A MODIFIED SCHMIDT GAME

AND A CONJECTURE OF MARGULIS

Dmitry Kleinbock, Brandeis University

and

Barak Weiss, Ben Gurion University

1. General set-up.

Let (X, μ, F) be a dynamical system, where

- X is a metric space
- $F = \{g_t\}$ a (semi)group of maps $X \to X$
- μ a finite F-invariant measure

If μ is ergodic, almost all F-orbits are dense.

Specifically, for a subset Z of X, denote by $E(F, Z)$ the set of points of X with F-orbits avoiding Z, i.e.

$$E(F, Z) \stackrel{\text{def}}{=} \{x \in X \mid \overline{Fx} \cap Z = \varnothing\}\,.$$

Also, if X is noncompact, consider

$$E(F, \infty) \stackrel{\text{def}}{=} \{x \in X \mid Fx \text{ is bounded}\}\,,$$

$$E(F, Z \cup \infty) \stackrel{\text{def}}{=} E(F, Z) \cap E(F, \infty)\,.$$

All of these are sets of measure zero.

Examples.

If g_t is an ergodic translation of a torus, then all orbits are dense $\Rightarrow E(F, Z) = \varnothing$ for all Z.

More generally, if $X = G/\Gamma$, where G is a Lie group and $\Gamma \subset G$ a lattice, and $F = \{g_t\} \subset G$ is *unipotent*, it follows from the work of Ratner and Dani-Margulis that the set

$$\bigcup_{z \in X} E(F, \{z\})$$

of all nondense orbits is a countable union of proper submanifolds of X (*singular sets*).

On the other hand, for partially hyperbolic flows the situation is different: the sets $E(F, Z)$ are very big (although still of measure zero). Say that $Y \subset X$ is *thick* if $\forall$ open $U \subset X$, $\dim(U \cap Y) = \dim(U)$.

(dim = Hausdorff dimension)

In what follows we will assume that g_t is partially hyperbolic and let $F = \{g_t \mid t \geq 0\} \subset G$.

Theorem 1.1. [K-Margulis 1996] *Let G be a semisimple Lie group, Γ an irreducible lattice in G. Then the set $E(F, \infty) \subset X = G/\Gamma$ of bounded orbits is thick.*

Theorem 1.2. [K 1997] *Let G be a Lie group, Γ a lattice in G, Z finite (or, more generally, a compact submanifold of small enough dimension transversal to the flow direction). Then $E(F, Z)$ is thick.*

Cf. similar results for expanding maps and Anosov diffeos/flows by Urbanski and Dolgopyat.

Theorems 1.1 and 1.2 were proved in response to conjectures made by Margulis at his Kyoto ICM address (1990):

Conjecture (A). [= Theorem 1] *$E(F, \infty)$ is thick.*

Conjecture (B). [$\supset$ Theorem 2] *Z finite*

$$\Rightarrow E(F, Z \cup \infty) \text{ is thick.}$$

(i.e. there are many bounded orbits avoiding Z)

Unfortunately, the methods of proof of Theorems 1.1 and 1.2 are not helpful when it comes to intersecting the two thick sets $E(F, \infty)$ and $E(F, Z)$.

An important remark: in the proof of both theorems, as well as in our new results, all the work is done in the group

$$H = \{h \in G \mid g_{-t} h g_t \to e \text{ as } t \to +\infty\},$$

called *expanding horospherical* with respect to F

(H-orbits = unstable leaves)

That is, it is proved that for any $x \in X$, the sets

$$\left\{h \in H \;\middle|\; hx \in E(F,\infty)\right\} \quad \text{or} \quad \ldots E(F,Z)\right\}$$

are thick in H.

Sketch of proof of Theorem 1.2. Fix $z \in X$ and $T > 0$; then $hx \in E(F, \{z\}) \iff$ the trajectory

$$\{(g_T)^n hx \mid n = 1, 2, \dots\}$$

stays away from the curve $Z_T \stackrel{\text{def}}{=} \{g_t z \mid -T \leq t \leq 0\}$.

Now one can choose a <u>small enough</u> ball $V \subset H$ such that for any $y \in X$, the intersection of $g_T V g_{-T} y$ with Z_T consists of at most one point.

Then inside V one can construct a Cantor subset of large dimension whose points will avoid a small neighborhood of Z_T. $\square$

Sketch of proof of Theorem 1.1. Instead of a neighborhood of a point, we need to avoid the complement of a big compact $K \subset X$.

