Currents and measures in parameter space

(joint work with Charles Favre)

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Introduction

1 dim. dynamics

- canonical invariant measure
- equidistribution of preimages of pts and periodic orbits

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higher dim. dynamics

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- Invariant measure: equidistribution of periodic orbits

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higher dim. parameter space

- which objects?
(cf. DeMarco, Bassanelli-Berteloot)
- which equiditribution?

Positive closed currents

Let Ω be a complex manifold.

- Currents of dimension k are dual to differential forms of degree k:
$$\varphi \mapsto \langle T, \varphi \rangle.$$
- Natural d operator: $\langle dT, \varphi \rangle = (-1)^{\deg \varphi} \langle T, d\varphi \rangle$. T is closed if $dT = 0$.
- