Some Open Problems for Complex Hénon Diffeomorphisms

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The dynamics of polynomial diffeomorphisms of $\mathbb{C}^2$ can be thought of as a bridge between one dimensional complex dynamics and higher dimensional complex dynamics.

Polynomial diffeomorphisms of $\mathbb{C}^2$ are some of the simpler systems in multivariable dynamics. Results which can be proved for these systems call out for extension to higher dimensions.

Because stable and unstable manifolds of saddle points are one dimensional polynomial diffeomorphisms of $\mathbb{C}^2$ are perhaps not too far from one dimensional systems.

The simplest interesting family of polynomial diffeomorphisms is the complex Hénon family.

$$f_{bc}(x,y) = (x^2 + c - by, x)$$

I want to advertise some unsolved problems and research directions connected with symbolic dynamics and complex Hénon maps.

This is a theme first suggested by John Hubbard.

From the point of view of dynamical systems there are a number of reasons to pay attention to elegant work in dynamics in one complex variable. Here are some:

- There are interesting symbolic dynamic models.

- There is a very good interaction with computer experiments.

- There is a great collection of concrete examples.

To my mind these are all important issues in dynamical systems in general. Do these solutions developed in the theory of dynamics in one complex variable extend to other families of maps?

Comments on symbolic dynamics.

If you understand a a dynamical system then it is a good exercise to attempt to describe it by means of symbolic dynamics.

Hyperbolic dynamical systems have Markov partitions and these can be used to give symbolic descriptions.

For example the doubling map on the circle can be described in terms of one sided infinite sequences of 0's and 1's.

In other cases there are symbolic descriptions more adapted to the system at hand.

Hubbard trees provide an example. These can be used to give neat descriptions of the topological dynamics of complex quadratic polynomial maps with periodic or preperiodic critical points.

John Milnor, "Periodic Orbits, External Rays and the Mandelbrot Set ..." Asterisque 261 (2000) 'Geometrie Complexe et Systemes Dynamiques', pp. 277-333 [Stony Brook IMS Preprint 1999 no. 3]

Symbolic descriptions of Julia sets in one dimension often involve external rays.

External rays are lines of steepest decent for the rate of escape function

$$G(z) = \lim_{n\to\infty} \frac{1}{2^n} \log |f^n(z)|.$$

If J is connected and f is well behaved all external rays land and the landing map is a continuous map from the space of external rays to the Julia set. In fact it is a semi-conjugacy between the doubling map on the circle of external rays and the map f restricted to J.

External rays in two dimensions

As Hubbard observed, in two variables the circle of external rays should be replaced by a solenoid.

"A compactification of Henon mappings in $\mathbb{C}^2$ as dynamical systems", John Hubbard, Peter Papadopol, Vladimir Veselov. Stony Brook IMS 1997/11. Dynamical Systems 9/15/97

The solenoid can be realized as the inverse limit of the doubling map on S^1.

A point on the solenoid can be coded by a bi-infinite sequence of 0's and 1's.

The solenoid was introduced in dynamics as an example of a hyperbolic attractor in $\mathbb{R}^3$.

There are other approaches to external rays in two dimensions.

Unstable manifolds of periodic saddle points are complex one dimensional manifolds. Let $\phi : \mathbb{C} \to \mathbb{C}^2$ be a parametrization.

Consider the forward rate of escape function:

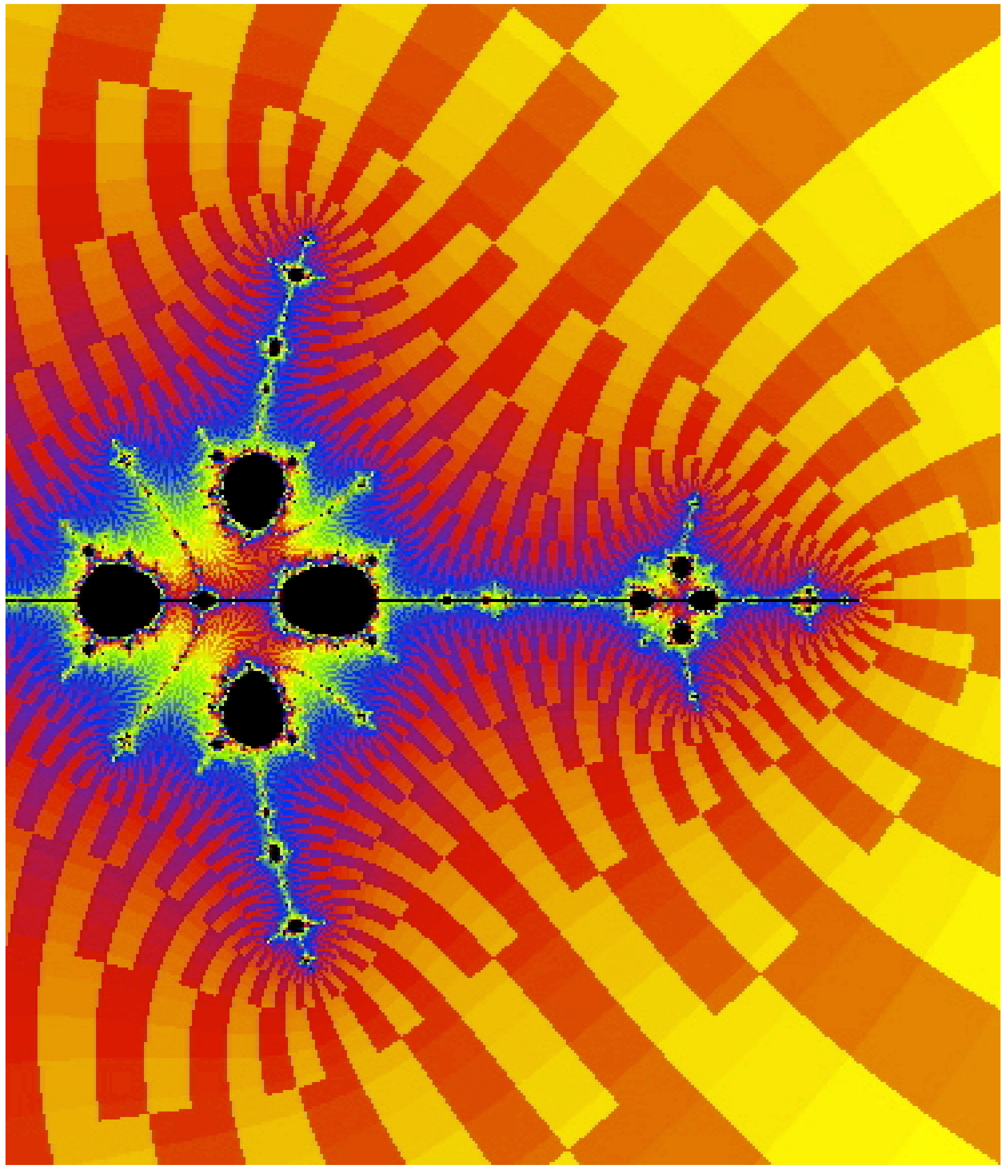
$$G^+(x,y) = \lim_{n\to\infty} \frac{1}{2^n} \log|\pi_x(f^n(x,y))|$$

Define external rays to be lines of steepest decent of $G^+ \circ \phi$.

