The Ending Lamination Conjecture: the Proof of the Model Theorem

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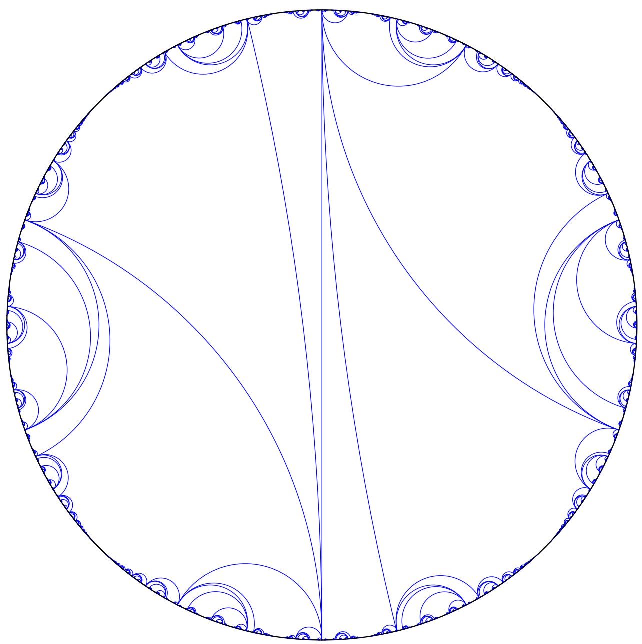


Joint work with

R. Canary – Michigan
Y. Minsky – Yale

Main Goal

To describe the model manifold and some elements of the proof of the model theorem.

Model Theorem. (B-C-M) *If a complete hyperbolic 3-manifold* $M \cong S \times \mathbb{R}$ *has end-invariant* ν*, there is a model* M_ν *depending only on* ν *and a piecewise* L_S*-bi-Lipschitz diffeomorphism* $f : M_\nu \to M$*.*

Models and Rigidity

- How does a bi-Lipschitz model help with ELC?
- Any two manifolds M and M' with

$$\nu(M) = \nu = \nu(M')$$

  are bi-Lipschitz to the model M_ν.

- Hence, there is a bi-Lipschitz diffeomorphism $\phi : M \to M'$.
- By Sullivan's rigidity theorem, ϕ is homotopic to an isometry.

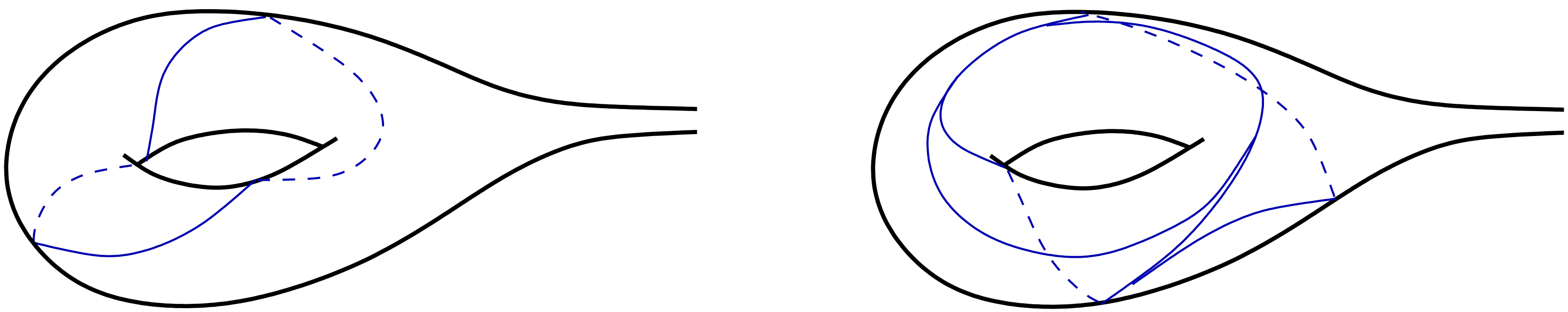
The General Case

- In the general case we pass to a cover corresponding to each end and obtain a model for this end.
- The bi-Lipschitz model maps extend across the remaining compact piece.
- This proves the ending lamination conjecture.

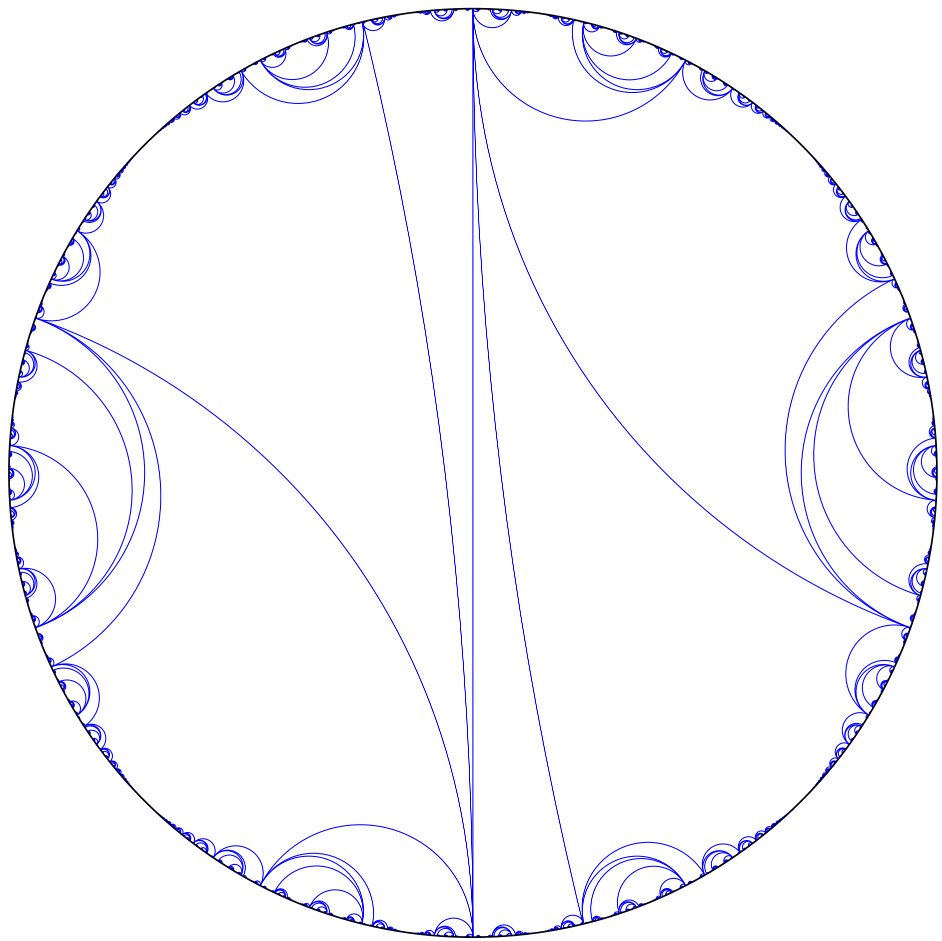
□

The Punctured Torus (Minsky)

- When T is a punctured torus, $\pi_1(T) = \langle a, b \rangle$, require $\rho([a,b])$ to be parabolic.
- The end-invariants lie in $\mathbb{H}^2 \cup \mathbb{R} \cup \infty$.
- $\nu \in \mathbb{Q}$ represents a cusp, $\nu \in \mathbb{R} \setminus \mathbb{Q}$ represent laminations of "slope" ν.



