Geometry and Combinatorics of Arborescent Link Complements

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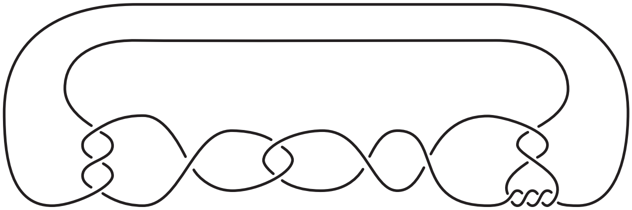
Goals

Our goal is to relate the geometry and topology of a knot or link complement to the combinatorics of a projection diagram.

Specific questions:

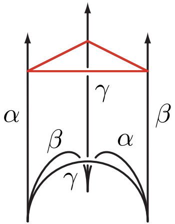
1. Can we tell from the diagram if K has a hyperbolic complement?
2. If K is hyperbolic, estimate its volume, lengths of geodesics, etc.
3. What surfaces are contained in the link complement?



We approach these questions by studying *angled ideal triangulations*.

Angled ideal tetrahedra

- Tetrahedra whose vertices have been removed
- Modeled on positively oriented ideal tetrahedra in $\mathbb{H}^3$



$$\alpha + \beta + \gamma = \pi$$

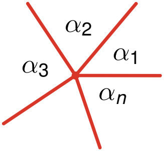
If we truncate an ideal vertex, we get a Euclidean triangle.
So around each ideal vertex, the three dihedral angles sum to π.

One can also work with larger polyhedra, also modeled on ideal convex polyhedra in $\mathbb{H}^3$.

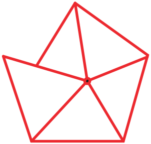
Angled triangulations

We construct a 3–manifold *M* by gluing up angled tetrahedra and ensuring that the angles line up around each edge of *M*.

The truncated vertices fit together to tile ∂M.



$$\sum_{i=1}^{n} \alpha_i = 2\pi$$



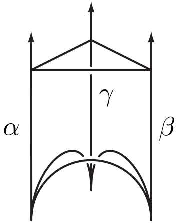
This does not quite give us a hyperbolic metric on *M* because there may still be *shearing singularities* along the edges.

This type of incomplete structure is called an *angle structure*.

Why study angle structures? Because we can.

To get a hyperbolic metric on M by gluing tetrahedra:

- Solve a non-linear system of complex–valued equations
- At most one solution (by Mostow–Prasad rigidity)



$\alpha + \beta + \gamma = \pi$ in each tetrahedron

$\sum_{i=1}^{n} \alpha_i = 2\pi$ around each edge

$\alpha_i > 0$ for all angles

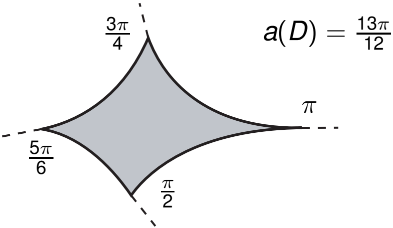
To get an angle structure on M:

- Solve a linear system of real–valued equations and inequalities
- The solution set (if non-empty) is a convex polytope in $\mathbb{R}^m$

Angle structures provide control over surfaces

When a surface $S \subset M$ intersects a tetrahedron in a disk D, the disk has a well-defined *combinatorial area*

$$a(D) = \sum_{\partial D} \epsilon_i - 2\pi.$$



This matches the formula for area of hyperbolic polygons.

Lemma (Casson)

$$a(S) = -2\pi\chi(S).$$

This has useful applications to Dehn surgery.

Angle structures tell us M is hyperbolizable

Theorem (Casson, Lackenby)

If an orientable manifold $(M, \partial M)$ has an angled triangulation, M admits a hyperbolic metric.

Proof idea.

Thurston's hyperbolization theorem says M is hyperbolic iff M contains no essential spheres, disks, tori, or annuli.
Use combinatorial area to rule them out.

- Spheres and disks have positive Euler characteristic, so negative area. This cannot happen.
- Tori and annuli have zero area, and can also be controlled. They are boundary–parallel or compressible. $\square$

Angle structures can lead to a hyperbolic structure

A tetrahedron T with angles α, β, γ has hyperbolic volume

$$\mathcal{V}_{\alpha\beta\gamma} = Л(\alpha) + Л(\beta) + Л(\gamma), \qquad Л(x) = \int_0^x -\log|2\sin t|\, dt.$$

Let P be the polytope of angle structures for a particular topological triangulation. Every angle structure $\tau \in P$ has an *angled volume* $\mathcal{V}(\tau)$, obtained by summing over all tetrahedra.

