

$$T_+ \wedge T_- ?$$

R vertical closed  
S horizontal closed.



$$\langle R \wedge S, \varphi \rangle := \overline{\text{Im}} \langle T' \wedge S', \varphi \rangle$$

$$\begin{array}{l} R' \rightarrow R \\ S' \rightarrow S \end{array}$$

$$(R, S, \varphi) \rightarrow \psi(a, a') = \langle T_a \wedge T_{a'}, \varphi \rangle$$

$$\langle R \wedge S, \varphi \rangle = \overline{\text{Im}}_{(a, a')} \langle T_a \wedge T_{a'}, \varphi \rangle$$

$(a, a') \rightarrow T_a \wedge T_{a'}$   
continuous for the fine topology.

$R$  vertical  $C_r'(D)$ .

$$\psi_n(\theta) = \langle R_\theta, \frac{(f^n)^* \phi}{d^n} \rangle = \langle \frac{(f^n)^* R_\theta}{d^n}, \phi \rangle$$

$$\psi_n(0) = \psi_n(\theta) \rightarrow M_\phi$$

$$\overline{\text{lin}} \psi_n \subseteq M_\phi.$$

"geometry of discs"

$$\psi_n(1) \rightarrow M_{\mathbb{F}}.$$

How to define  $\mu$ ?

$F = (f, f^{-1})$  is horizontal  
of degree  $d^2$

$$T_n = \mathbb{R}^n \otimes S_n$$

$$\langle (F^n)^* T_n, \psi[\Delta] \rangle$$

So  $d^{-2n} (f^n)^* \mathbb{R}^n \wedge (f^n)^* S_n \rightarrow \mu.$

$\phi$   $(k-p, k-p)$  form.  $dd^c \phi \geq 0$

?  $\left\langle \frac{(f^n)^* R_n}{d^n}, \phi \right\rangle \rightarrow$  ?

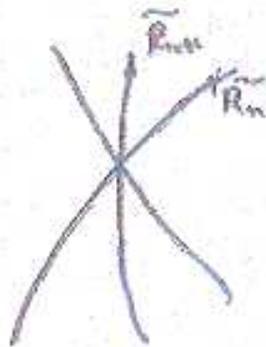
$\|R_n\|_v = 1.$

$k=2$  use of potentials of  $R_n$ .  
 $R_n = i\partial\bar{\partial}u_n$ .

1) Prove  $\lim \left\langle \frac{(f^n)^* R_n}{d^n}, \phi \right\rangle \leq M\phi$

Modify  $R_n$  so that

$\left\langle \frac{(f^n)^* \tilde{R}_n}{d^n}, \phi \right\rangle \rightarrow M\phi$



structural discs of smooth currents except at  $\tilde{R}_n$  passing through the same current.

$\psi_n(\theta) = \left\langle \tilde{R}_n, \theta \right\rangle, \frac{(f^n)^* \phi}{d^n}$   
 is subharmonic

$\psi_n(1) \rightarrow M\phi$  . . .  $\psi_n \rightarrow M\phi$  in  $L^1_{loc}$

"geometry of discs"

$\psi_n(0) \rightarrow M\phi$

How to move

$$\frac{(\int^n)^* R_n}{d^n} \rightarrow T_+$$

$R_n \geq 0$  closed bidim.  $(k-p, k-p)$   
slice mass 1.

How to define  $T_+ \wedge T_-$  ?

Structural disc of currents (in  $C_0(D)$ )

$\mathcal{R}$  current of bidim  $(k-p+1, k-p+1)$   
in  $V \times D$  :  $\text{supp}(\mathcal{R}) \cap \pi_V^{-1}(\text{compact})$

$(R_a)_{a \in V}$  slices of  $\mathcal{R}$   $\downarrow \pi_M$   
relatively compact in  $M$

$a \mapsto R_a$  structural disc.

Tool:  $\phi$   $(k-p, k-p)$  form on  $V \times D$   
 $dd^c \phi \geq 0$   $\pi_M(\text{supp} \phi) \ll N$

Then

$$a \mapsto \langle R_a, \phi \rangle$$

is p.s.h.