This is definition is more directly accessible via computer calculation.

This is exploited in the computer program SaddleDrop created by Karl Papadontanakis and John Hubbard.

below: (c=-1.44, a=-.2). Karl Papadantonakis found these "cockroaches". The (
enables this coloration).



The checkerboard pattern on unstable manifolds corresponds to the coding of the solenoid by 0's and 1's. If we consider the regions through which a ray travels we get a bi-infinite sequence of 0's and 1's.

How do these unstable manifolds sit in $\mathbb{C}^2$?

Define K^- to be the set of points with bounded backward orbits. Let J^- be the boundary of this set. If we assume that the $|b| < 1$ then $J^- = K^-$.

If W^u is an unstable manifold of a periodic point then $W^u \subset J^-$ and W^u is dense in J^-¿

Define J^-_+ to be the collection of points with bounded backward orbit but unbounded forward orbit.

This is a natural place to look for unstable manifolds. In order to find them we need some laminar structure for J^-_+.

Theorem 1 (BS7). *If f is unstably connected and hyperbolic then the space of external rays is topologically conjugate to the solenoid. Furthermore all external rays land and the landing map is continuous.*

Remark: If $|b| \leq 1$ and J is connected then f is *unstably connected.*

Caveat: This theorem says less than it seems to. It does not specify the identification between the space of external rays and the standard solenoid.

In particular it does not say that the natural “SaddleDrop coding” provides an isomorphism. This is unfortunate from the point of view of using computer data to derive conclusions.

Problem: Resolve this.

There was an attempt to integrate the unstable manifold picture with the J_{+}^{-} picture in Ricardo Oliva's thesis.

Ricardo observed identifications of external rays in unstable manifolds and attempted to extrapolate all identifications.

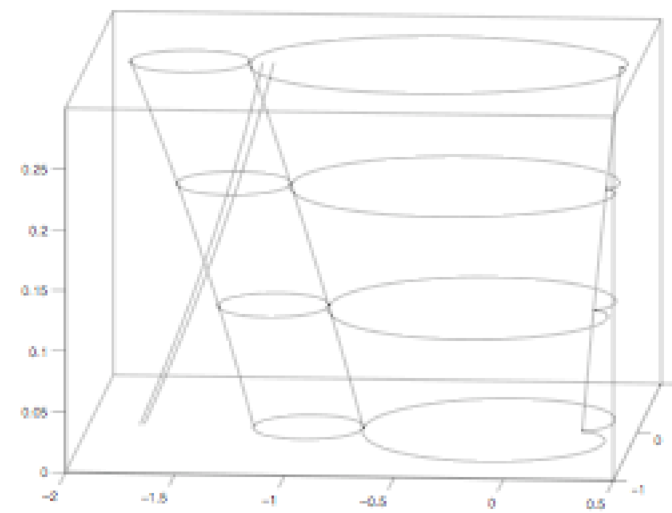


Figure 4.3: Sink loci in $\mathbf{C} \times \mathbf{R}$.

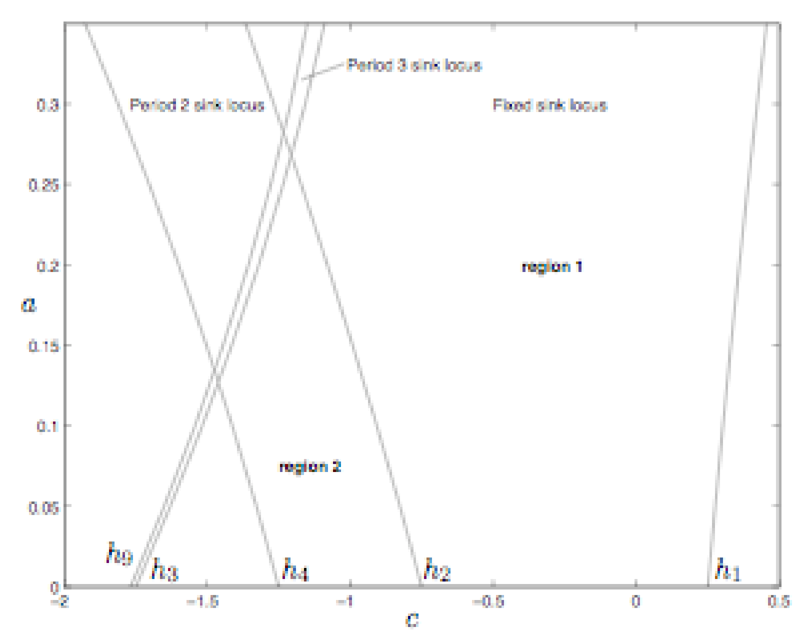


Figure 4.4: Sink loci in $\mathbf{R} \times \mathbf{R}$.

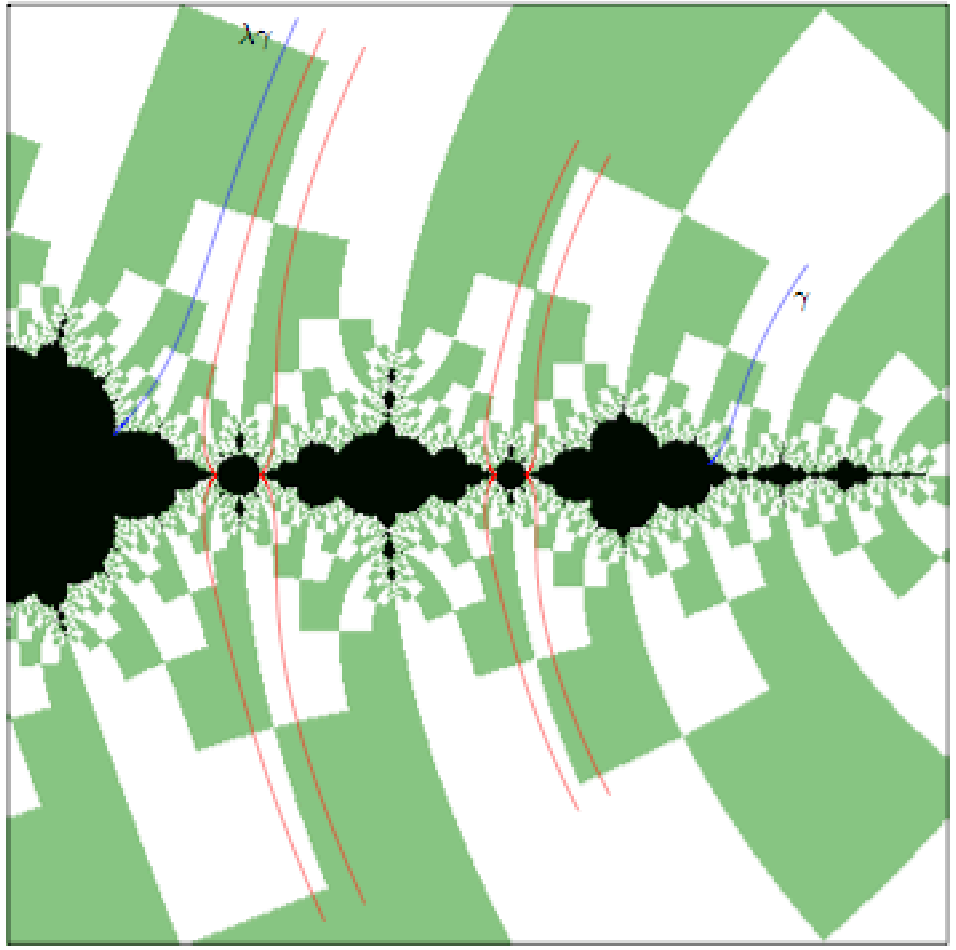


Figure 2.4: W^u_β picture with four real identifications in a fundamental domain.

