Axes in Outer Space

Michael Handel and Lee Mosher

May 19, 2006

Dictionary of Spaces:

Group:	$\text{Isom}(\mathbf{H}^n)$	$\mathcal{MCG}(S)$	$\text{Out}(F_n)$
Space:	Hyperbolic	Teichmüller	Outer
Notation:	$\mathbf{H}^n$	$\mathcal{T}(S)$	$\mathcal{X}_n$

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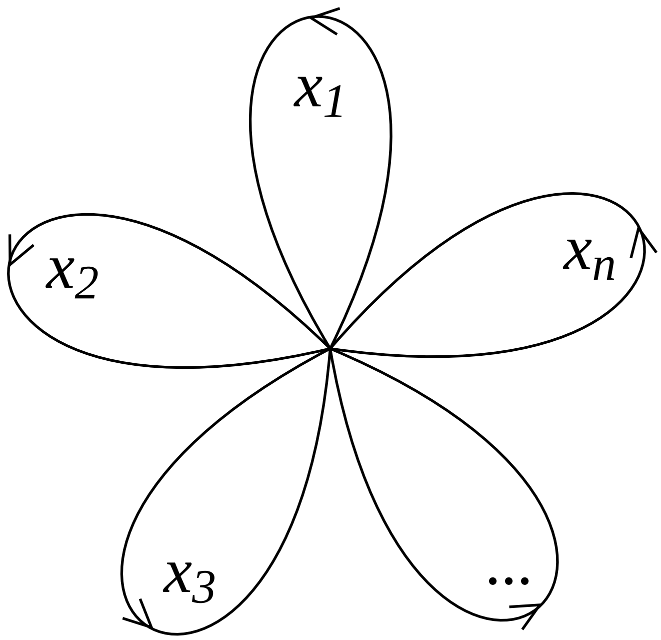
Axis for North-South elements:	$\surd$	$\surd$	???

Outer Space $\mathcal{X}_n$:

The moduli space of marked graphs (of rank n)



R_n = the rose of rank n

F_n = the free group of rank n

$= \langle x_1, x_2, x_3, ..., x_n \rangle$

$= \pi_1(R_n)$

$$\mathrm{Out}(F_n) = \mathrm{Aut}(F_n) / \mathrm{Inn}(F_n)$$

$$= \frac{\{\text{homotopy equivalences of } R_n\}}{\text{homotopy}}$$

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$\phi =$ an element of $\mathrm{Out}(F_n)$

$\Phi =$ a representative in $\mathrm{Aut}(F_n)$

or a representative homotopy equivalence $R_n \mapsto R_n$

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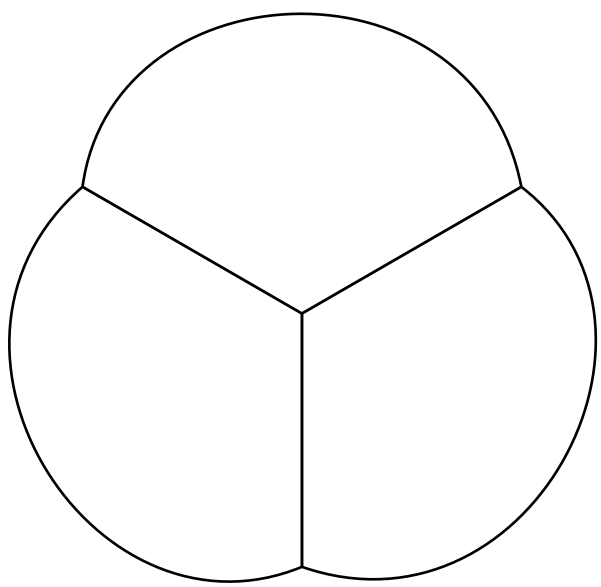
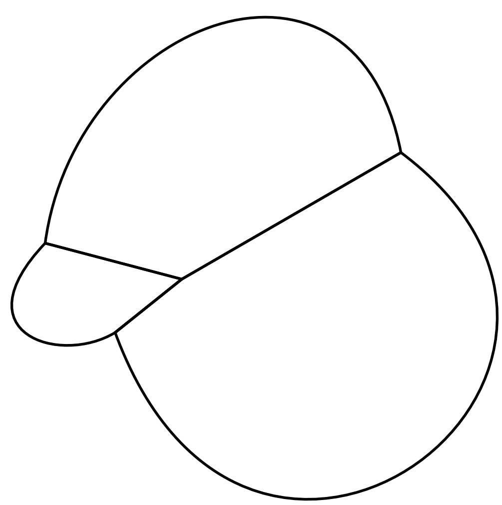
Try to prove:

- No nontrivial conjugacy class is periodic (easy).
- In particular, no rank 1 free factor has a periodic conjugacy class.
- No rank 2 free factor has a periodic conjugacy class (trickier).

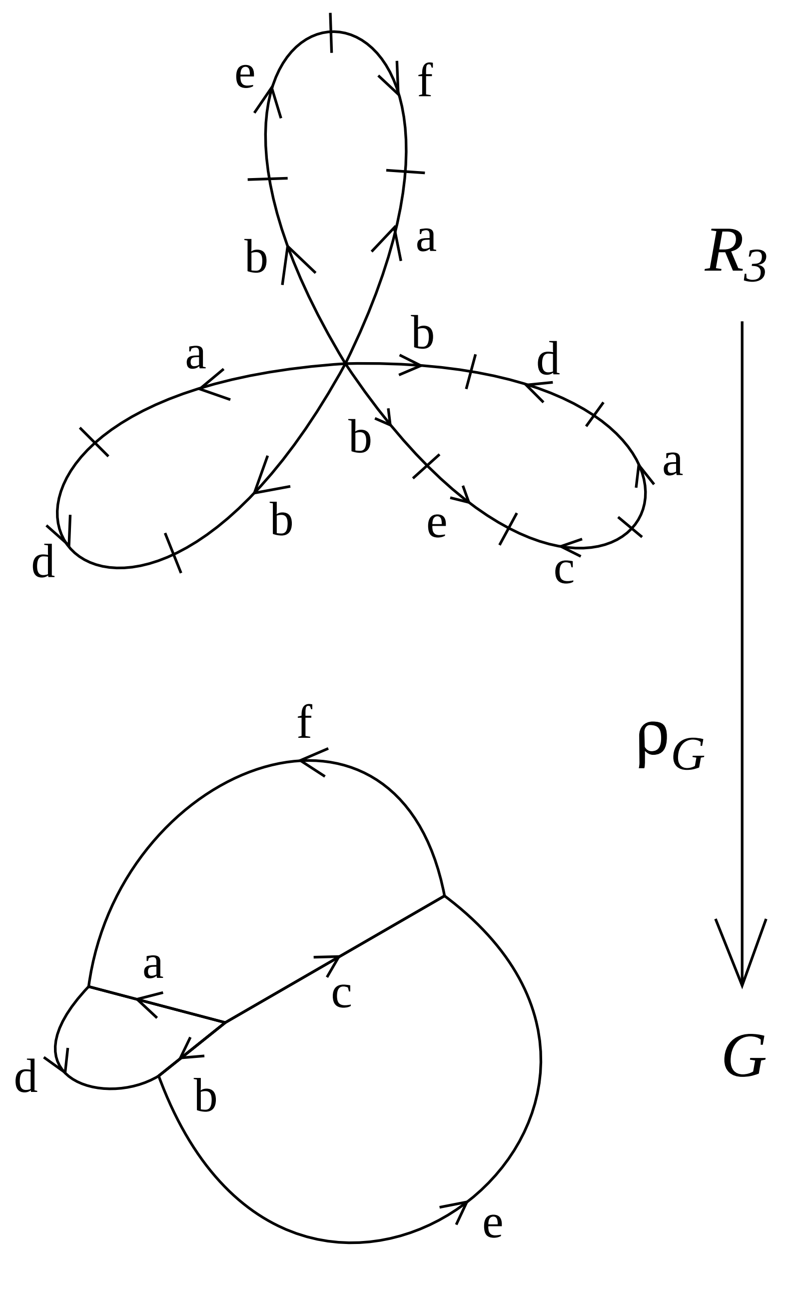


A graph of rank 3



G

A graph of rank 3
with a metric



A graph of rank 3
with a metric
and a marking

Definition of a marked graph (of rank n):
A graph G with no vertices of valence 1, equipped with:

- A geodesic metric, determined (up to isotopy) by assigning a length to each edge.

