The Weil-Petersson Geometry in the Interior of Teichmüller space

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1 Background

In this talk, S is a closed, orientable Riemann surface of genus $g > 1$ without punctures. Many aspects of Teichmüller theory study different structures on Riemann surfaces.

We study **asymptotic geometry** of Teichmüller space in this talk. However, we will not go to the infinity. There will be no curve pinching on the surface. We consider the asymptotics of the curvatures in the **interior** of Teichmüller space when the genus of the surface gets large.

Teichmüller space $\mathcal{T}_g$ is the space of conformal structures (or hyperbolic metrics for $g > 1$) on a closed surface S of genus g where two structures are equivalent if there is a biholomorphic map, homotopic to the identity, between them, i.e., Teichmüller space is defined as

$$\mathcal{T}_g = M_{-1}/Diff_0(S)$$

where M_{-1} is the space of hyperbolic metrics on S, and $Diff_0(S)$ is the group of orientation preserving diffeomorphisms homotopic to the identity.

Teichmüller space $\mathcal{T}_g$ is a $3g-3$ dimensional complex manifold if $g \geq 2$ (Ahlfors-Bers).

Different metrics defined on Teichmüller space reflect different perspectives of the structures of Teichmüller space. The Weil-Petersson metric is one of the most natural ones.

The Weil-Petersson metric d_{WP} on Teichmüller space is obtained by duality from the L^2-norm on the cotangent space (the space of holomorphic quadratic differentials):

$$\|\phi\|_{WP}^2 = \int_S \frac{|\phi|^2}{\sigma} dz d\bar{z}$$

where $\sigma dz d\bar{z}$ is the hyperbolic metric on S.

The geometry of $(\mathcal{T}_g, d_{WP})$ is very interesting, and there are many deep connections with the study of d_{WP} and hyperbolic 3-manifolds, which is the theme of this workshop.

Mysterious geometry of d_{WP}:

- (*Chu, Masur, Wolpert; 1976*) d_{WP} is **incomplete**;

- (*Tromba, Wolpert; 1986*) $K_{d_{WP}} < 0$;

- (*Wolpert; 1986*) The holomorphic sectional curvature and Ricci curvature are pinched from above by $\frac{-1}{2\pi(g-1)}$;

- (*Wolpert; 1987*) $(\mathcal{T}_g, d_{WP})$ is geodesically convex;

- (*Brock-Farb; 2000*) d_{WP} is δ-hyperbolic if and only if $dim_{\mathbb{C}}\mathcal{T} \leq 2$;

- (*Brock; 2000*) The volume of the convex core of the quasi-Fuchsian hyperbolic 3-manifold $Q(S_1, S_2)$ with S_1 and S_2 in the boundary is comparable to $d_{WP}(S_1, S_2)$;

- (*H-; 2002*) The sectional curvature $K_{d_{WP}}$ is not negatively pinched.

2 Systole and Genus

Throughout the talk, we denote l_σ as the **systole**, the length of the shortest closed geodesic on S with respect to the hyperbolic metric σ. We assume there is a positive lower bound r_0 on the injectivity radius $int_\sigma(S)$.

We call the compact subset of the moduli space where $int_\sigma(S) \geq r_0 > 0$ **the thick part**, i.e., *the thick part of moduli space consists of fat surfaces.*

As convenient constants, the systole and genus show up regularly at many compactness arguments and estimates. It is very important to understand their roles in the study of the Weil-Petersson geometry. In this section, we closely exam this aspect which motivated current work.

- On the surface S, the (hyperbolic) area is given by the Gauss-Bonnett formula: $Area(S) = 4\pi(g-1)$;

- (*Bers; 1972*) The hyperbolic diameter satisfies

$$diam(S) \leq (g-1)\frac{l_\sigma}{sinh^2(\frac{l_\sigma}{2})},$$

  where $\frac{l_\sigma}{sinh^2(\frac{l_\sigma}{2})} \approx \frac{4}{l_\sigma}$ for l_σ sufficiently small;

- Bers constant satisfies $\sqrt{6g} - 2 \leq L_g \leq 26(g-1)$;

- (*Wolpert; 1977*) The moduli space has finite (bound depends on the surface) Weil-Petersson diameter;

- The frontier space of $\mathcal{T}_g$ consists of noded surfaces, thus can be described as the set $\{\sigma : l_\sigma = 0\}$;

- (*Wolpert; 2003*) The Weil-Petersson injectivity radius of the moduli space is comparable to $\sqrt{l_\sigma}$, with constant independent of the genus.

