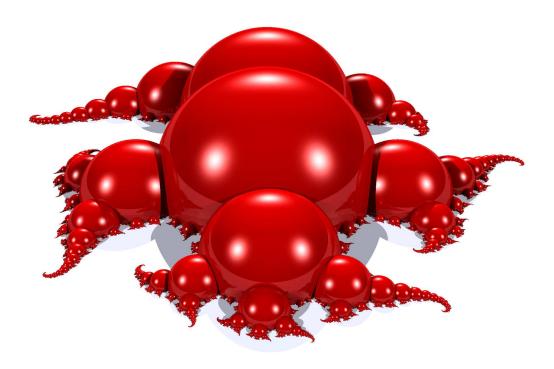
Introduction to hyperbolic 3-manifolds and their classification

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A minicourse on recent developments in hyperbolic 3-manifolds

J. Brock - Brown

K. Bromberg - Utah

R. Canary - Michigan

J. Souto - U. Chicago and Michigan

Broad Goal

I. To give an overview of the classification of all hyperbolic 3-manifolds with finitely generated fundamental group,

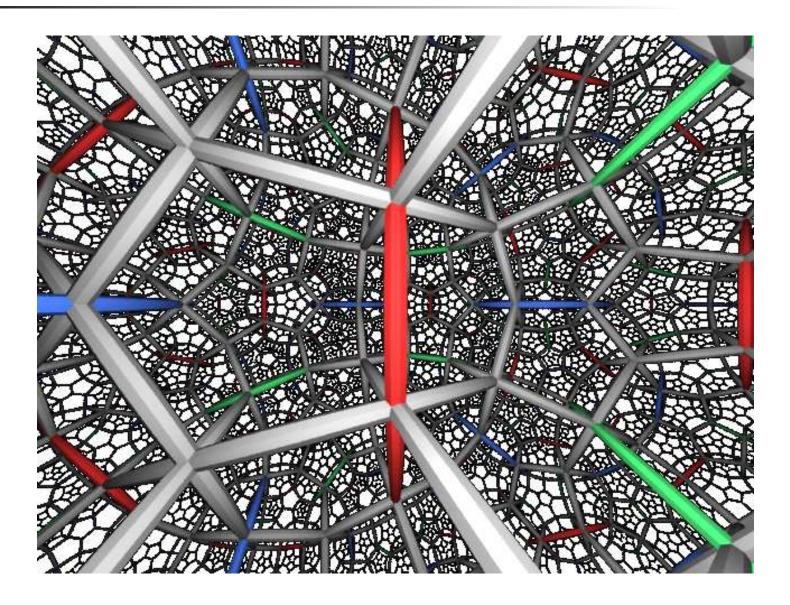
II. outline other recent developments in our understanding of dynamical features of Kleinian groups, and

III. describe new directions for future research.

Hyperbolic 3-manifolds

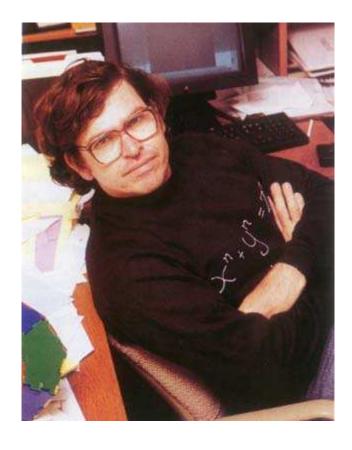
- A Riemannian 3-manifold *M* is *hyperbolic* if it admits a complete metric of constant sectional curvature −1.
- The *hyperbolic 3-space*, \mathbb{H}^3 is the unique simply connected hyperbolic 3-manifold.

HYPERBOLIC SPACE



Geometry Center graphic

Geometrization



"...yeah - I thought it might go something like that..."

- W. Thurston conjectured all
 3-manifolds admit a
 geometric decomposition.
- Most are hyperbolic.
- Our motivation: to classify hyperbolic manifolds using simple invariants.