- A homotopy equivalence

$$\rho_G \colon R_n \to G$$

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Definition of a marked homotopy equivalence:
A homotopy equivalence $f \colon G \to G'$ of marked graphs such that

$$\begin{array}{ccc} & & G \\ & \overset{\rho_G}{\nearrow} & \\ R_n & & \downarrow f \\ & \underset{\rho_{G'}}{\searrow} & \\ & & G' \end{array}$$

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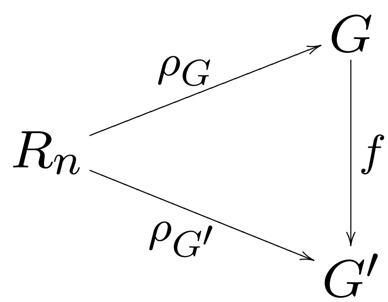

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Types of marked homotopy equivalences:

- Marked isometry (preserves metric).
- Marked homothety (multiplies metric by a constant).
- Marked homeomorphism (preserves topology).

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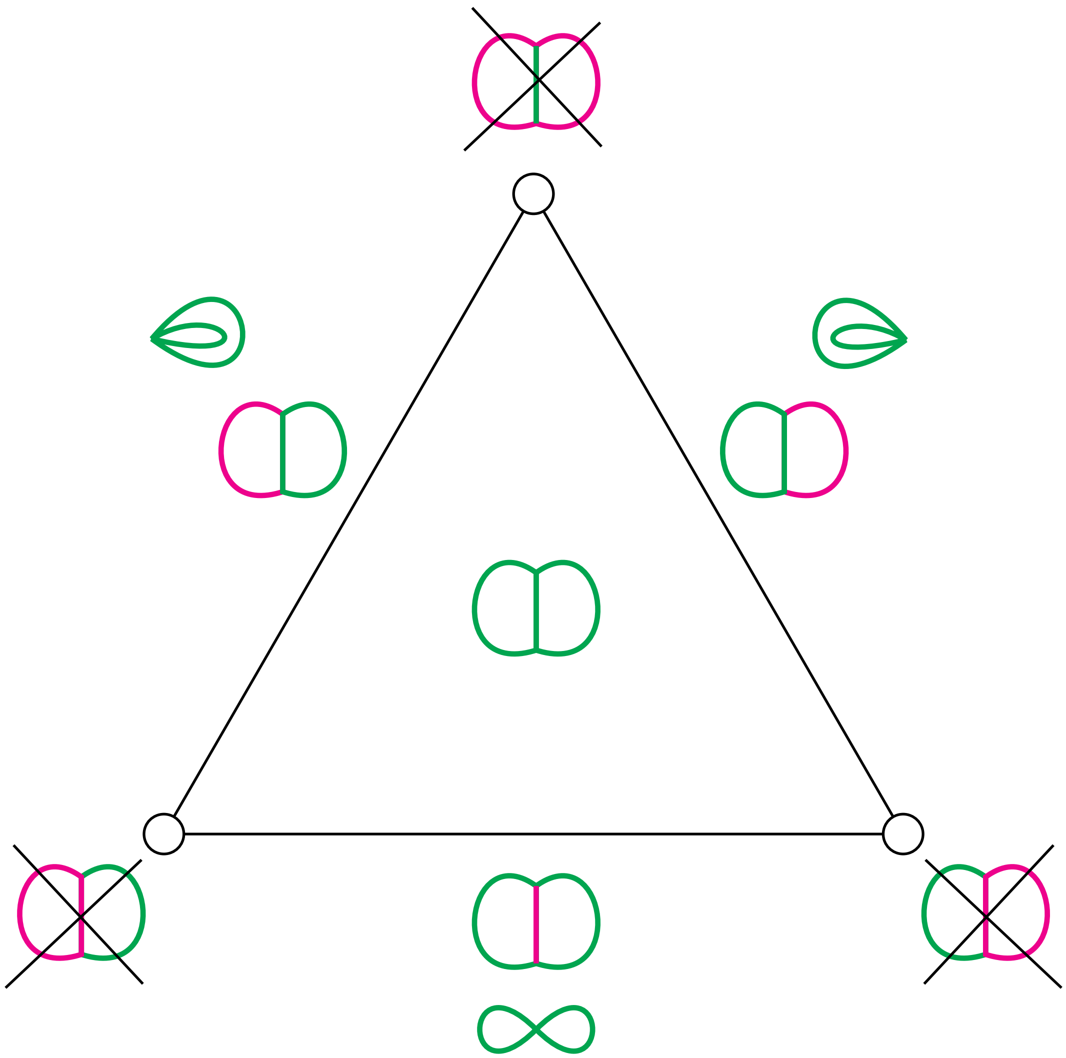
Cone stratification of $\widehat{\mathcal{X}_n}$,
Simplex stratification of $\mathcal{X}_n$:

- One cone (simplex) for each marked graph G, up to marked homeomorphism.



Cone stratification of $\widehat{\mathcal{X}_n}$,
Simplex stratification of $\mathcal{X}_n$:

- One cone (simplex) for each marked graph G, up to marked homeomorphism.

- Open cone parameterized by $(0, \infty)^k$, one coordinate for each of the k edges of G.

- Closed cone parameterized by $[0, \infty)^k$ minus one face for each nonforest subgraph of G:

  - Edge lengths may be zero on a subforest, collapsing each component of the subforest to a point.

  - Edge lengths *may not* be zero on a loop, else the homotopy type changes.

Topology of $\widehat{\mathcal{X}_r}$:

- Closed cones inherit topology from parameterization.

- Weak topology w.r.t. collection of closed cones.

Topology of $\mathcal{X}_r$:

- Quotient topology w.r.t. projection $\widehat{\mathcal{X}_r} \to \mathcal{X}_r$.

- Quotients of open cells form open simplex decomposition.

- Quotients of closed cells form closed (but non-compact) simplex decomposition

- Weak topology w.r.t. collection of closed simplices.

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- Define marked graph $\rho_{G \cdot \phi} = \rho_G \circ \Phi \colon R_n \to G$

$$\begin{array}{ccc} R_n & \xrightarrow{\rho_{G\cdot\phi}} & G \\ {\scriptstyle\Phi}\downarrow & \nearrow_{\rho_G} & \\ R_n & & \end{array}$$

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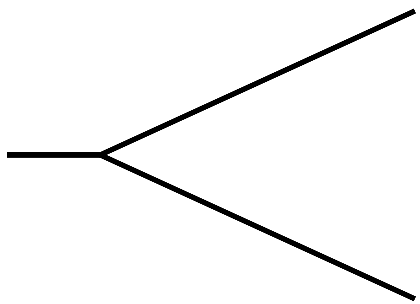
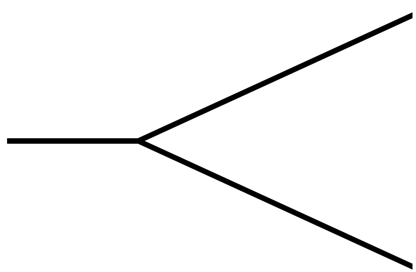
Fold line in Outer Space:

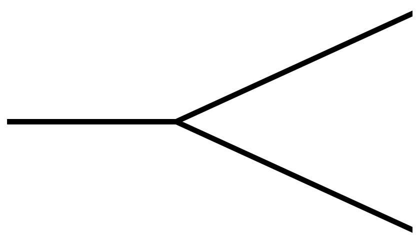
Continuous path in $\mathcal{X}_r$ interpolated by edge isometries

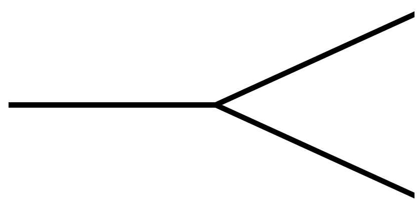
Periodic fold lines:

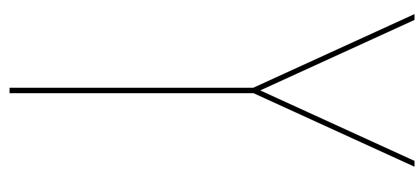
Arise from train track maps











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$$h_{ts} \colon G_s \to G_t, \qquad -\infty < s < t < +\infty$$

satisfying the semiflow identity

$$h_{ts} \circ h_{sr} = h_{tr}, \qquad -\infty < r < s < t < +\infty$$

$$G_r \xrightarrow{h_{sr}} G_s \xrightarrow{h_{ts}} G_t, \qquad G_r \xrightarrow{h_{tr}} G_t$$

Train track maps: Given:

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- $\exists \lambda > 1$ such that $\forall i \geq 1$ and $\forall E$ an edge of G,

$g^i \Big| \mathrm{int}(E)$ is a local homothety with stretch λ^i

Example of a train track map:
On the 3-petalled rose:

$$\Phi\colon \begin{cases} A & \to B \\ B & \to C \\ C & \to \overline{B}A \end{cases}$$

Only "illegal turns" (turns which are not locally injective under all powers) are:

$$AC, \overline{CA}, CB, \overline{BC}, BA, \overline{AB}$$

No edge has an image containing one of these illegal turns.
So, Φ is a train track map.

$$\Phi\colon \begin{cases} A & \to B \\ B & \to C \\ C & \to \overline{B}A \end{cases}$$

Sufficient condition for $\phi \in \mathrm{Out}(F_3)$ to be fully irreducible:

Check that the map M_Φ on $H_1(G;\mathbf{Z})$ induced by the train track map Φ

$$M_\Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

has no eigenvalue on the unit circle.

(This sufficient condition only works in rank 3).

$$\mathrm{Eigenvalues}(M_\Phi) = \{0.6823278040, \\ -0.3411639019 \pm 1.161541400i\}$$

Theorem (Bestvina, Handel). *For every fully irreducible* $\phi \in \mathrm{Out}(F_r)$ *there exists an affine train track representative*

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From this we shall obtain:

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For $i < j < k \in \mathbf{Z}$, the identity

$$g^{k-j} \circ g^{j-i} = g^{k-i}$$

induces the *semiflow identity*

$$h_{kj} \circ h_{ji} = h_{ki}$$

From this “discrete” fold line:

- $G_i = \frac{1}{\lambda^i} G \cdot \phi^i, \qquad i \in \mathbf{Z}$

- $h_{ji} \colon G_i \to G_j$ for $i < j \in \mathbf{Z}$

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To get a continuous fold line:

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To get a continuous fold line:

- Interpolate $h_{i+1,i} \colon G_i \to G_{i+1}$ by a fold path.

- Fit together to give a ϕ-periodic fold line.

$\mathcal{A}_\phi$

The Axis Bundle for ϕ:

Periodic fold lines for powers of ϕ
all bundled together

The axis bundle:

- Choices in the construction of a periodic fold line for power of ϕ:
  - Positive power ϕ^k
  - Train track map $g\colon G \to G$ representing ϕ^k
  - The interpolations of $h_{i+1,i}\colon G_i \to G_{i+1}$.

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Define the *axis bundle* $\mathcal{A}_\phi \subset \mathcal{X}_r$ to be:

- The closure of the union of all ϕ^i-periodic fold lines for $i \geq 1$.

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Natural dependence implies:

- $\mathcal{A}_{\phi^i} = \mathcal{A}_\phi$ *for* $i \geq 1$

Theorem. *For each fully irreducible $\phi \in \mathrm{Out}(F_n)$, with source $T_- \in \partial\mathcal{X}_r$ and sink $T_+ \in \partial\mathcal{X}_r$:*

- $\mathcal{A}_\phi$ *is proper homotopy equivalent to* $\mathbf{R}$
- *The two ends of* $\mathcal{A}_\phi$ *converge in* $\overline{\mathcal{X}}_r$ *to* T_-, T_+.
- $\mathcal{A}_\phi$ *depends naturally on* (T_-, T_+)

Natural dependence implies:

- $\mathcal{A}_{\phi^i} = \mathcal{A}_\phi$ *for* $i \geq 1$
- *If* $\psi \in \mathrm{Out}(F_n)$ *commutes with a (nonzero) power of* ϕ *then*

$$\psi(\mathcal{A}_\phi) = \mathcal{A}_\phi$$

Natural dependence on (T_-, T_+):