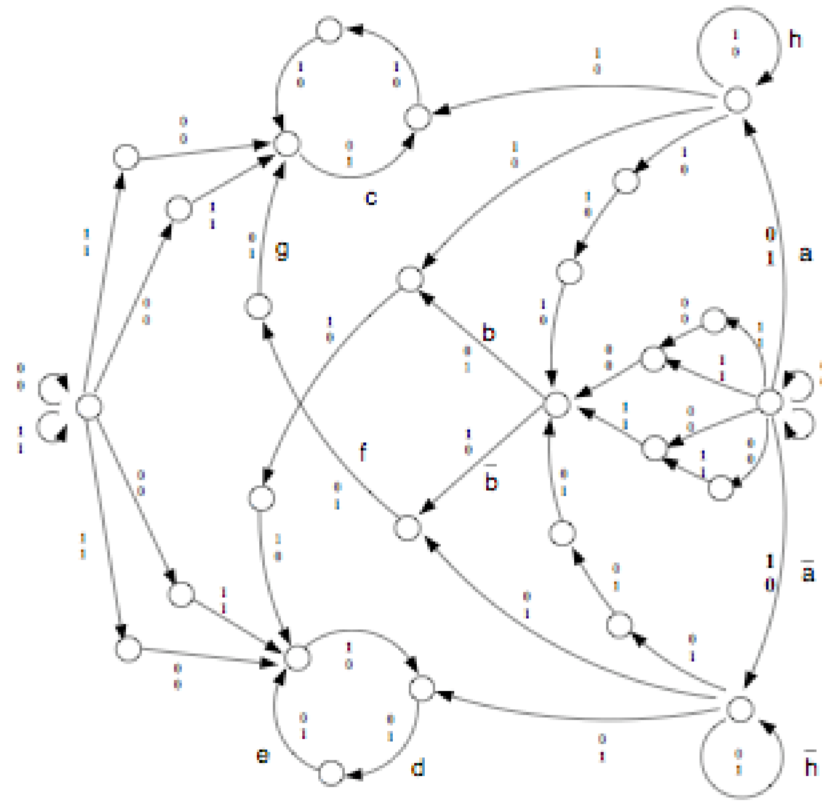


Figure 3.13: The graph $\Gamma_{(3,1)}$ with marked edges referenced in the proof of Lemma 22.

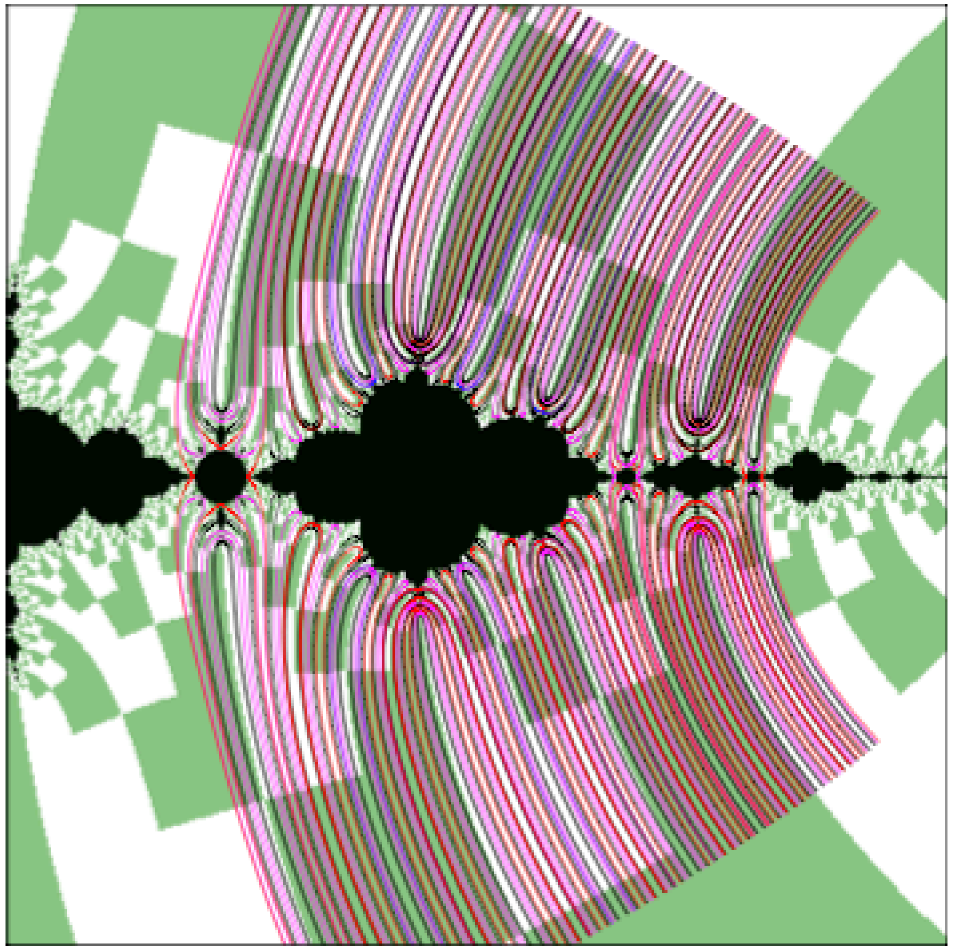


Figure 4.17: 252 pairs of rays generated by the computer from the graph $\Gamma_{(3,1)}$ of figure 3.11. The landing point of each pair was verified by the computer to be the same (to machine accuracy).

While this analysis carries a great deal of information it may not be in a particularly usable form.

Let us focus on one aspect of the 3-1 map: the immediate basin of the fixed sink.

In one variable the immediate basin of a sink is a circle and the first return map is a doubling map.

The fact that a sink basins has the same dynamics as the collection of all external rays is connected with combinatorial renormalization.

It would be reasonable to surmise that for Hénon maps the immediate basin should be a solenoid.

Hubbard and Oliva observed that the symbol sequences corresponding to external rays that land in the immediate basin appear to be all sequences which have no subsequences consisting of three 0's and no sequences of three 1's.

This leads to the following conclusions:

In the 3-1 map the immediate basin of the fixed sink is an expanding attractor in the sense of Williams.

It is the inverse limit of an expanding map on a branched one-manifold.