The Punctured Torus



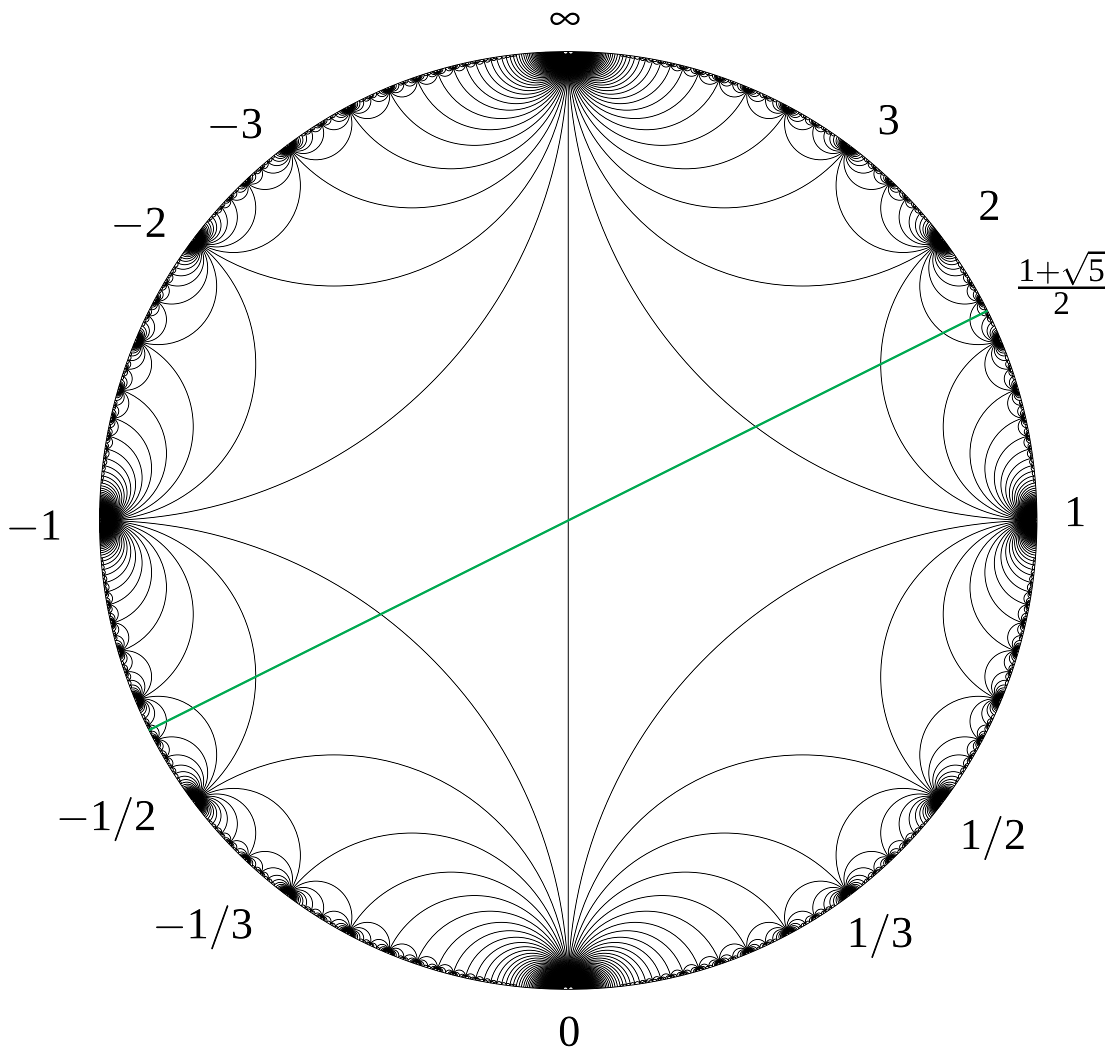
The lift of a lamination on the punctured torus to $\mathbb{H}^2$.

Laminations

Our end-invariants will be simple closed curves or laminations.

Definition. *A* geodesic lamination *on a hyperbolic surface X is a closed subset foliated by geodesics.*

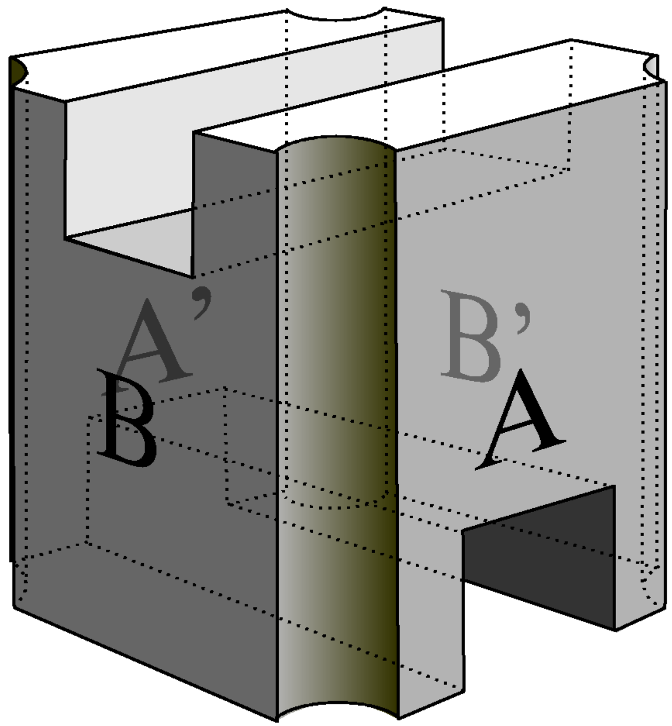
The Farey Graph



The Farey graph encodes the model.

Building Blocks

Blocks like these are stacked end to end.



After pairing sides, the remaining tori are Dehn-filled using continued fractions for ν.

The General Case

- What should the right model be in general?
- Thin parts can have a non-uniform shape, since there are geometric limits of quasi-Fuchsian manifolds with *new* infinite volume ends (B-'97).
- This phenomenon leads to polynomial volume growth of $M_{\geq \varepsilon}$.