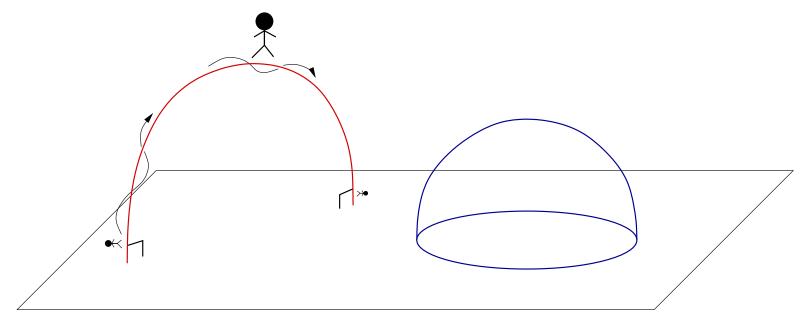
Topology

Geometry

Conformal Dynamical Systems

Hyperbolic Isometries

 $\phi \in \text{Isom}^+(\mathbb{H}^3) = \text{PSL}_2(\mathbb{C})$ is either



- loxodromic \sim glide rotation.
- parabolic \sim single fixed point at ∞ .
- \blacksquare elliptic \sim rotation about an axis.

The Deformation Space

When N^3 is compact and int(N) admits some complete hyperbolic metric,

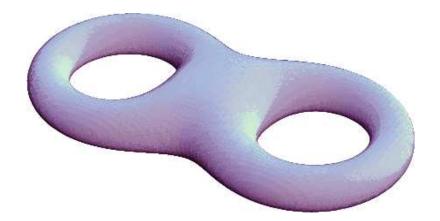
consider the space

$$\mathscr{V}(N) = \frac{\{\rho \colon \pi_1(N) \to \text{Isom}^+ \mathbb{H}^3 \mid \rho \text{ disc., } 1\text{-}1\}}{\text{conjugacy}}$$

- Then $M = \mathbb{H}^3/\rho(\pi_1(N))$ is homotopy equivalent to N for $\rho \in \mathscr{V}(N)$.
- If *N* closed or $\partial N = \sqcup T^2$, $\mathscr{V}(N) = \{*\}$ (Mostow, Prasad), so these are classified.

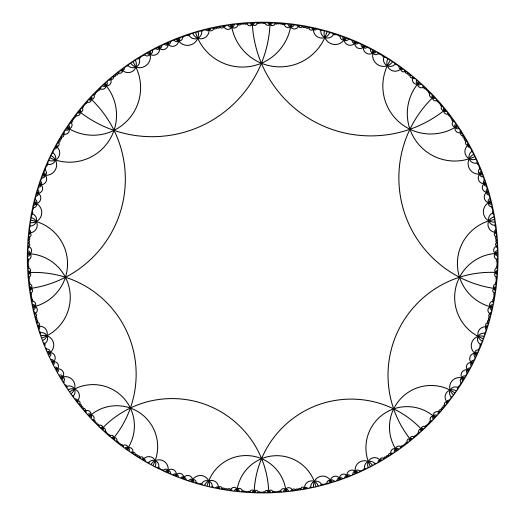
The Infinite Volume Case

- When *M* has infinite volume, there is a rich deformation theory.
- Points in the interior of $\mathscr{V}(N)$ are parameterized by Teichmüller spaces of ∂N .
- A simple example:

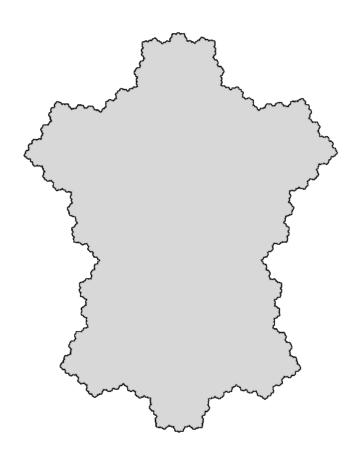


Fuchsian Manifolds

A Fuchsian group $\Gamma_X \subset \operatorname{Isom}^+\mathbb{H}^2$ determines a Fuchsian 3-manifold $M_X = \mathbb{H}^3/\Gamma_X$.