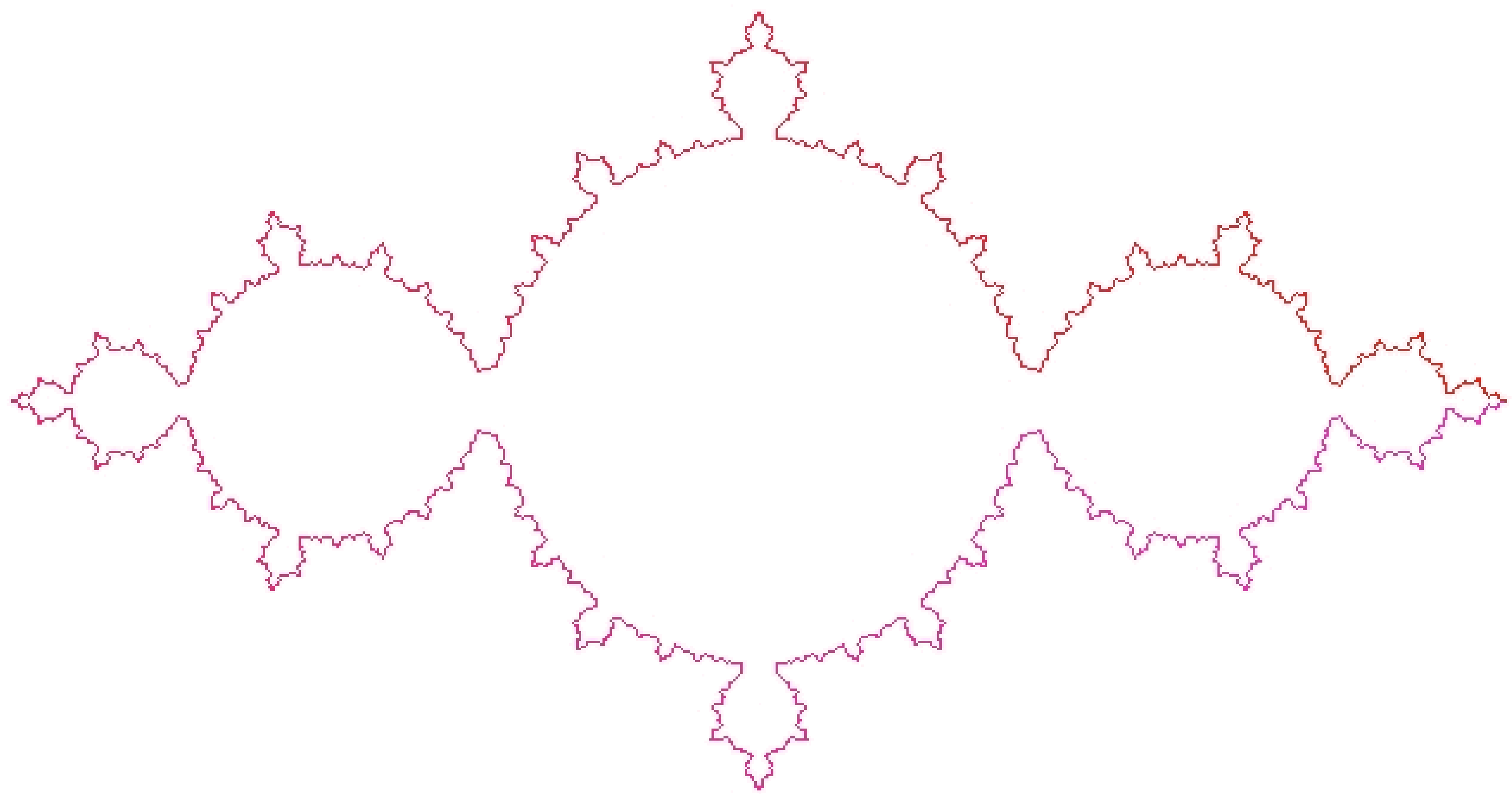
This set is not a solenoid.

The entropy of its first return map is the log of the golden ration.

It would be nice to have a computer algorithm that leads more directly to geometric information.

Philip Mummert, drawing on ideas from Biham-Wenzel and Sterling-Meiss has a method of showing approximations to the Julia set of complex Hénon maps.

a=-0.9+0i, b=0+0i, pt dist=5e-05, r=3



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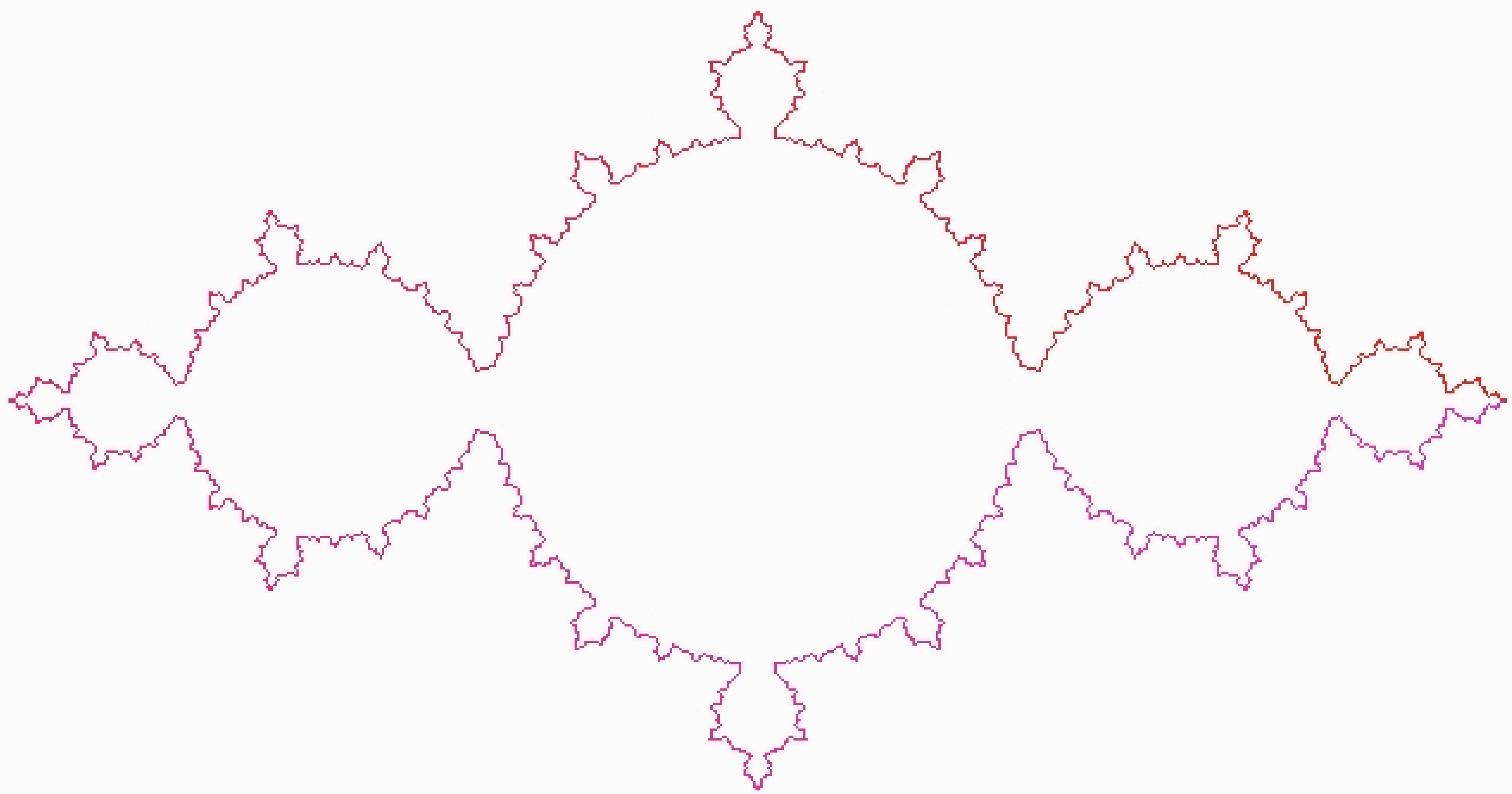