Examples

Mapping Tori:

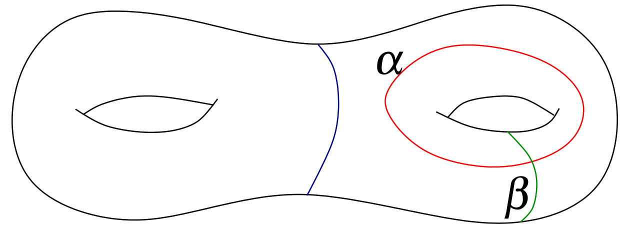
- When $\psi\colon S \to S$ is pseudo-Anosov,

$$T_\psi = S \times [0,1]/(x,0) \sim (\psi(x),1)$$

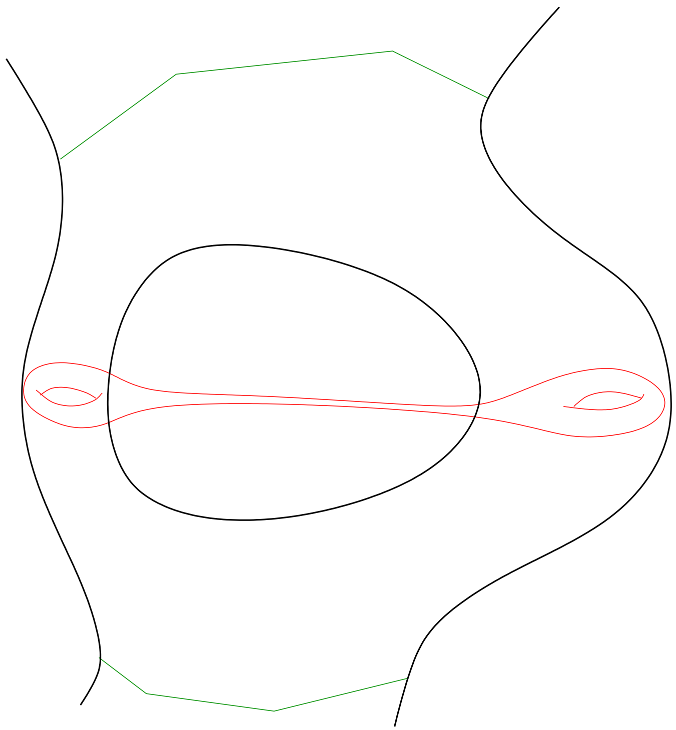
  is hyperbolic (Thurston).

- Cover M_ψ corresponding to S, is *periodic*.
- Model is quasi-isometric to $\mathbb{R}$.

Iteration on a Bers slice



$$\varphi = \tau_\alpha \circ \tau_\beta$$



$Q(\varphi^n(X), X)$

- But a non pseudo-Anosov φ can produce non-uniform geometry.
- Surfaces X and $\varphi^n(X)$ are close on one subsurface and far on another.
- Perverse geometric limits.

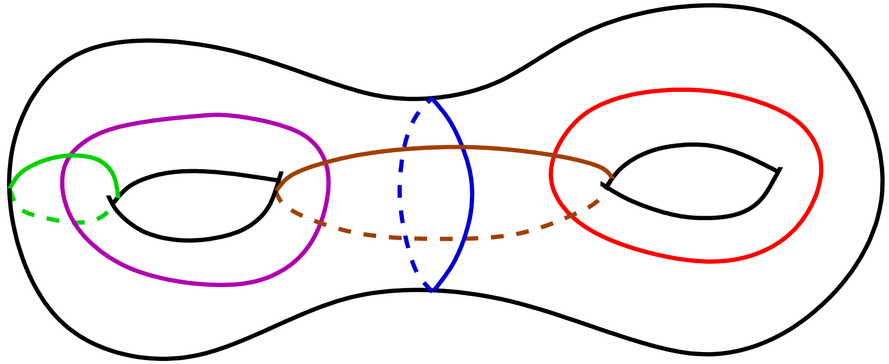
The General Case

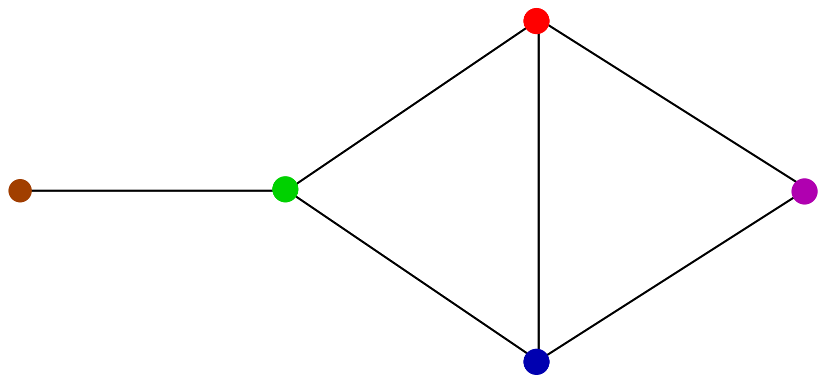
- To account for non-uniformity, one employs a kind of generalized continued fractions.
- Each essential subsurface of S inherits a projection coefficient via the geometry of the complex of curves.
- Work of Masur and Minsky organizes these data into a *hierarchy* H_ν of geodesics in *curve complexes* $\mathscr{C}(T)$, $T \subseteq S$.
- H_ν is the combinatorial blueprint for the model.

The Complex of Curves

- The *complex of curves* $\mathscr{C}(S)$ organizes the essential simple closed curves on S.
- *0-skeleton:* isotopy classes of essential simple closed curves on S.
- *k-simplices:* spanned by families of $k+1$ curves $\alpha_1, \ldots, \alpha_{k+1}$ with $i(\alpha_i, \alpha_j) = 0$.

The Complex of Curves

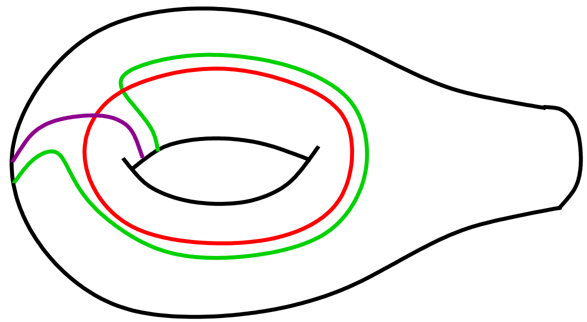




Simplices in the complex of curves.

Exceptional case

- When S is a one-holed torus or four-holed sphere, edges connect vertices whose curves intersect minimally.