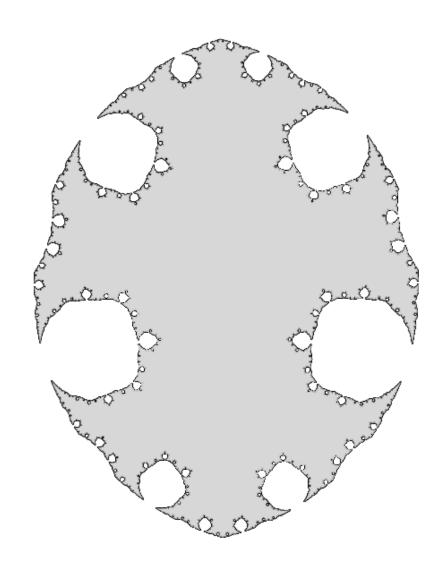


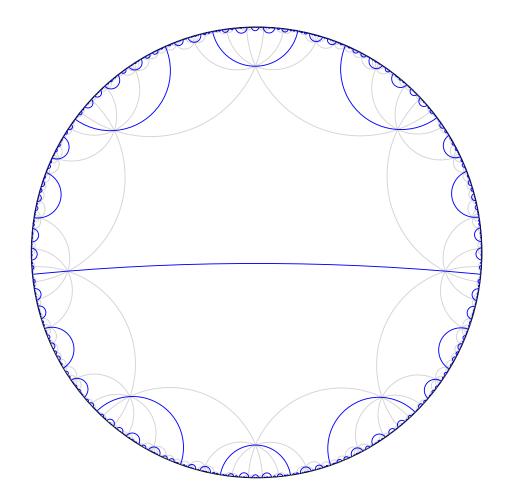
- Γ_X is the image of ρ : $\pi_1(S) \to \text{Isom}^+(\mathbb{H}^3)$, where $\rho \in \mathscr{V}(S) = \mathscr{V}(S \times I)$.
- A slight perturbation of ρ to ρ' yields a quasi-Fuchsian manifold.



- Each quasi-Fuchsian group Γ preserves a quasi-circle Λ, a fractal Jordan curve.
- Bowen: $H.dim(\Lambda) \in (1,2)$.
- $\Omega_+/\Gamma = X$ and $\Omega_-/\Gamma = Y$.
- The manifold Q(X,Y) interpolates from X to Y, its conformal boundary.

A more severe quasi-circle.





Equivariantly bend along the lifts.

Quasi-Fuchsian Space

■ The space QF(S) of quasi-Fuchsian representations sits in $\mathcal{V}(S)$.

Theorem. (Bers) The simultaneous uniformization

$$Q: \operatorname{Teich}(S) \times \operatorname{Teich}(S) \to QF(S)$$

is a homeomorphism.

■ The quasi-Fuchsian representations are parameterized by their conformal boundary.

Central Question:

How does the pair (X,Y) determine Q(X,Y)?

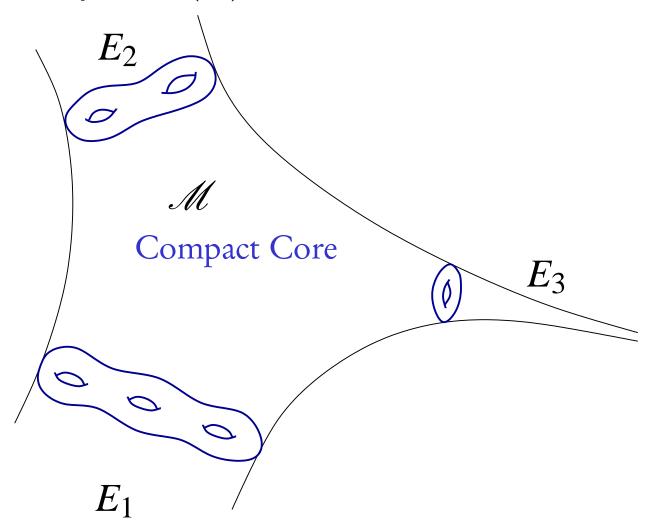
Topology

What is the topological type $\mathbb{H}^3/\rho(\pi_1(S))$?

Theorem. (Marden) Each Q(X,Y) is homeomorphic to $S \times \mathbb{R}$.

Topology

What about $\rho \in \mathscr{V}(N)$?



The Tameness Conjecture

Marden's Conjecture. Each hyperbolic 3-manifold M with finitely generated fundamental group is homeomorphic to the interior of a compact 3-manifold.

Such an M is said to be topologically tame – in particular, M is homeomorphic to int(\mathcal{M}).