Theorem (Rivin, Chan–Hodgson)

$\mathcal{V}$ has a critical point at $\tau \in P \quad \Leftrightarrow \quad \tau$ is a geometric triangulation that gives the complete hyperbolic metric on M.

Thus, if $\mathcal{V}$ attains a maximum inside P, we have a lot of information:

- Any angle structure gives a lower bound on the volume of M.
- All edges of the triangulation are hyperbolic geodesics.

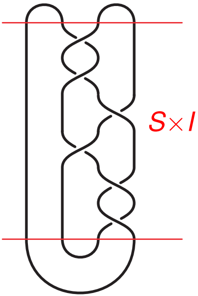
A program for studying the geometry of a link

1. Use a nice diagram to construct an ideal triangulation. Typically, the edges are arcs in the projection plane.
2. Parametrize the polytope P of linear solutions, and prove it is non-empty.
3. Prove that the volume function $\mathcal{V}$ is maximized inside P, using Euclidean geometry of the cusp tori.
4. **Bonus:** Computations in the Minkowski space model of $\mathbb{H}^3$ can tell us that the triangulation is in fact *geometrically canonical*, i.e., dual to the Ford–Voronoi domain. This seems very hard to prove without data about dihedral angles.

For different families of links, we can get through different stages of this program.

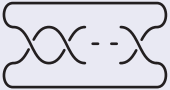
Two–bridge links

A *two–bridge link* L in S^3 has a projection with two maxima, two minima, and a 4–string braid in between. The complement of the braid part is a *product region* $S \times I$, where S is a 4–punctured sphere.



Theorem (Menasco)

All two–bridge links are hyperbolic, except these:



We re-prove this theorem using angled triangulations. All parts of the program work: the triangulations give the hyperbolic metric on $S^3 \setminus L$, and are geometrically canonical.

Two–bridge corollaries

Theorem

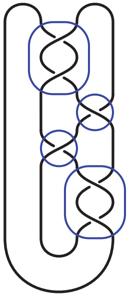
In any reduced alternating diagram of a two-bridge link, the arcs at the crossings are short geodesics. All these arcs are edges of the canonical triangulation.

This confirms a conjecture of Thistlethwaite in the case of two–bridge links.

Theorem

For a reduced alternating diagram of a two-bridge link,

$$\frac{v_8}{2}\,(tw\,D(K)-1) \;\le\; Vol(S^3 \setminus K) \;\le\; 10v_3\,(tw\,D(K)-1).$$

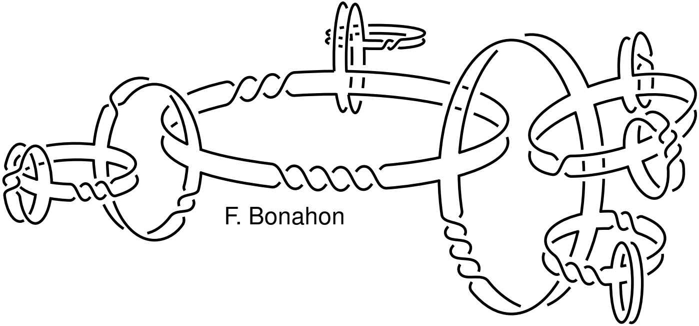


$\mathrm{tw}\,D = 4$

The upper bound is due to Agol–D. Thurston.
The lower bound is a small improvement over Agol–Storm–W. Thurston.

Arborescent links

Arborescent links are constructed by plumbing together twisted, unknotted bands.

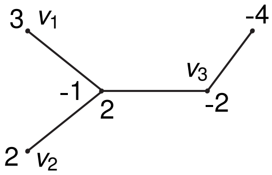


F. Bonahon

This large family includes all two–bridge, pretzel, and Montesinos links.