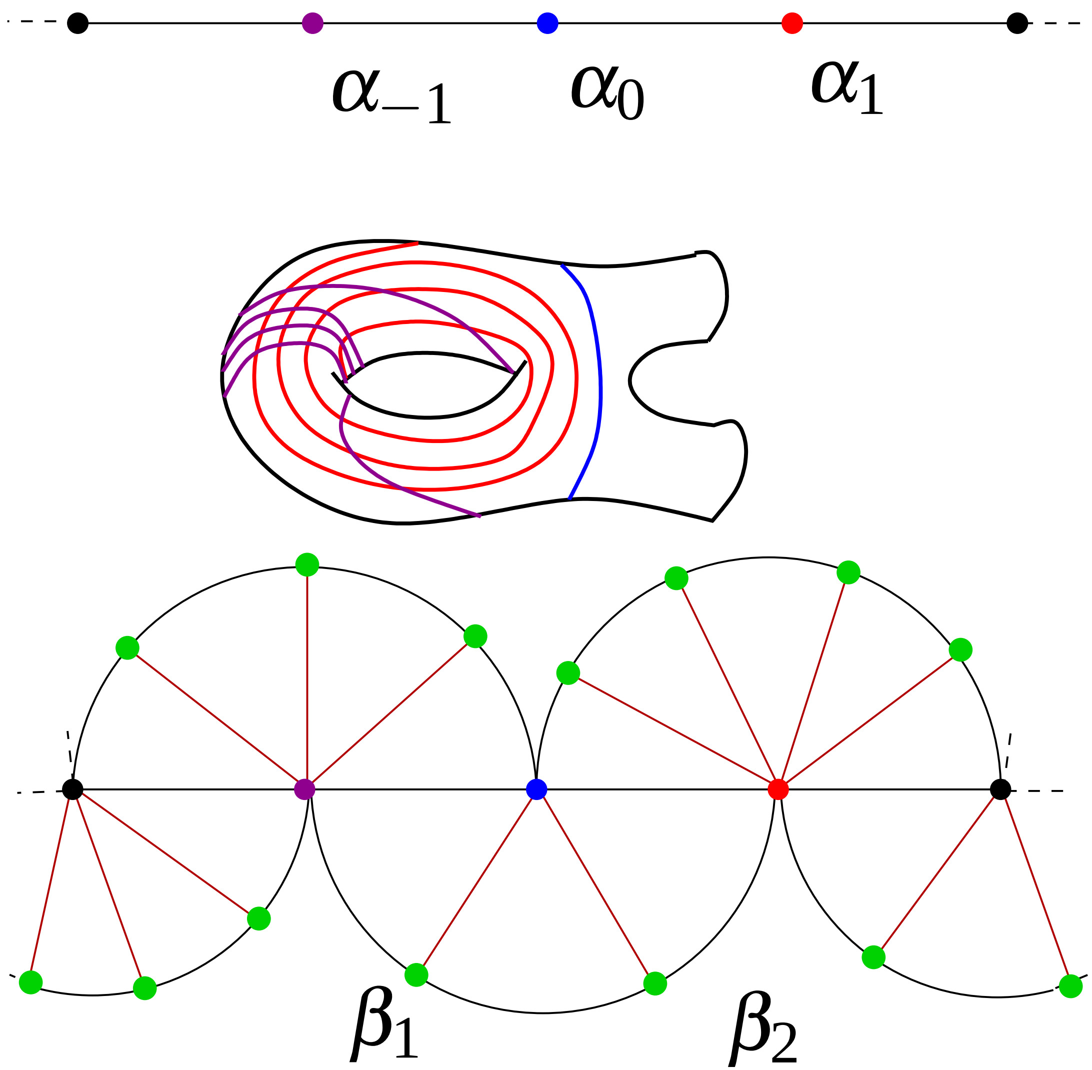
Theorem. (Masur-Minsky) *In each case, $\mathcal{C}(S)$ is negatively curved in the sense of Gromov.*

Theorem. (Klarreich) $\partial\mathcal{C}(S) = \mathcal{E}\mathcal{L}(S)$.

Hierarchies

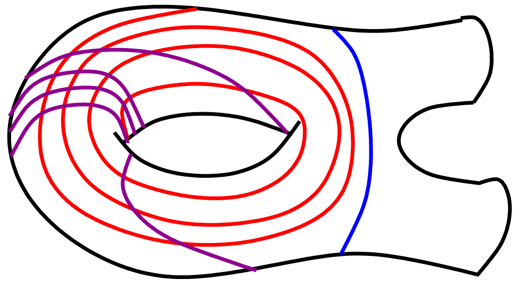
- Masur-Minsky exhibit a geodesic g_ν in $\mathscr{C}(S)$ associated to $\nu = (\nu^-, \nu^+)$ joining ν^- to ν^+.
- A *hierarchy* H_ν of geodesics in $\mathscr{C}(T)$, $T \subset S$, gives a kind of thickening of g_ν in $\mathscr{C}(S)$.
- The model manifold M_ν is built from blocks, one for each edge of a geodesic in H_ν whose underlying surface is a one-holed torus or four-holed sphere.

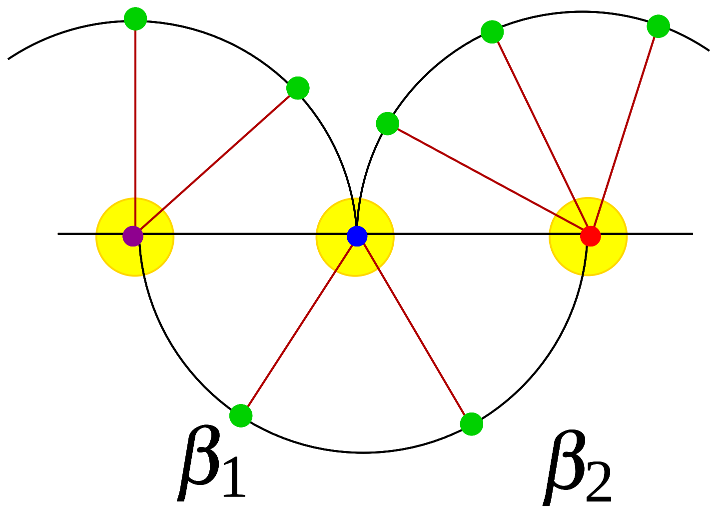
Hierarchies



A hierarchy in the two-holed torus case

Hierarchies and Pants





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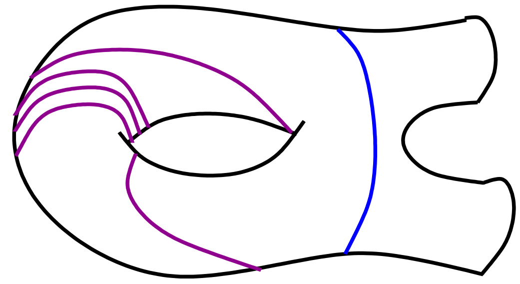

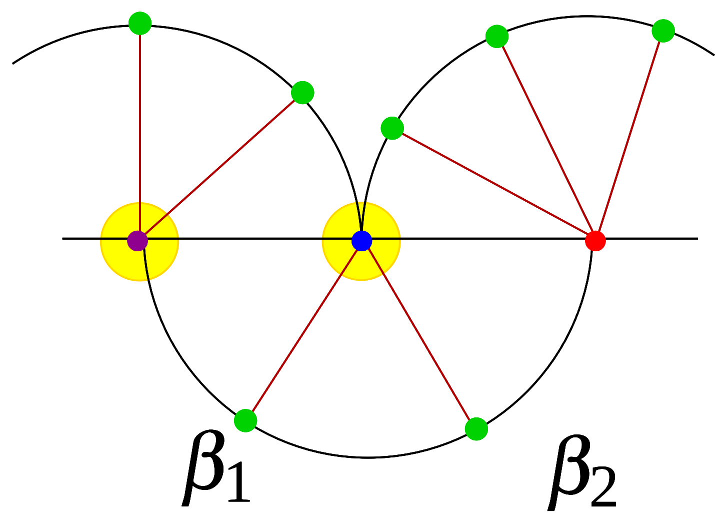

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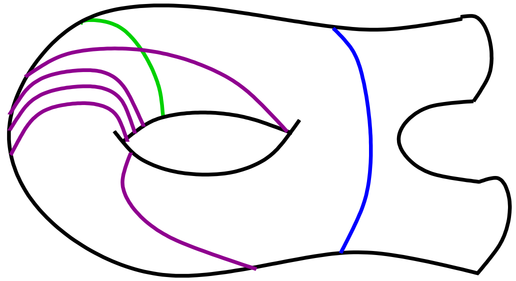