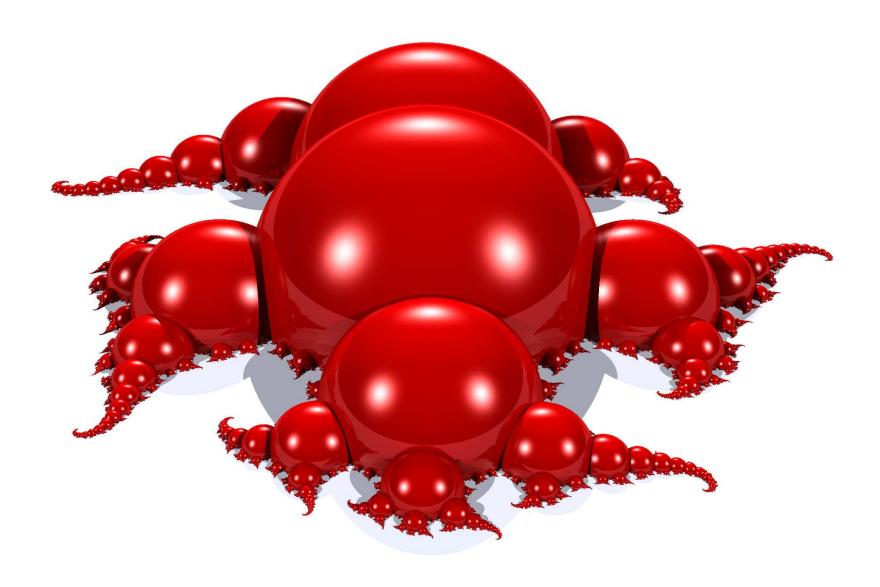
The Convex Core Boundary

- The invariant quasi-circle for the quasi-Fuchsian group Γ represents the limit set $\Lambda(\Gamma)$, the set of limit points of the action of Γ on $\widehat{\mathbb{C}}$.
- The *convex core* is the quotient

$$\text{hull}(\Lambda(\Gamma))/\Gamma$$
.

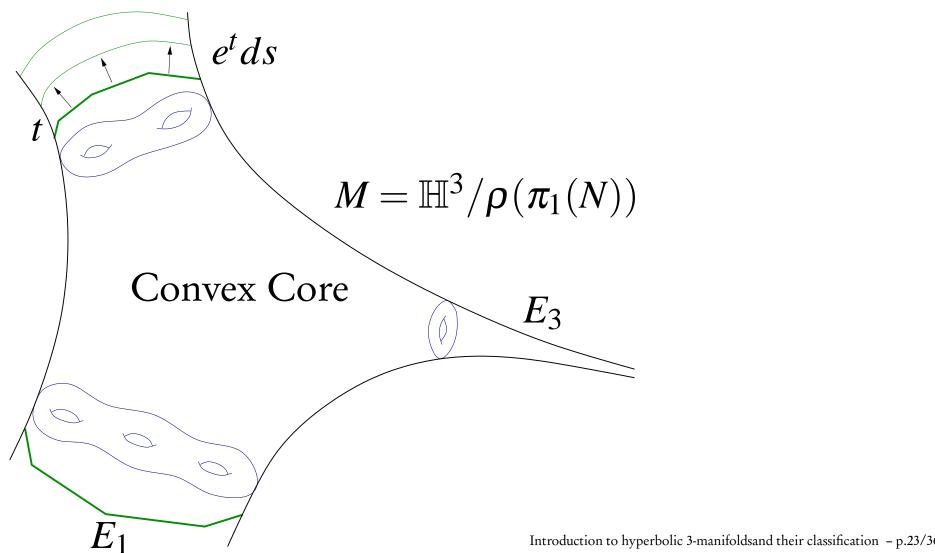
■ The boundary ∂ hull($\Lambda(\Gamma)$) is a pleated surface in \mathbb{H}^3 .

The Convex Core Boundary



Geometric Finiteness

 $M = \mathbb{H}^3/\rho(\pi_1(N))$ is **geometrically finite** if the **convex core** has finite volume.



Danger at the Boundary

■ Geometrically finite manifolds are parameterized by Teich($\partial_0 N$).

(Ahlfors, Bers, Abikoff, Maskit, Marden, Kra, Sullivan)

- The normal projection from the convex core boundary gives a product structure for the ends, guaranteeing tameness (Marden), *but*...
- A sequence with divergent parameters in Teich($\partial_0 N$) can converge to $\rho_\infty \in \mathscr{V}(N)$.