Thm  $f : D = M \times N \longrightarrow \mathbb{C}^k$  horizontal

$R$   $(p, p)$  form vertical support (not nec. closed)

$S$   $(k-p, k-p)$  form horizontal support (not nec. closed)

Then  $d^{-1}(f^n)^* R \longrightarrow \lambda_R T_+$   $\lambda_R \in \mathbb{R}$

$d^{-1}(f^n)_* S \longrightarrow \lambda_S T_-$   $\lambda_S \in \mathbb{R}$ .

$T_{\pm}$  are positive closed currents **independent of  $R$  and  $S$ .**

$$f^* T_+ = d T_+ \quad f_* T_- = d T_-$$

$$\mu = T_+ \wedge T_- \quad \text{probability measure on } \partial K = \partial K_+ \cap \partial K_-$$

Mixing of maximal entropy  
log d.

- Random iteration

- Speed of convergence.  
Solve  $i\partial\bar{\partial}$ .

- Bedford-Lyubich-Smillie Hénon maps

Fornaess-S  
Regul. autom.  $\mathbb{C}^k$   $S$ .

$\mathbb{R} : 1$

$C_n(D)$

horizontal currents bidim  $(p, p)$   
**positive closed**

$$f_* = (P_{21r})_* \circ (P_{11r})^*$$

acts continuously on  
horizontal currents

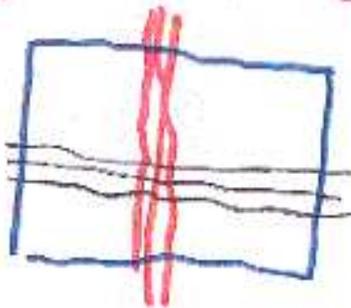
There is an integer  $d \geq 1$

$$\|f_*(S)\|_n = d \|S\|_n$$

$E_{nr}(D)$  vertical currents

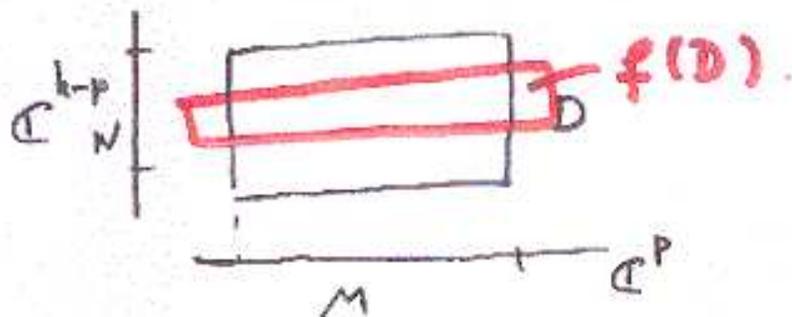
bidim  $(k-p, k-p)$

**positive closed**



$$\|f^*R\|_r = d \|R\|_r$$

## Horizontal like maps.



$$K_+ = \bigcap_n f^{-n}(D) \quad K_- = \bigcap_n f^n(D).$$

- 1) A horizontal like map is a graph  $\Gamma$   
 $\Gamma$  irreducible manifold of  $D \times D$
- 2)  $p_1|_{\Gamma}$  injective,  $p_2|_{\Gamma}$  has finite fibers
- 3)  $\Gamma$  does not intersect  $\overline{\partial_{in} D} \times \overline{D}$   
 nor  $\overline{D} \times \partial_{in} D$ .

Ex: i) Hénon maps in  $\mathbb{C}^2$   
 on a large polydisc.

ii) Regular automorphism of  $\mathbb{C}^k$   
 i.e. polynomial autom.

$$I_+ \cap I_- = \emptyset$$

" - - - - - hation -

Thm Assume  $d^* \ll d$   $f: U \rightarrow V$ .

i)  $\exists \mathcal{E}$  finite or countable union of analytic sets

$$z \notin \mathcal{E} \iff \frac{1}{d_t^n} \sum_{f^n(w)=z} \delta_w \rightarrow \mu.$$

ii)  $\exists x_{i,n}$  repelling period. pts of order  $n$

$$\frac{1}{d_t^n} \sum_i \delta_{x_{i,n}} \rightarrow \mu.$$

iii) Lyapunov exp.  $> 0$ .

iv)  $\mu$  is exponentially mixing for DSH or other spaces (by interpolation)

v) C.L.T.  $\varphi$  not a cocycle

$$\int \varphi d\mu = 0 \quad \varphi \text{ DSH...}$$

$$\lim_{N \rightarrow \infty} \mu \left( \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (\varphi \circ f^n) \in [a, b] \right) = \frac{1}{\sqrt{2\pi\sigma}} \int_a^b e^{-\frac{x^2}{2\sigma^2}} dx.$$

