^t_{\mathbf{X}} = F^* \circ \phi^t_{\tilde{\mathbf{Z}}} \circ (F^*)^{-1}$. It suffices to show that the flow $\phi^t_{\tilde{\mathbf{Z}}}$ has non-zero Lyapunov exponents a.e.

A direct calculation shows that

$$\tilde{\mathbf{Z}}(x,y,t) = \alpha(x,y,t)\mathbf{Z}^*(x,y,t),\ (x,y,t) \in K^*.$$

Hence, the flows $\varphi^t_{\tilde{\mathbf{Z}}}$ and $\varphi^t_{\mathbf{Z}^*}$ have the same orbits and the flow-stable and flow-unstable invariant subspaces $E^{ts}_{\mathbf{Z}^*}(x,y,t)$ and $E^{tu}_{\mathbf{Z}^*}(x,y,t)$ are also invariant under the flow $\phi^t_{\tilde{\mathbf{Z}}}$. Note that the flow $\phi^t_{\mathbf{Z}^*}$ has non-zero Lyapunov exponents a.e. Chose a point $(x_0,y_0,t_0) \in K$ and a vector $v \in E^u_{\mathbf{Z}^*}(x_0,y_0,t_0)$. Note that for a.e. (x_0,y_0,t_0) (with respect to volume) the proportion of time the trajectory $\{\phi^t_{\tilde{\mathbf{Z}}}(x_0,y_0,t_0)\}$ spends in the set $\{(x,y,t)\colon x \notin U\}$ is strictly positive. It follows that the Lyapunov exponent at (x_0,y_0,t_0) with respect to the flow $\phi^t_{\tilde{\mathbf{Z}}}$ is positive.

The map $f = \varphi_X^t$ is partially hyperbolic. Two points $z, z' \in N$ are *accessible* if there are points $z = z_0, z_1, \ldots, z_{\ell-1}, z_\ell = z'$, $z_i \in N$ s.t. $z_i \in W_X^u(z_{i-1})$ or $z_i \in W_X^s(z_{i-1})$ for $i = 1, \ldots, \ell$. The collection of points $[z, z'] = [z_0, z_1, \ldots, z_\ell]$ is called a *path* connecting z and z'. Accessibility is an equivalence relation. The map f has *accessibility property* if the partition into accessibility classes is trivial (i.e. any two points z, z' are accessible) and has *essential accessibility property* if the partition into accessibility classes is ergodic (i.e. a measurable union of equivalence classes has zero or full measure).

Lemma 3. (1) For every t the time-t map of the flow $\varphi_{\mathbf{X}}^t$ has essential accessibility property. Moreover, for any set E of zero measure and almost any two points $z, z' \notin E$ one can find a path $[z, z'] = [z_0, z_1, \ldots, z_\ell]$ s.t. each $z_i \notin E$.

(2) The flow $\varphi_{\mathbf{X}}^t$ is Bernoulli.

By identifying some boundary points, one can show that the manifold N can be mapped onto the n-dimensional disc B^n via a map $\phi : N \to B^n$ s.t. $\phi(N) = B^n$ and $\phi|int(N)$ is a diffeo (Brin). Since $X|\partial N = 0$, we have that $d\phi(X)$ is smooth on B^n. There is also an embedding $\psi : B^n \to M$ (Katok), and the vector field $d\psi d\phi(\mathbf{X})$ generates the flow with the desired properties.

Proof of the theorem: $\dim M = 3$

Consider the suspension flow over g with the roof function $H = 1$. The suspension manifold K is diffeomorphic to $N = D^2 \times S^1$ and the vector field of the suspension flow $\mathbb{Z} = (0, 1)$.

Let $F : K \to N$ be given by $F(x,t) = (G(x,t), t)$. Define the vector field $\mathbf{X}$ on N by

$$\mathbf{X}(G(x,t), t) = \left(\frac{\partial G}{\partial t}(x,t), \alpha(G(x,t))\right),$$

where $\alpha(x)$ is a C^∞ function satisfying (A1) – (A3). The vector field $\mathbf{X}$ is divergence-free and the flow $\phi^t_{\mathbf{X}}$ has all the desired properties.

Proof of the theorem: $\dim M = 4$

We begin with a Bernoulli map with non-zero Lyapunov exponents on a 3-manifold. Let

$$T(x,y) = (g(x), T_{\gamma(x)}y) : D^2 \times S^1 \to D^2 \times S^1,$$

where $T_{\gamma(x)}$ is rotation by $\gamma(x)$ and γ is a non-negative C^∞ function, which is zero in a small neighborhood of the discontinuity set

$$Q = \{q_1, q_2, q_3, \partial D^2\} \times S^1$$

and is positive elsewhere.

One can choose γ s.t. T is *robustly accessible*, i.e., any C^1 perturbation R of T is accessible provided R coincides with T in a small neighborhood of Q. There is a perturbation R of T with nonzero Lyapunov exponents.

Set

$$H = \{(x, y, t) : x \in D^2, y \in S^1, t \in [0, 1]\}/ \sim_1$$

with the identification

$$\sim_1: (x, y, 1) = (T(x, y), 0)$$

and

$$K = \{(x, y, t) : x \in D^2, y \in S^1, t \in [0, 1]\}/ \sim_2$$

with the identification

$$\sim_2: (x, y, 1) = (R(x, y), 0).$$

Let $S = g \times id$ and K' the suspension manifold of the suspension flow over S. The manifold K' is diffeomorphic to $N = D^2 \times S^1 \times S^1$ and there is a diffeo $F : H \to N$.

Let $\mathbf{Z} = (0, 0, 1)$ be the vector field on H of the suspension flow over R; it is divergence free. The vector field $\mathbf{X}$ on N, given by

$$\mathbf{X} = \left(\frac{\partial G}{\partial t}(x, t), 0, \alpha(x) \right),$$

is divergence free and the flow generated by $\mathbf{X}$ has all the desired properties.