Hyperbolicity has appeared as an assumption in much of the work I have described. This is reasonable from the point of view of dynamical systems theory but not so reasonable from the point of view of our knowledge of one variable complex dynamics.

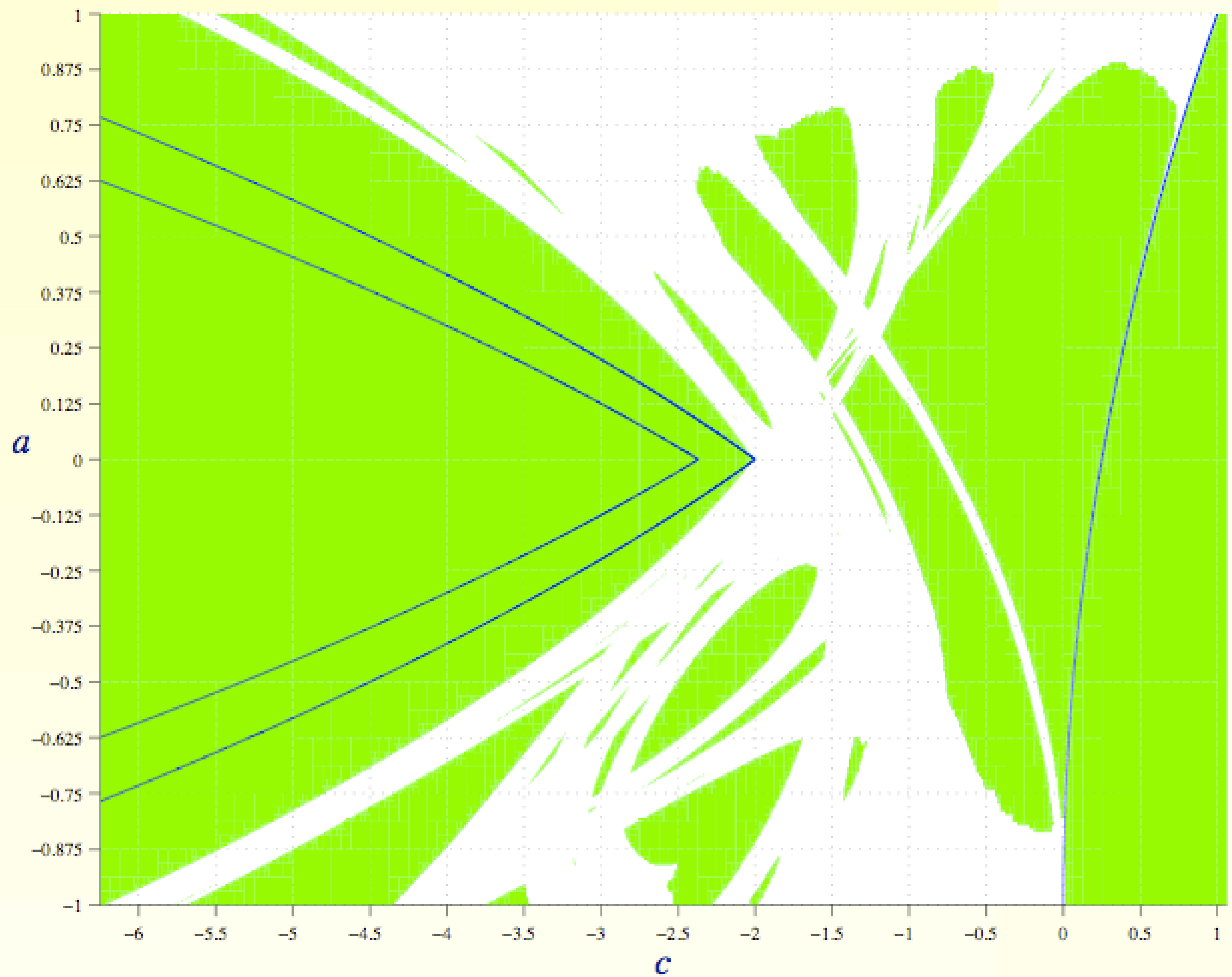


Figure 1: Hyperbolic plateaus of the real Hénon map

The Branner-Hubbard-Yoccoz theory of puzzles uses symbolic dynamics as a technique to analyze much larger classes of maps.

The landing patterns of external rays at periodic points give a decomposition of the complex plane which leads to a coding of orbits.

If we have to assume hyperbolicity perhaps we are assuming too much?

In Milnor's exposition of Yoccoz's proof of local connectivity of finitely renormalizable Julia sets he makes the following remark:

"The proof gives much more since it effectively describes the Julia set by a new kind of symbolic dynamics."

"Local connectivity of Julia sets: expository lectures" pp. 67-116 of "The Mandelbrot set, Theme and Variations" edit: Tan Lei, LMS Lecture Note Series 274 , Cambr. U. Press 2000

In the $\frac{1}{2}$ wake, $\mathcal{W}_{\frac{1}{2}}$, we can code points with "kneading sequences" of 0's and 1's.