1. Decay correlation in  $\mathbb{P}^k$ , and  $\mathcal{E}$  pluripot. Fornæss-S (1994)  
Mixing.
2. In  $\mathbb{P}^k$  endomorphisms. i)  $\mathcal{E}$  algebraic ii) iii) Brändén-Duval  
1999-2001.
3. Polyn. like maps Dink-S 2003.

$$\left| \int (\varphi \circ f^n) \psi d\mu - \left( \int \varphi d\mu \right) \left( \int \psi d\mu \right) \right| \leq \frac{1}{\lambda^n} \|\varphi\|_{DSH} \|\psi\|_{L^\infty}.$$

$$\Lambda = \frac{1}{d_t} \mathbb{1}^*$$

$\varphi$  p.s.h.  $\varphi \in L^2(U)$

$\varphi = u + v$ .  $u \in \mathcal{H}$  pluriharmonic form  
 $v \in \mathcal{H}^\perp$  p.s.h.

$$\Lambda \varphi = (\underbrace{\Lambda_1 u}_{\mathcal{H}} + \underbrace{\Lambda_2 v}_{\mathcal{H}^\perp}, \Lambda_3 v)$$

$$dd^c \Lambda \varphi = dd^c(\Lambda_3 v) \quad *$$

$\Lambda_3 v$  is the best solution

of \*.

$$\|\Lambda_3 v\|_{L^2(U)} \leq c \|dd^c(\Lambda \varphi)\| \sim c \frac{d^*}{d_t}$$

$$d^* < d_t$$

$$\|\Lambda_3^n v\|_{L^2(U)} \leq c^n \quad c < 1.$$

$$\Lambda^n \varphi = (\Lambda_1^n u + \Lambda_1^{n-1} \Lambda_2 v + \Lambda_1^{n-2} \Lambda_2 \Lambda_3 v + \dots + \Lambda_2 \Lambda_3^{n-1} v, \Lambda_3^n v)$$

$$\|\Lambda^n \varphi - c \varphi\| \rightarrow 0 \quad \text{exponent. fast.}$$

Speed of convergence.

decay of correlation for  $\mu$ .

$$d^* = \sup_{\varphi} \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \|\varphi^n\|_{dd^c \varphi}^2, \|\varphi\|_{dd^c \varphi} \leq 1, dd^c \varphi \geq 0 \right\}$$

For compact Kähler manifolds  
 $d^*$  computed cohomologically.

Assumption  $d^*(f) < d_t(f)$ .

Condition open with respect to  $f$ .

Spectral properties of

$$\Lambda = \frac{1}{d_t} f_* \quad \text{in } L^2(\mu).$$

$\|\Lambda^n \varphi - c_\varphi\| \rightarrow 0$  exponentially fast

Space DSH.

stable  $f_x$  + good.  
compactness properties.

TOPOLOGY on DSH.

$$\|\phi^+\| + \|\phi^-\| + \|\Omega^+\| + \|\Omega^-\|.$$

bounded + convergence as currents -

Thm  $f: U \rightarrow V$  polynomial like  
 $\Omega$  smooth probability measure on  $V$ .

$$\frac{1}{d_t^n} (f^n)^* \Omega \rightarrow \mu.$$

$\mu$  probab. on  $\partial K$ . indep. of  $\Omega$   $f_* \mu = d_t \mu$ ,  $f_n^* \mu = \mu$

$\mu$  is of maximal entropy  $h_t(f) = h(\mu) = \log d_t$

$(f, \mu)$  is  $K$ -mixing.