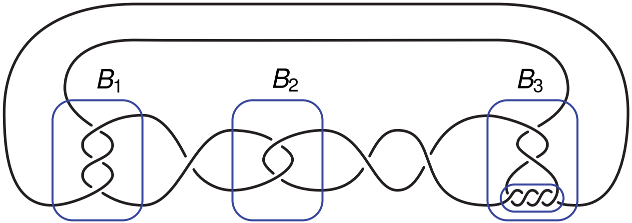
A tree represents an arborescent link diagram

We can encode the combinatorics by a *weighted planar tree* T.



vertices	$\leftrightarrow$	bands
edges	$\leftrightarrow$	plumbing of bands
weights	$\leftrightarrow$	half-twists in the bands

The plumbing of bands can be flattened to an *arborescent diagram* $D_T(K)$.



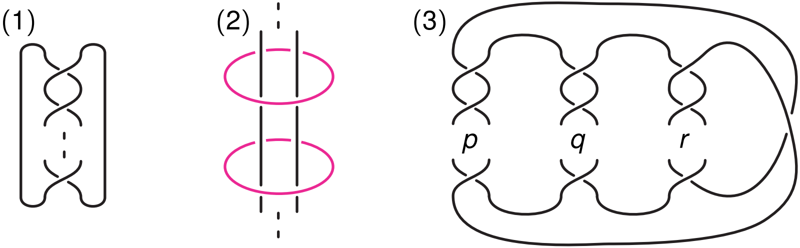
Which arborescent links are hyperbolic?

Theorem (Bonahon–Siebenmann)

All arborescent links are hyperbolic, except the following families:

1. *K has only one band,*
2. *K has two isotopic components, each of which bounds a twice–punctured disk,*
3. *K is the pretzel link $P(p, q, r, \text{-}1)$, $\quad p, q, r \geq 2, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq 1.$*

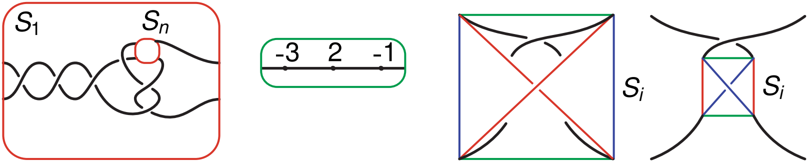
Furthermore, an effective algorithm decides if K is an exception.



Constructing the triangulation

We re-prove this theorem by subdividing the link complement into ideal tetrahedra, as well as some larger blocks.

Key step: Focus on the *branches* (linear subtrees) of T. Each branch defines a 4–string braid, hence a *product region* $S \times I$.



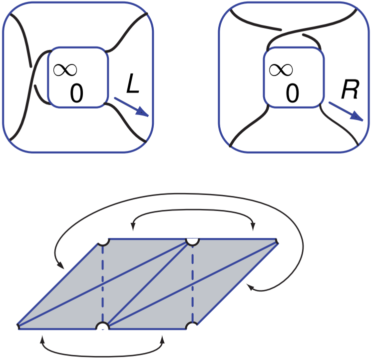
Each crossing in the braid corresponds to a 4–punctured sphere S_i.

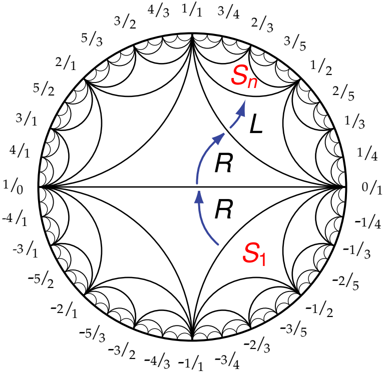
S_i is a *pleated surface*, subdivided into ideal triangles by arcs that look vertical or horizontal immediately before or after the crossing.

Tetrahedra fill the spaces between neighboring pleated surfaces.

Diagonal exchanges as seen in the Farey complex

The *Farey complex* $\mathcal{F}$ encodes triangulations of a 4–punctured sphere. Vertices of $\mathcal{F}$ correspond to *slopes* (pairs of arcs), edges of $\mathcal{F}$ to disjoint slopes, and triangles of $\mathcal{F}$ to ideal triangulations.





Each edge between S_1 and S_n defines a *diagonal exchange* between two pleated surfaces, giving rise to a layer of two tetrahedra.

Folding a 4–punctured sphere produces a clasp

To close off a braid (e.g. at the bridge regions of a two–bridge link), we fold the pleated surface S_{n-1} along its *peripheral edges*.

This closes off the braid while producing the last two crossings.