Theorem. *For each fully irreducible $\phi \in \mathrm{Out}(F_n)$, with source $T_- \in \partial\mathcal{X}_r$ and sink $T_+ \in \partial\mathcal{X}_r$, $\mathcal{A}_\phi$ is*

Natural dependence on (T_-, T_+):

Theorem. *For each fully irreducible* $\phi \in \mathrm{Out}(F_n)$, *with source* $T_- \in \partial\mathcal{X}_r$ *and sink* $T_+ \in \partial\mathcal{X}_r$, $\mathcal{A}_\phi$ *is the union of* all *fold lines (periodic or not), whose negative end converges to* T_-, *and whose positive end converges to* T_+.

So far two characterizations of when $G \in \mathcal{X}_r$ is in the axis bundle $\mathcal{A}_\phi$:

- (3) G is a limit of points on ϕ^i-periodic fold lines, $i \geq 1$.

- (1) G is on a fold line with ends T_-, T_+.

These are both "global" or "extrinsic" conditions: existence of certain fold lines passing through or near G.
Need a "local" or "intrinsic" condition on G.

- (3) G is a limit of points on ϕ^i-periodic fold lines, $i \geq 1$.

- (1) G is on a fold line with ends T_-, T_+.

For the cognoscenti, alternate condition depends on the lamination theory of Bestvina, Feighn, and Handel:

$\Lambda_- = \Lambda_-(\phi)$, the expanding lamination of ϕ:

- Limits of iterates of edges of a train track map.

- Minimal lamination whose leaves have length zero in T_-.

- (3) G is a limit of points on ϕ^i-periodic fold lines, $i \geq 1$.

- (1) G is on a fold line with ends T_-, T_+.

Alternate characterization of $G \in \mathcal{A}_\phi$:

- (2) $G \in \mathcal{X}_r$ is a *weak train track*, meaning G has a normalization in $\widehat{\mathcal{X}_r}$ so that:

  - there exists an edge isometry $\widetilde{G} \to T_+$ such that every leaf of Λ_- realized in $\widetilde{G}$ embeds in T_+.

Main point to take from this definition: each weak train track has a "canonical" normalization, and hence has a "canonical" length.

Where's the hard work?

- For all $G \in \mathcal{X}_r$, TFAE:

  (1) G is on a fold line with ends T_-, T_+

  (2) G is a weak train track.

  (3) G is a limit of points on ϕ^i-periodic fold lines, $i \geq 1$.

- Proving that $\mathcal{A}_\phi$ is proper homotopy equivalent to $\mathbf{R}$.

Main technical result from which others flow:

Proposition 1 (The $S - T$ Lemma (Prop. 5.4 but it doesn't look like this))**.**
For every $L > 0$ there exists a train track S of length $> L$ such that for each weak train track T of length $< L$, there is an edge isometry $S \mapsto T$.

In other words: From a certain sufficiently long S, can fold to anything sufficiently short.

Proof that (2) $\Longrightarrow$ (1): every weak train track lies on a fold line going from T_- to T_+.

- Every train track S is on a periodic fold line (proved earlier).

- Every periodic fold line goes from T_- to T_+ (apply Source-Sink dynamics).

Construction of a fold line from T_- through T to T_+:

- Fold from T_- to S by part of a periodic fold line;

- Fold from S to T by interpolating the edge isometry $S \mapsto T$;

- Fold from T to T_+ by interpolating the edge isometry $T \mapsto T_+$.

Proving that $\mathcal{A}_\phi$ is proper homotopy equivalent to $\mathbf{R}$:

- Length map $\mathcal{A}_\phi \to (0, \infty)$ is a proper map:
  - Use the $S-T$ lemma: those T, foldable from S, with length in a compact interval $[a, b]$, forms a compact set.

- Skora's method, which has been used to prove contractibility of $\mathcal{X}_r$, $\overline{\mathcal{X}}_r$, etc., can also be used to prove that the length map $\mathcal{A}_\phi \to (0, \infty)$ is a proper homotopy equivalence.