Now if $T > 0$ is fixed, $y \in K$, and a ball $V \subset H$ is not too small,

then mixing of the g_t-action on X will force most of $g_T V y$ to come back to K, and a similar Cantor set construction can be carried out. $\square$

Remark: the building blocks of Cantor sets in the above construction are preimages of balls (cubes) under contractions induced by the conjugation $h \mapsto g_{-t}hg_t$, and, unless the above contractions are conformal, will resemble narrow degenerate rectangles, such as e.g. in the self-affine set below:

2. The Schmidt game. [Wolfgang Schmidt, 1966]

B: picks an arbitrary closed ball B_0 in E,

W: $W_1 \subset B_0$ of radius α times the radius of B_0,

B: $B_1 \subset W_1$ of radius $\beta\times$ the radius of W_0, etc.

W *wins* if the point of intersection

$$\cap_{i=1}^{\infty} W_i = \cap_{i=0}^{\infty} B_i$$

lies in S. S is called (α, β)-*winning* if **W** can win no matter how **B** plays. S is called α-*winning* if it is (α, β)-winning for all $\beta > 0$, and *winning* if it is α-winning for some $\alpha > 0$.

The following was proved by Schmidt:

Theorem 2.1. *$\{S_k\}_{k=1}^{\infty}$ is a sequence of α-winning sets for some α $\Rightarrow$ $\cap_{k=1}^{\infty} S_k$ is also an α-winning set.*

Theorem 2.2. $E = \mathbb{R}^n$, $\alpha > 0 \Rightarrow$ *any α-winning subset $S \subset E$ is thick.*

Theorem 2.3. *The set $\mathcal{BA}$ of <u>badly approximable</u> matrices (systems of linear forms) $A \in M_{m\times n}(\mathbb{R})$ is α-winning for any $0 < \alpha < 1/2$.*

Corollary 2.4. *For any $\{A_k\}_{k=1}^{\infty} \subset M_{m\times n}(\mathbb{R})$, the set*

$$\cap_{k=1}^{\infty}(A_k + \mathcal{BA})$$

is thick.

(this generalized earlier work by Cassels–Davenport)

Then in 1985 S.G. Dani showed

Proposition 2.5. *$A \in M_{m\times n}(\mathbb{R})$ is badly approximable $\iff$ $\{g_t L_A x_0\}$ is bounded, where*

- $G = \mathrm{SL}_{m+n}(\mathbb{R})$
- $\Gamma = \mathrm{SL}_{m+n}(\mathbb{Z})$
- $x_0 = \Gamma \in X = G/\Gamma$
- $L_A = \begin{pmatrix} I_m & A \\ 0 & I_n \end{pmatrix}$
- $g_t = \mathrm{diag}(e^{t/m}, \ldots, e^{t/m}, e^{-t/n}, \ldots, e^{-t/n})$.

By a lucky chance, $H = \{L_A \mid A \in M_{m\times n}(\mathbb{R})\}$ is exactly the expanding horospherical subgroup with respect to F; hence in this case the set $\big\{h \in H \bigm| hx_0 \in E(F,\infty)\big\}$ happens to be winning, which gives rise to a possibility of intersecting it with other winning sets!

Moreover, in another paper written in 1985, Dani showed that whenever G has $\mathbb{R}$-rank 1,

$$(*)\qquad \begin{gathered}\text{for any } x \in G/\Gamma, \text{ the set} \\ \bigl\{h \in H \bigm| hx \in E(F,\infty)\bigr\} \text{ is winning}\end{gathered}$$

This suggests the following nice way to attack Conjecture (B):

An Unfounded Claim. $(*)$ *always holds, as well as its counterpart with* ∞ *replaced by* $\{z\}$.

If true, it would:

- imply a stronger form of Conjecture (B) (with countable Z instead of finite, and winning, not just thick, exceptional sets)
- allow considering actions by different subgroups F simultaneously.

What we do not know:

- whether the sets $\{h \in H \mid F(hx) \text{ is bounded}\}$ are always winning (most likely not)

What we know:

- to make the above sets winning, the rules of the Schmidt game have to be modified (adjusted according to F).

[difficult, by using a precise description of compact subsets of G/Γ instead of mixing]

- the sets $\{h \in H \mid F(hx) \text{ stays away from } z\}$ are winning according to both original and modified rules

[not so difficult, by a modification of proof of Theorem 1.2]

3. A modified Schmidt game.

$\{D_t \mid t > 0\}$: nested closed subsets, $\mathrm{diam}(D_t) \to 0$ as $t \to \infty$

B: picks $B_0 =$ a right-translate of D_{t_0},

W: $W_1 \subset B_0$ a right-translate of $D_{t_0+\boldsymbol{a}}$,

B: $B_1 \subset W_1$ a right-translate of $D_{t_0+\boldsymbol{a}+\boldsymbol{b}}$, etc.

Again, **W** *wins* if the point of intersection

$$\cap_{i=1}^{\infty} W_i = \cap_{i=0}^{\infty} B_i$$

lies in S.