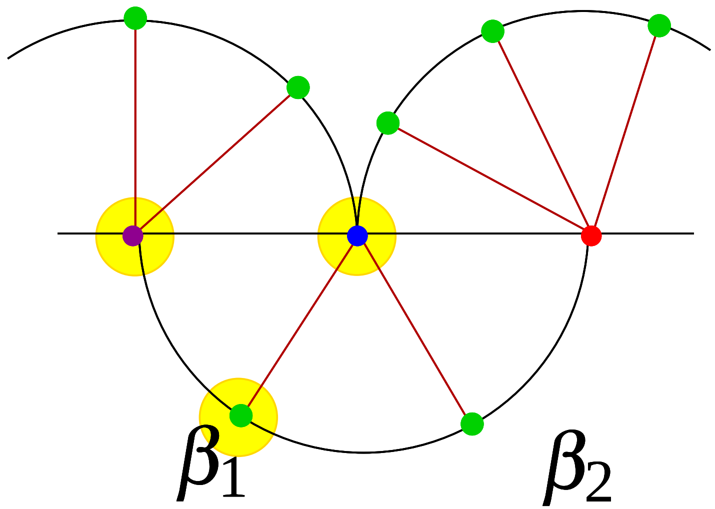

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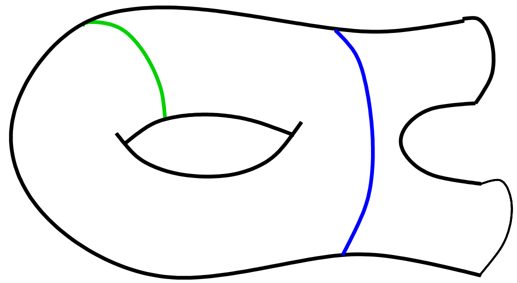

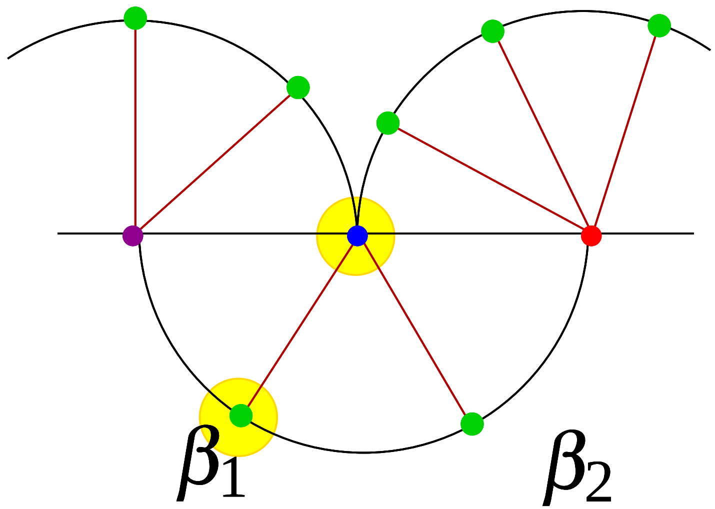

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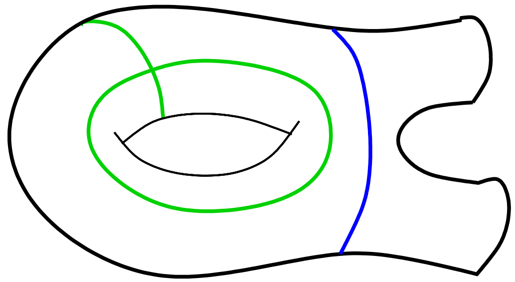

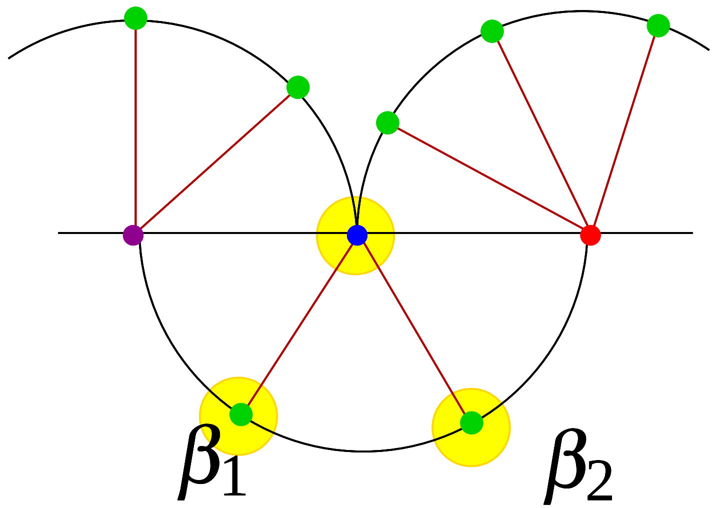

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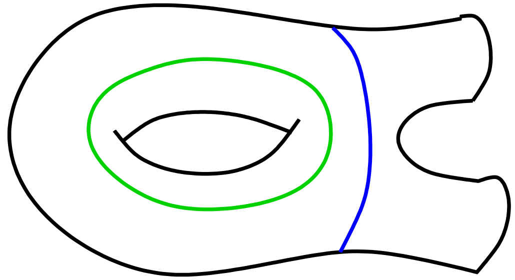

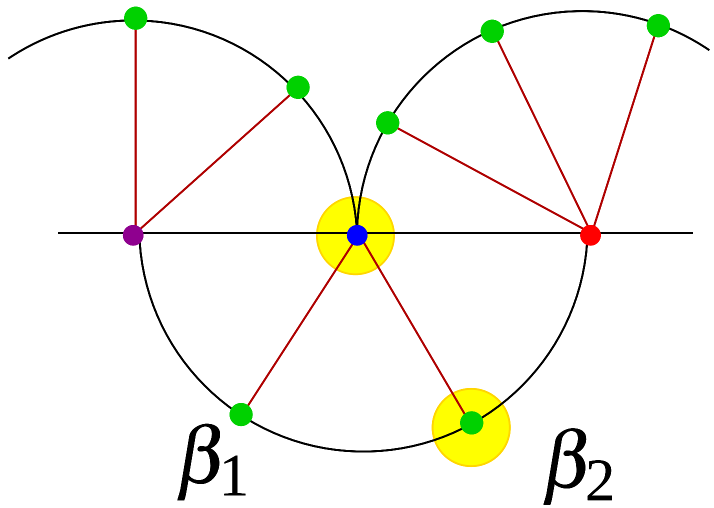

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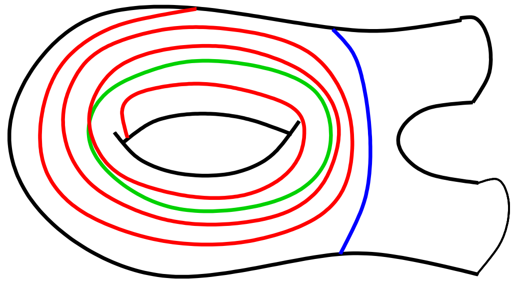