Ending Laminations

Thurston: What is the quotient

$$M = \mathbb{H}^3/\rho_{\infty}(\pi_1(N))$$

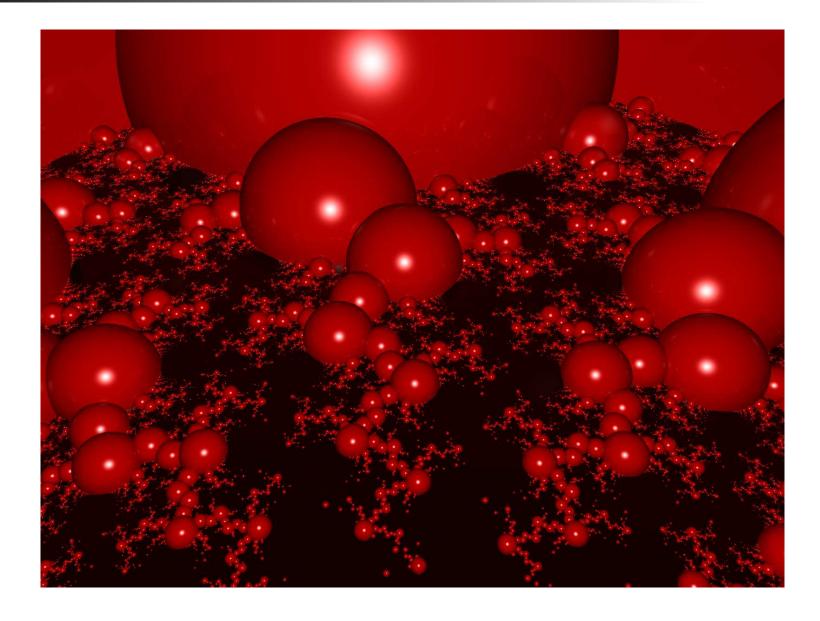
topologically, and what are the correct new parameters for its ends?

Ending Lamination Conjecture. Hyperbolic structures are determined by their topology and their end-invariants...?

Degenerate Ends

- When $vol(core(M)) = \infty$, we say M is degenerate.
- If there are geodesics exiting an end homotopic to *simple* curves on $\partial \mathcal{M}$ their limit on $\partial \mathcal{M}$ is an ending lamination.
- Thurston showed such ends are products, or *tame* when $\pi_1(N)$ is indecomposable.
- In principle, ends of *M* could fail to be products (Whitehead manifolds).

Inside the Degenerate Core



Inside the convex core of a degenerate manifold.

Ends are Tame

Theorem. (Bonahon '86) *If* $\pi_1(N)$ *is indecomposable each* $M \in \mathcal{V}(N)$ *is tame.*

And more recently,

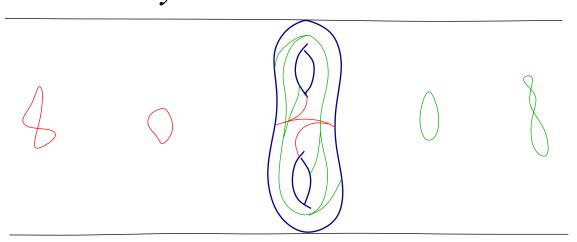
Theorem. (Agol..., Calegari-Gabai '04) *Each* $M \in \mathcal{V}(N)$ is tame.

Applying a theorem of Canary, all tame ends have well-defined end-invariants (surfaces/laminations).

Ending Laminations

Theorem. (B-Canary-Minsky - Model Theorem)

A hyperbolic structure on $S \times \mathbb{R}$ has a bi-Lipschitz model determined by its end-invariants.



The end-invariant v(E) is a limit of bounded length geodesics in E, pulled back to $\partial \mathcal{M}$.

Models and Rigidity

- How does a bi-Lipschitz model help with ELC?
- \blacksquare Any two manifolds M and M' with

$$\nu(M) = \nu = \nu(M')$$

are bi-Lipschitz to the model M_V .

- Hence, there is a bi-Lipschitz diffeomorphism $\phi: M \to M'$.
- By Sullivan's rigidity theorem, ϕ is homotopic to an isometry.

The General Case

- In the general case we pass to a cover corresponding to each end and obtain a model for this end.
- The bi-Lipschitz model maps extend across the remaining compact piece.
- This proves the ending lamination conjecture.

Classification

Combining the ending lamination and tameness theorems, we have..

Theorem. (The Classification Theorem) *Each* hyperbolic 3-manifold with finitely generated fundamental group is determined up to isometry by the topology of its compact core, its cusps, and its end invariants.

This classifies finitely generated discrete subgroups of $PSL_2(\mathbb{C})$, Kleinian groups.

Dynamical Consequences

Combining classification with work of Kleineidam-Souto, Namazi-Souto and Lecuire...

Theorem. (The Density Conjecture) *Each* $M \in \mathcal{V}(N)$ is a limit of geometrically finite manifolds.

Previously, the indecomposable cusp-free case was obtained by **B-Bromberg**...

...and more recently, **Bromberg-Souto** gave complete proof along these lines.

Geometric Consequences

The model verifies a conjecture of McMullen.

Theorem. (Volume Growth) Let $B_{\text{thick}}(x,R)$ denote the ball of radius R in the path metric on the Margulis thick part of the convex core of $M = \mathbb{H}^3/\rho(\pi_1(S))$. Then

$$\operatorname{vol}(B_{\operatorname{thick}}(x,R)) \leq CR^d$$

where C and d depend only on S.

Here, the *model* reveals new effective geometric information.