As before, define $(\boldsymbol{a}, \boldsymbol{b})$-*winning*/$\boldsymbol{a}$-*winning*/*winning* sets of a $\{D_t\}$-Modified Schmidt Game.

Theorem 3.1. $\{S_k\}_{k=1}^{\infty}$ *is a sequence of* $\boldsymbol{a}$-*winning sets of a* $\{D_t\}$-*MSG* $\Rightarrow$ *so is* $\cap_{k=1}^{\infty} S_k$.

Proof. Exactly the same as Schmidt's:

Theorem 3.2. *Assume that:*

(1) $m(D_t) = e^{-\delta t} m(D_0)$, *some* $\delta > 0$

($m =$ Haar measure on H)

(2) $\operatorname{diam}(D_t) \le \mathrm{const} \cdot e^{-\sigma t}$, *some* $\sigma > 0$

(3) $\limsup_{t\to\infty} \frac{\log N(t)}{t} = \delta$, *where*

$$N(t) = \max \left\{ N \;\middle|\; \begin{array}{l} \forall\, s > 0 \quad \exists\; x_1, \ldots x_N \in H \\ \textit{s. t. the sets } D_{t+s} x_i \textit{ are} \\ \textit{pairwise disjoint and } \subset D_s \end{array} \right\}.$$

Then any winning set of a $\{D_t\}$*-MSG is thick.*

Proof. Not quite the same as Schmidt's – uses Mass Distribution Principle to create a measure which bounds the Hausdorff dimension from below.

Back to the original set-up:

G a Lie group, $\Gamma \subset G$ a lattice, $X = G/\Gamma$;

$F = \{g_t \mid t > 0\} \subset G$ partially hyperbolic;

$H =$ expanding horospherical w.r.t. F;

D_0 a neighborhood of $e \in H$, $D_t = g_{-t} D_0 g_t$.

Theorem 3.3. $\big\{h \in H \;\big|\; F(hx) \in E(F, \{z\})\big\}$ *is a winning set of a* $\{D_t\}$*-MSG.*

Theorem 3.4. *G semisimple, Γ irreducible*

$\Rightarrow \big\{h \in H \;\big|\; F(hx) \in E(F, \infty)\big\}$ *is too.*

Corollary 3.5. *Conjecture (B) of Margulis (with countable Z).*

Sketch of proof of Theorem 3.3.

Pick a nice $D_0 \subset H$ and large enough $\boldsymbol{a}$

given $\boldsymbol{b}$, choose $T = \boldsymbol{a} + \boldsymbol{b}$,
and consider $Z_T = \{g_t z \mid 0 \leq t \leq T\}$.

choose $s > T$ such that $\forall\, y \in X$, $D_s\, y \cap Z_T$ is at most one point.

WLOG can assume that $t_0 \geq s + T$ (otherwise make random moves until this is the case)

<u>The winning strategy</u>: for $k = 0, 1, \ldots$ choose W_{k+1} such that $g_{t_0 - s + kT}$ does not bring it too close to Z_T. $\square$

Sketch of proof of Theorem 3.4.

The case $\mathrm{rank}_{\mathbb{R}}(G) = 1$ was settled by Dani; by Margulis' Arithmeticity, can assume that G is semisimple real algebraic/$\mathbb{Q}$ and $\Gamma = G(\mathbb{Z})$.

The proof uses a compactness criterion (description of bounded subsets of G/Γ) from a paper by Tomanov–Weiss:

Proposition 3.6. *$K \subset X$ is precompact* $\iff$ $\exists$ *a neighborhood U of* 0 *in* $\mathfrak{g}$ =Lie(G) such that $\forall\, x \in K$, no subset of $U \cap \mathrm{Ad}(x)(\mathfrak{g}_{\mathbb{Z}})$ spans a unipotent radical of a maximal $\mathbb{Q}$-parabolic subgroup.

Consequently, informally speaking, an excursion of $g_t hx$ outside of a compact subset of $G/\Gamma \longleftrightarrow$ a small value of some polynomial (in $h \in H$) from a fixed <u>finite</u> family $\mathcal{P}$ (dependent on x and t).

The game: choose a nice D_0, and $\boldsymbol{a}$ such that whenever a polynomial $f \in \mathcal{P}$ has a coefficient $\geq \varepsilon_1$, its values are $\geq \varepsilon_2$ on some translate of $D_{\boldsymbol{a}} \subset D_0$.

B chooses $\boldsymbol{b}$, $T = \boldsymbol{a} + \boldsymbol{b}$,

Then choose K
(determined by $\varepsilon_1, \varepsilon_2, t_0, T$)
such that as long as $x \in K$,
every polynomial from that
finite family has a coefficient $\geq \varepsilon_1$.

The winning strategy:
choose W_k such that g_{t_0+kT}
sends it to a translate of $D_{\boldsymbol{a}} \subset D_0$
as above. $\square$