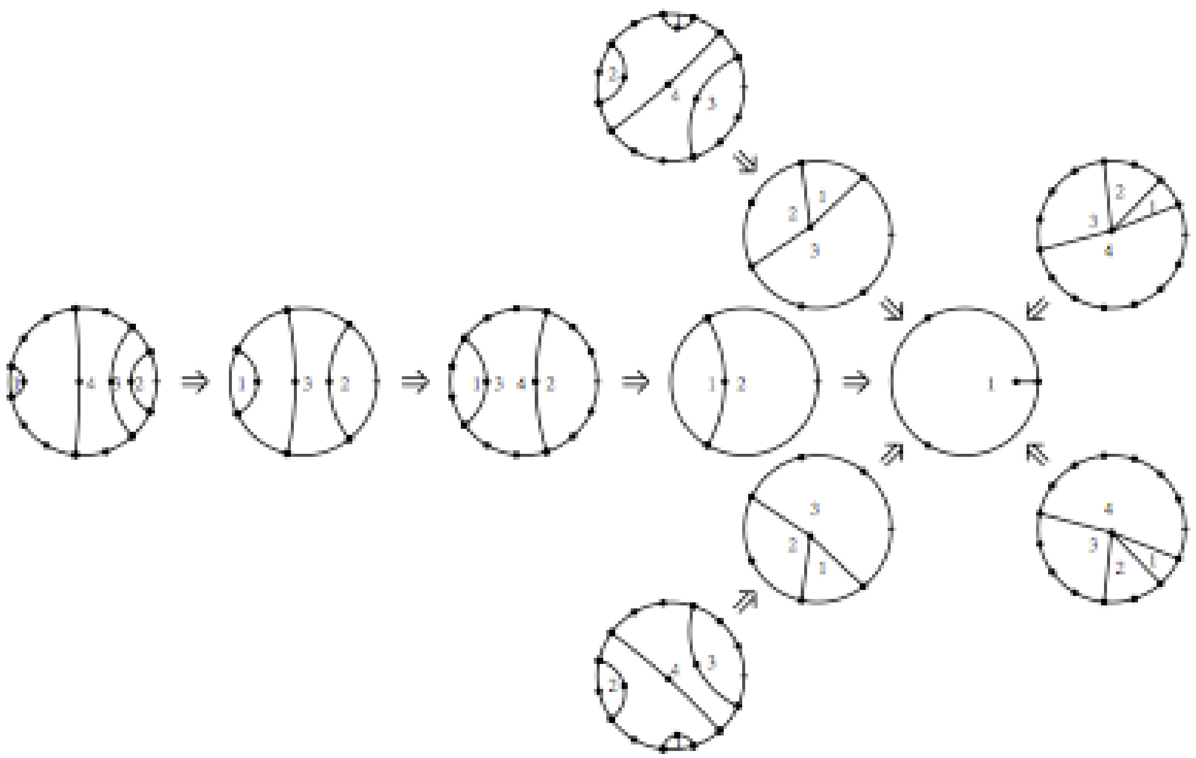


Figure 5. Forcing tree for the non-trivial orbit portraits of ray period $n \le 4$. Each disk in this figure contains a schematic diagram of the corresponding orbit portrait, with the first n forward images of the critical value sector labeled. (Compare Figure 4; and compare the "disked-tree model" for the Mandelbrot set in Douady [D5].)

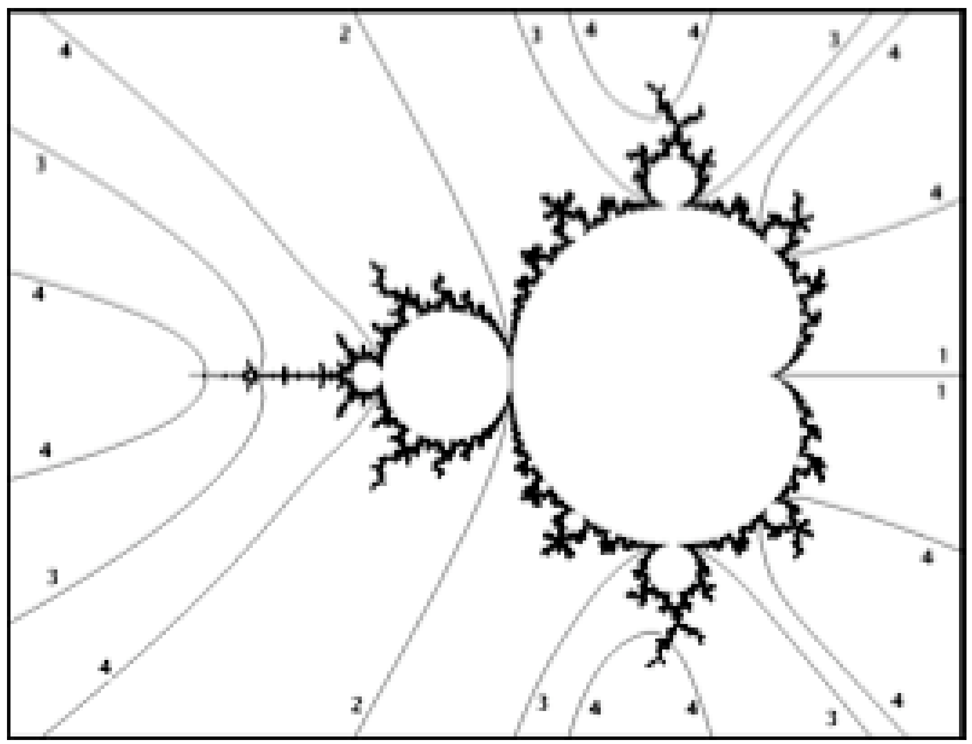


Figure 4. Boundaries of the wakes of ray period four or less.

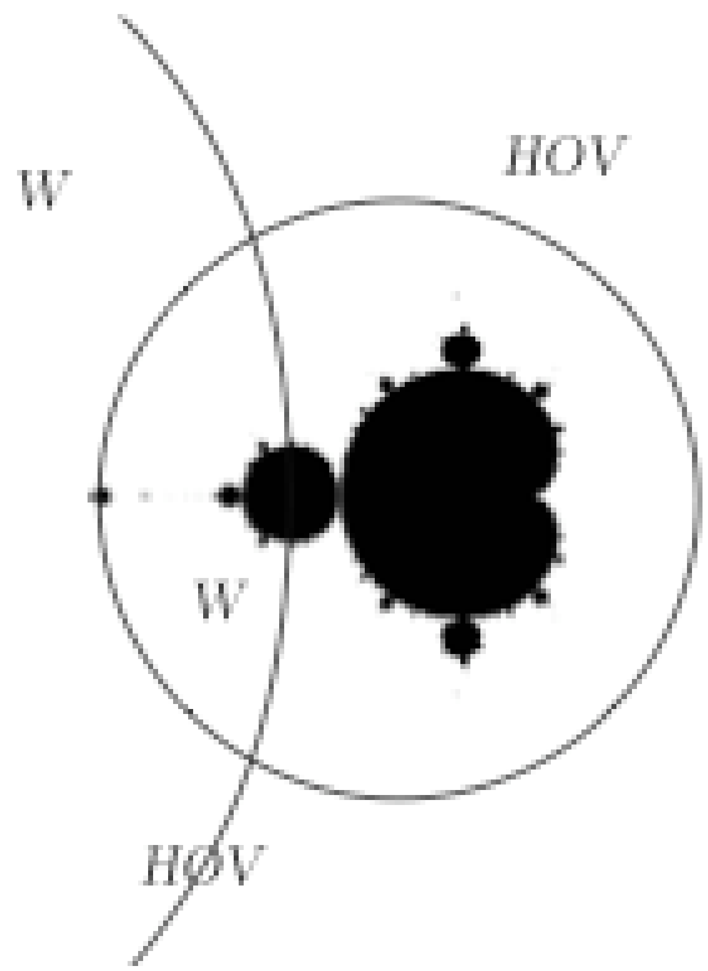


Figure 6. HOV, $\mathcal{W}$, and M

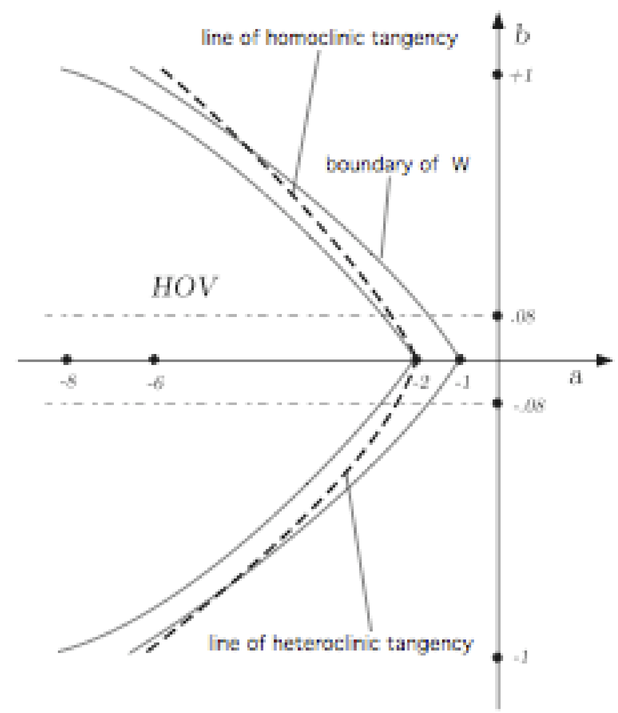


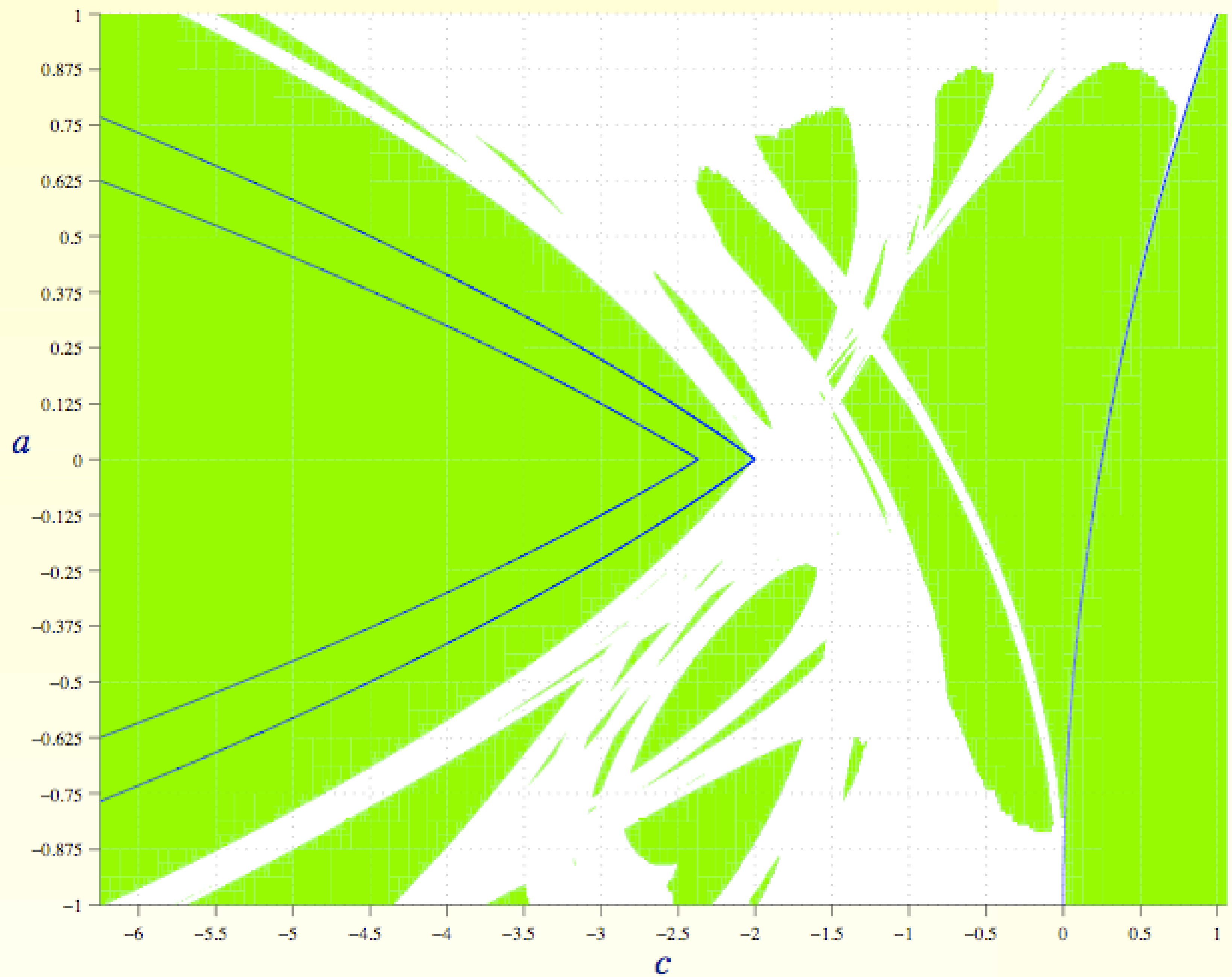
Figure 9. The real slice $\mathbf{R}^2 \cap \mathcal{W}$

In the $\frac{1}{2}$ wake, $\mathcal{W}_{\frac{1}{2}}$, we can code points with "kneading sequences" of 0's and 1's. Bedford and I extend this wake to a region in two variable parameter space which we call $\mathcal{W}_*$

Theorem 2 (BS). *Let f_{bc} be a real Hénon map with (b,c) in the region $\mathcal{W}_*$. Consider the collection of (real) periodic points with the coding sequence* $\ldots 0010100 \ldots$. *There are at most 4 such points.*

- *If there are 4 such points then f_{bc} is a hyperbolic horseshoe.*
- *If there are 3 such points then f_{bc} has a quadratic tangency.*
- *If there are less than 3 such points then f_{bc} has entropy less than* $\log 2$.

Totally disconnected case. Hubbard's vision of the prevalence of complex horseshoes in parameter space.