3 Asymptotics in the thick part

An important yet very difficult consideration is the calculation of the Weil-Petersson integrals on the moduli space $\mathcal{M}_g$. A direct application is to determine their Weil-Petersson volumes.

Considering the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$, its Weil-Petersson volume is a rational multiple of π^{3g-3+n} (*Wolpert*, 1983). Recently Mirzakhani has worked out a recursive formula on the Weil-Petersson volumes.

It is natural to ask how geometrical preperties of $\mathcal{M}_g$ vary if the genus of the underlying surface gets large.

As an example, it is known (*Wolpert*, 1986) that the holomorphic sectional curvature and Ricci curvature are bounded from above by $\frac{-1}{2\pi(g-1)}$, i.e., large genus implies almost vanishing Ricci ~~curvatures~~ curvature bound.

Before we move towards the thick part of the moduli space, we consider the Weil-Petersson geometry near the compactification divisor:

- (*Schumacher 1986, Trapani 1992, H-, 2003*) The Weil-Petersson holomorphic sectional curvature and Ricci curvature are bounded from below by $-Cl_\sigma^{-1}$; moreover (*H*-, 2003), there exists a family of tangent planes (towards the compactification divisor) with curvatures of the order $-l_\sigma^{-1}$. Holomorphic sections are the only ones with "extremely hyperbolic" curvatures. The Weil-Petersson sectional curvature has no negative lower bound;

- (*H*-, 2002, 2003) There exist families of tangent planes (towards the compactification divisor) with absolute values of the curvatures of the order $O(l_\sigma)$. In other words, near the compactification divisor, tangential directions are asymptotically flat. Therefore, the Weil-Petersson sectional curvature has no negative upper bound.

We now assume the surface S is in the thick part of the moduli space.

First observation: because of the compactness of the thick part in $\mathcal{M}_g$, all Weil-Petersson curvatures are bounded. However, the bounds depend on the genus, and/or other invariants.

Theorem 1. *There exists a positive constant C_1, independent of g, such that the Weil-Petersson holomorphic sectional curvature K_h satisfies that*

$$-C_1 < K_h < \frac{-1}{2\pi(g-1)},$$

in the thick part of the moduli space.

As for the general sectional curvature, we have

Theorem 2. *(Main theorem) In the thick part of the moduli space, there is a positive constant C_2, independent of the genus g, such that the sectional curvature K of the Weil-Petersson metric satisfies that*

$$-C_2 < K < 0.$$

Let us consider what need to be estimated to have curvature estimates as in the theorems.

The tangent space of Teichmüller space at S is equivalent to $HB(S)$, the space of *harmonic Beltrami differentials*. A harmonic Beltrami differential $\mu(z)\frac{d\bar{z}}{dz}$ is given as $\bar{\phi}(ds^2)^{-1}$, for ϕ a holomorphic quadratic differential with at most simple poles at the cusps and ds^2 the hyperbolic metric tensor. Since the surface S has no cusps and this quadratic differential $\phi = \phi(z)dz^2$ has no poles.

The Weil-Petersson holomorphic sectional curvature for a tangent vector μ is given by

$$K(\mu, J\mu) = \frac{-2\int_S D(|\mu|^2)|\mu|^2 dA}{\int_S |\mu|^2 dA},$$

where $D = -2(\Delta_\sigma - 2)^{-1}$ is a self-adjoint, compact operator, and $dA = \sigma|dz|^2$.

In general, the Weil-Petersson curvature tensor is given by the Tromba-Wolpert formula:

$$R_{1\bar{2}3\bar{4}} = \int_S D(\mu_1\bar{\mu}_2)\mu_3\bar{\mu}_4 dA + \int_S D(\mu_1\bar{\mu}_4)\mu_3\bar{\mu}_2 dA$$

Now if we choose μ to have a unit Weil-Petersson norm, i.e., $\int_S |\mu|^2 dA = 1$, then the holomorphic sectional curvature is a multiple of the integral $\int_S D(|\mu|^2)|\mu|^2 dA$.

The next lemma passes a L^2 bound to a pointwise bound:

Lemma 3. *For $\mu(z)\frac{d\bar{z}}{dz} \in HB(S)$ with $||\mu||_{WP} = 1$, there exists a positive constant h_0, independent of g, such that $|\mu(z)| \leq h_0$, for all $z \in S$, where the surface S is in the thick part of the moduli space.*

With this lemma, we see that:

$$\begin{aligned} |K_h| &= 2\int_S D(|\mu|^2)|\mu|^2 dA \\ &< 2h_0^2 \int_S D(|\mu|^2) dA \\ &= 2h_0^2 \int_S |\mu|^2 dA \\ &= 2h_0^2. \end{aligned}$$