Proof **Test functions  $\varphi$**

**p.s.h.** ( $i \partial \bar{\partial} \varphi \geq 0$ ,  $\varphi$  u.s.c.).

$$\varphi_n(z) := \frac{1}{d_t^n} \sum_{w \in f^{-n}(z)} \varphi(w).$$

$$\langle d_t^{-n} (f^n)^* \Omega, \varphi \rangle = \langle \Omega, d_t^{-n} (f^n)_* \varphi \rangle = \langle \Omega, \varphi_n \rangle$$

$$\varphi_{n+1}(z) = \frac{1}{d_t} \sum_{f(w)=z} \varphi_n(w)$$

$$\max_V \varphi_{n+1} \leq \max_V \varphi_n.$$

Max-principle + Hartogs.

$$\Rightarrow \varphi_n \rightarrow c_\varphi = \langle \Omega, c_\varphi \rangle =: \langle \mu, \varphi \rangle$$

Independent of  $\Omega$ !

$$d_t^{-n} (f^n)^* \Omega \rightarrow \mu.$$

Extension to several variables

DSH

$$f: U \rightarrow V \subset \mathbb{C}^k \quad U \Subset V$$

proper

$$\# f^{-1}(z) = d_f > 1.$$

$$K = \bigcap_{n \geq 1} f^{-n}(U).$$



"dynamics is repelling" critical pts.

Douady - Hubbard  $k=1$ .

Tool Measurable Riemann Mapping Thm.

On  $K$   $f$  is conjugate to a polynomial of degree  $d_f$ .

$k \geq 2$   $f$  not necess. conj. to polynomial.

Ex: i)  $f: \mathbb{P}^k \rightarrow \mathbb{P}^k$  endomorphism

$$F: \mathbb{C}^{k+1} \rightarrow \mathbb{C}^{k+1} \quad f(B_R) \xrightarrow{f} B_R$$

$R \gg 1.$

perturbo with "transcendental" map.

ii) Meromorphic maps region  $U \xrightarrow{f} V$   
polynomial like.

$\rightarrow$  In several variables polynomials  $\not\Rightarrow$  polynomial like maps

Two ideas useful to study these problems.

1. Good choice of space of test functions (test forms) to study convergence.  
dd<sup>c</sup> method.

$$i \partial \bar{\partial} \phi = \phi_2$$

$$i \int \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j = \phi_2 \quad (1,1) \text{ form.}$$

2. How to deal with convergence of positive closed currents of bidegree  $(p,p)$   $p > 1$ .

$$T_n \rightarrow T$$

Bidegree  $(1,1)$

$$u_n \xrightarrow{C^1} u \quad T_n = i \partial \bar{\partial} u_n \quad i \partial \bar{\partial} u = T$$

?

For ex:

$$[z:w:t] \rightarrow [z^2+awt:zt:t^2]$$

Hénon map.

Instead of  $\mathbb{P}^k$

$X$  algebraic manifold.

$f: X \rightarrow X$  a meromorphic  
correspondance.

Any algebraic manifold has  
dynamically interesting correspondance  
(positive entropy).

3. Automorphisms of compact  
Kähler manifolds.

4.  $X = \prod_{n=1}^{\infty} \{ \text{proj. space of homogeneous polys. of degree } n \text{ on } \mathbb{P}^k \}$

For which measures  $\nu$  on  $X$ .

$$\frac{[P_n'(0)]}{n} \rightarrow \omega \text{ (Kähler form on } \mathbb{P}^k)$$

$\nu$ . a. e.

at what speed?

# dd<sup>s</sup> method in dynamics.

with Tien Cuong Dinh.

Examples of dynamical systems.

1.  $X$  complex manifold (ex: open set of  $\mathbb{C}^k$ )  
 $f: U \subset X \rightarrow X$  holomorphic or meromorphic

$$f^n = f \circ \dots \circ f : U_n \rightarrow X$$

introduce the right objects to describe the dynamics.

2.  $f: \mathbb{P}^k \rightarrow \mathbb{P}^k$  meromorphic

$$\begin{array}{ccc} \mathbb{C}^{k+1} \setminus \{0\} & \xrightarrow{F} & \mathbb{C}^{k+1} \setminus \{0\} \\ \pi \downarrow & & \downarrow \\ \mathbb{P}^k & \xrightarrow{f} & \mathbb{P}^k \end{array}$$

$$[z_0 : z_1 : \dots : z_k] \xrightarrow{f} [P_0(z) : \dots : P_k(z)]$$

$P_j$  homogeneous,  $\deg P_j = d \geq 2$ .

$$I = \{z \mid P_j(z) = 0 \quad 0 \leq j \leq k\}$$

indeterminacy.

$\text{codim } I \geq 2$ .

$f$  endomorphism.  $I = \emptyset$ .

- Maps non continuous  $(z, w) \rightarrow z/w$

- non conformal