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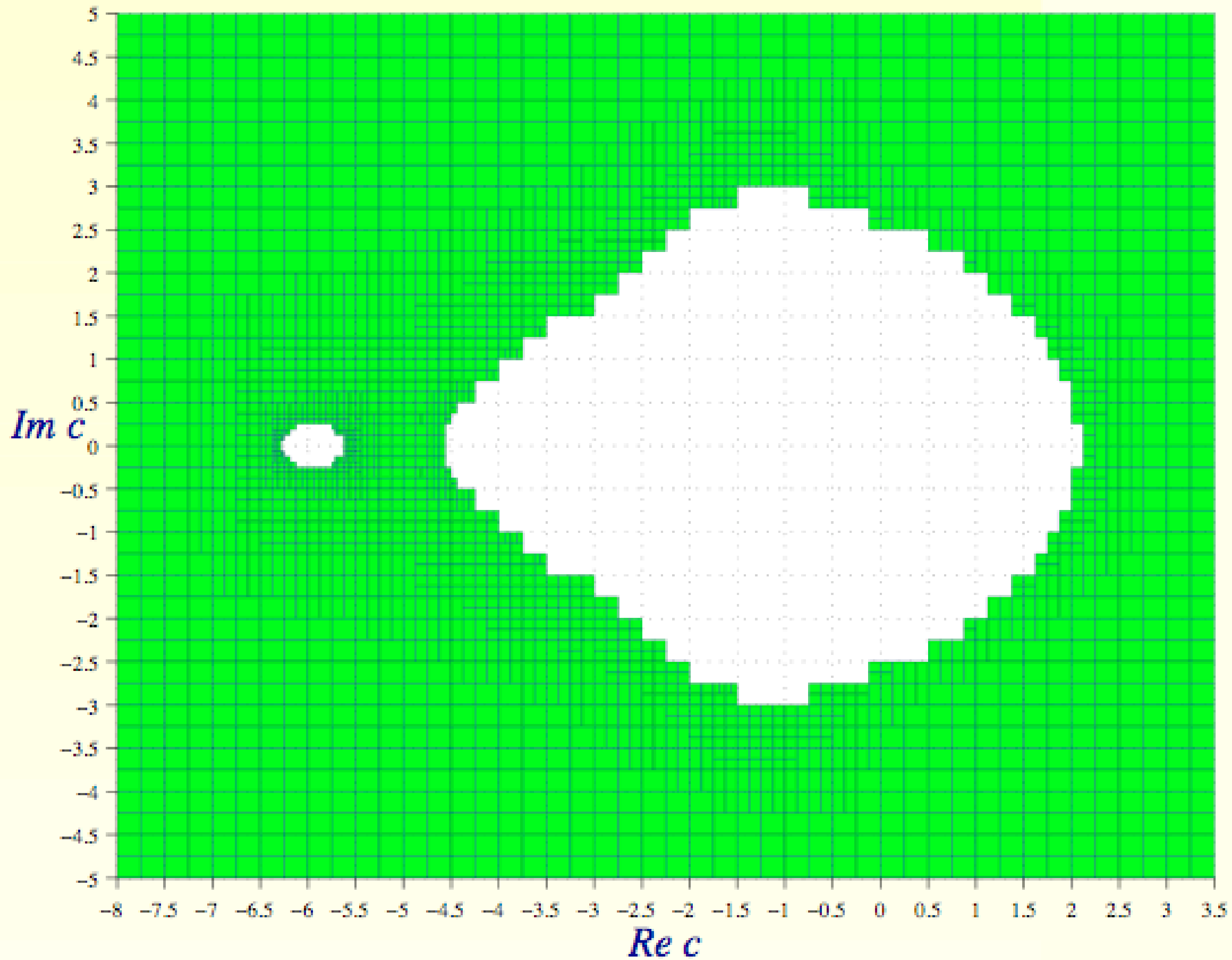

Figure 2: Hyperbolic Horseshoe Locus of $H_{-1,c}$

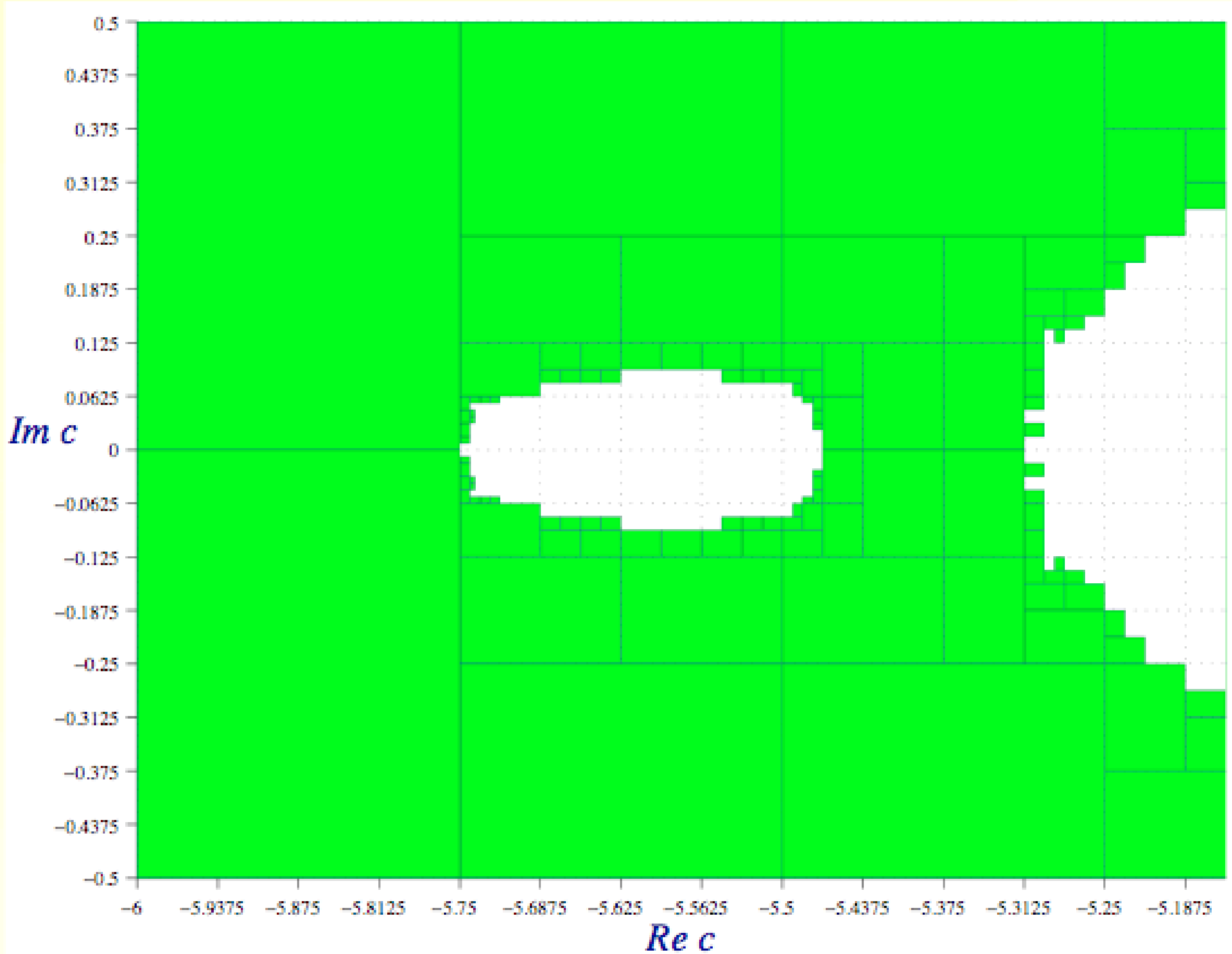

Figure 3: Hyperbolic Horseshoe Locus of $H_{1,c}$

Connection with the structure of parameter space.

Pick a loop in the constant Jacobian slice of parameter space which is symmetric with respect to complex conjugation. Say that this loop intersects the real axis at a point (b_0, c_0).

Bedford-Smillie observation: The monodromy around this loop is an involution of the full 2-shift whose fixed point set is the set of real points of f_{b_0,c_0}.

(Snowbird Conference Proceedings: 25 years after the discovery of the Mandelbrot set)