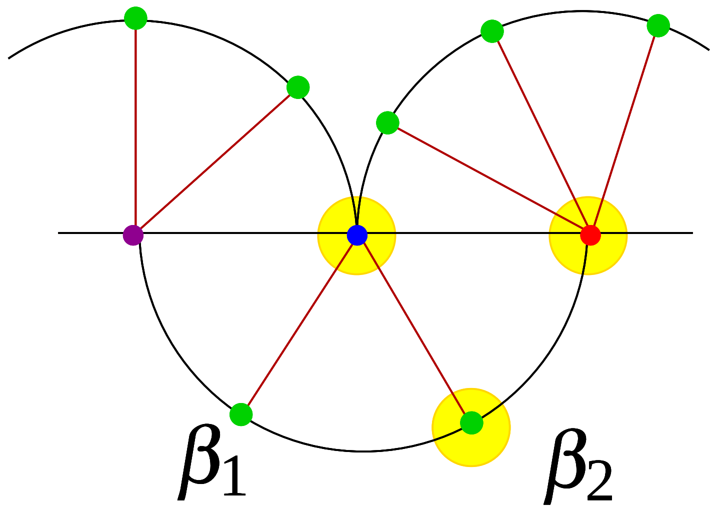

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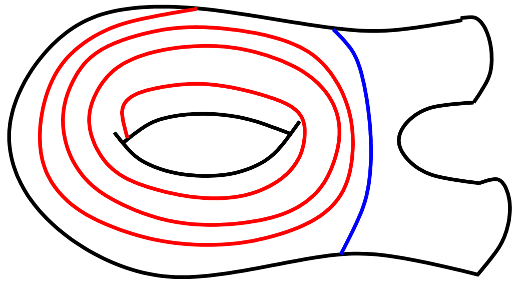

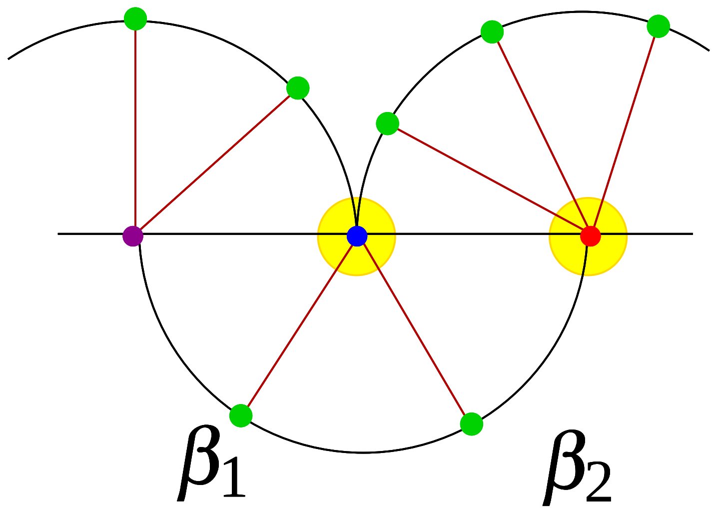

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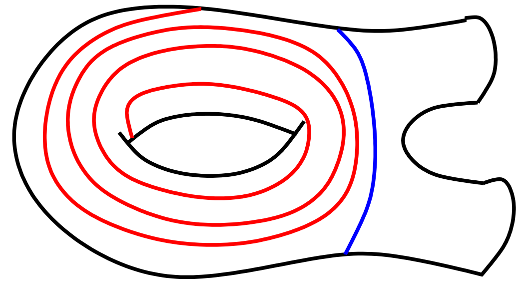
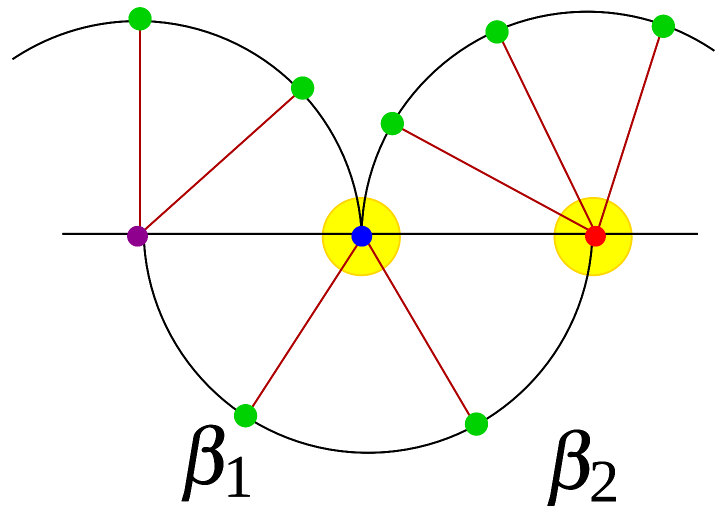
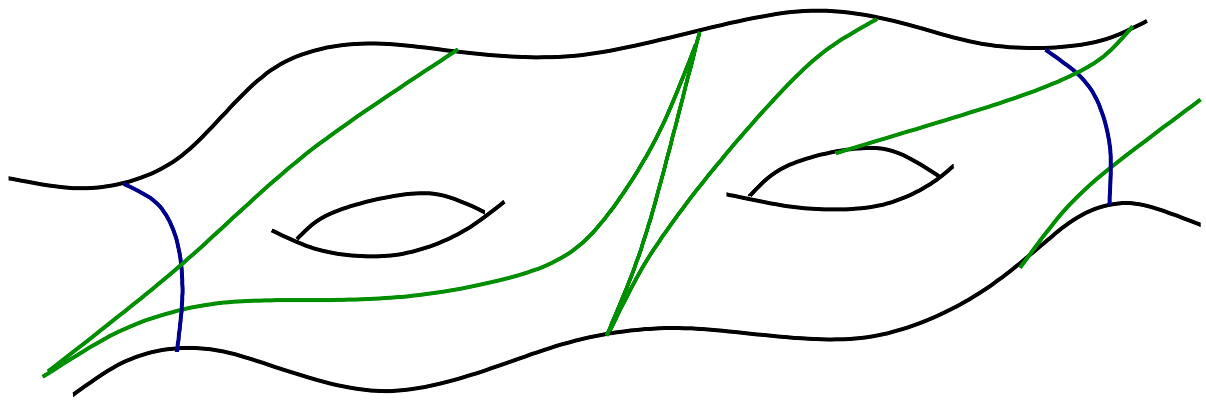




The *spokes* of the wheels are pants decompositions.

Subsurface Projections

Roughly speaking, the laminations determine curves in H via a surgery procedure: *subsurface projection*.



Subsurface Projections

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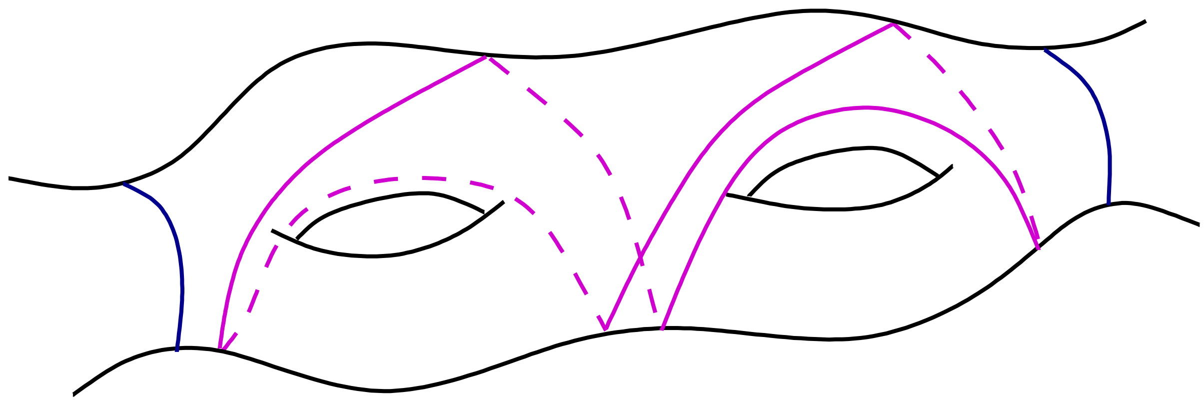
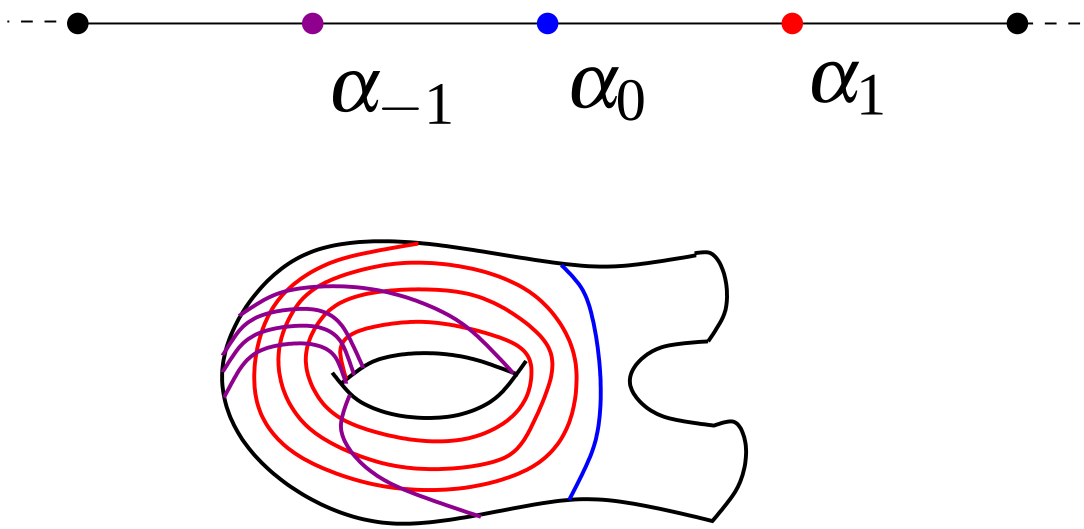


The result is the projection $\pi_Y(\nu)$ where $Y \subset S$ and ν is the ending lamination.

Structure of H_ν



- Here, if Y is the one-holed torus bounded by α_0,

$$\alpha_{-1} \sim \pi_Y(\nu^-) \quad \text{and} \quad \alpha_1 \sim \pi_Y(\nu^+)$$

where $\nu = (\nu^-, \nu^+)$ is the end-invariant.

Projection Distances

- Such *projections* control the structure of H_ν and the geometry of M.
- In particular each *vertex* $\gamma \in H_\nu$ has an associated coefficient:

$$\omega(\gamma) = [d_\gamma(\nu^-, \nu^+)]_\kappa + i \sum_{\substack{Y \subset S \\ \gamma \subset \partial Y}} [d_Y(\nu^-, \nu^+)]_\kappa$$

Such coefficients predict short lengths:

$$|\omega(\gamma)| \ \text{large} \implies \ell_M(\gamma) \ \text{small}.$$

Building the Model

- The combinatorics of the sequence of pants decompositions produced by H_ν give instructions to build a model manifold M_ν.
- One block is added for each *elementary move*.
- *a priori* bounds: each $\gamma \in H_\nu$ satisfies

$$\ell_M(\gamma) < L.$$

Theorem. (Minsky) *There is a Lipschitz model map*

$$f : M_\nu \to M.$$

Combinatorial Thick Part

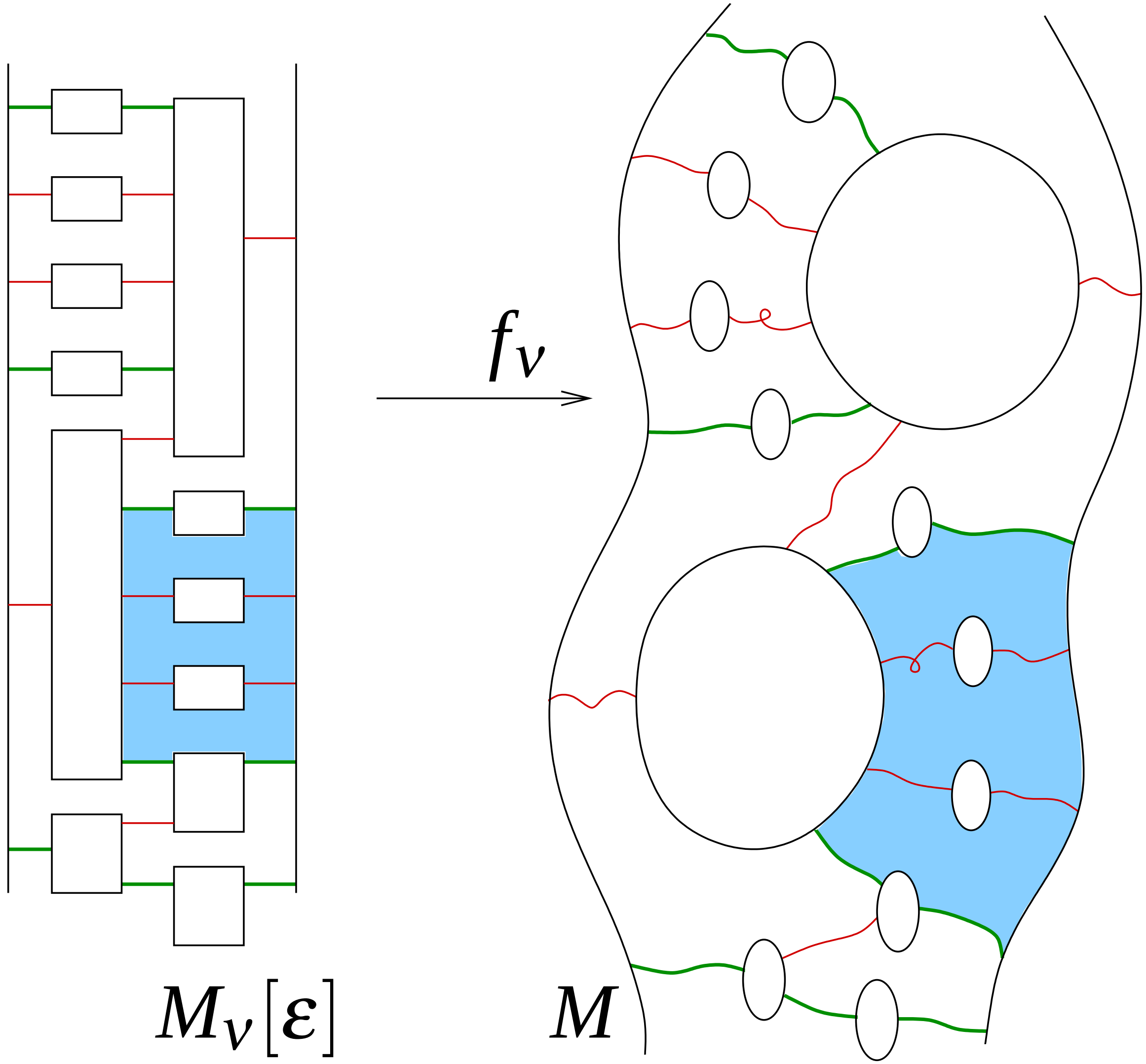
- Then one can form the *combinatorial* ε*-thick part* $M_\nu[\varepsilon]$ by filling in tubes for $\gamma \in H_\nu$ with

$$|\omega(\gamma)| < 1/\varepsilon.$$

- Minsky proves the model map can be made *tube preserving* on $M_\nu[\varepsilon]$.
- In fact one has more:

$$|\omega(\gamma)| \text{ large} \iff \ell_M(\gamma) \text{ small.}$$

Building the Model



Geometric limits control bounded regions.

Conclusion

- The combinatorial ε-thick part $M_\nu[\varepsilon]$ decomposes into regions $\{R_i\}$ consisting of a finite number of blocks – f_ν is homotopic to a map restricting to an embedding on ∂R_i.
- Using a geometric limit argument, we obtain bi-Lipschitz control on each region.
- The maps glue together to give a bi-Lipschitz map $F_\nu : M_\nu[\varepsilon] \longrightarrow M$ that is a bi-Lipschitz embedding on each region.

Conclusion

- By a degree argument, F_ν is a homeomorphism, and f_ν is homotopic to F_ν on ∂R. Thus f_ν is homotopic to F_ν.
- Finally, we extend F_ν over models for ε-Margulis tubes to obtain the bi-Lipschitz homeomorphism

$$\overline{F_\nu} \colon M_\nu \to M.$$

- This proves the Model Theorem. □

An Alternate Approach

(with Bromberg, Evans, Souto)

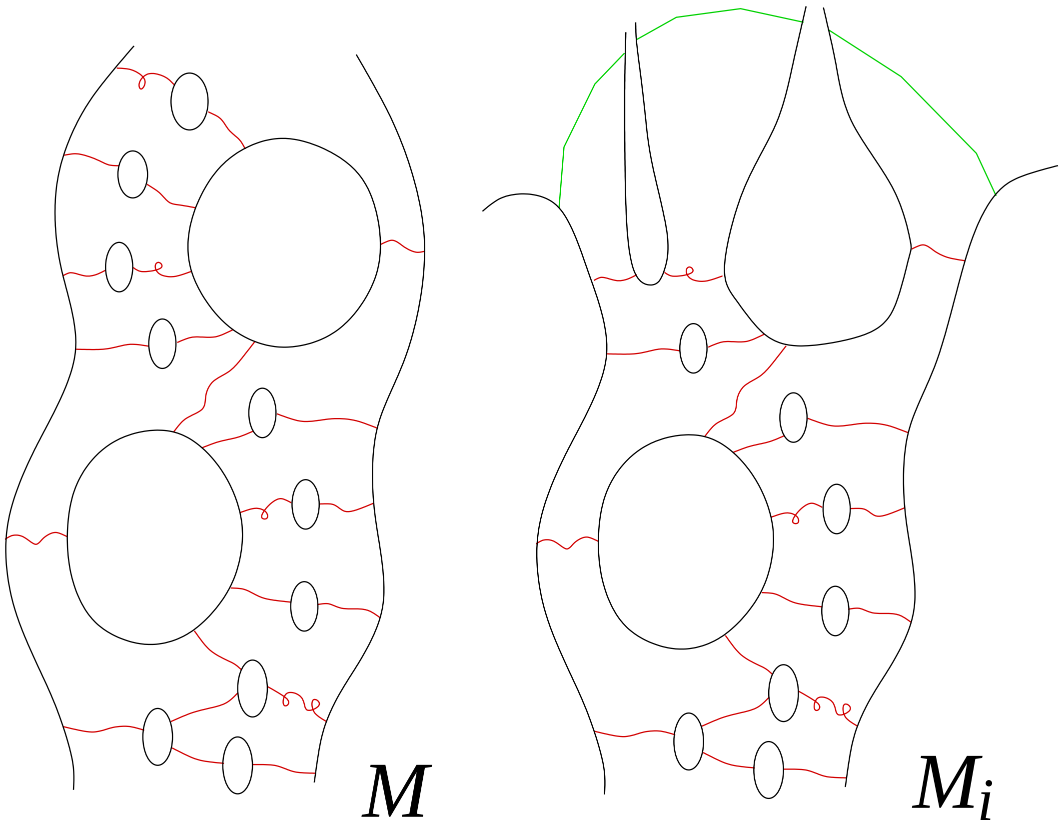
- Applying cone-manifold techniques of [B-Bromberg], each M can be realized as a limit of geometrically finite M_i in $\mathscr{V}(S)$.
- Given M, M' with $\nu(M) = \nu(M')$, suffices to find two sequences $M_i \to M$ and $M_i' \to M'$ with bi-Lipschitz diffeomorphisms

$$\phi_i : M_i \to M_i'$$

and extract a limit.

Drilling

- When the combinatorics determine arbitrarily short geodesics in H_v, we obtain unifomly bi-Lipschitz geometrically finite approximates.



Drilling

- A covering argument due to Bromberg-Souto makes these methods applicable to bounded geometry as well.
- Apparently, one sacrifices the model for this approach.
- Limit is realized as a Dehn-filling of a gluing, or combination of cusped quasi-Fuchsian bounded geometry manifolds determined by H_ν, so...
- model may be recoverable.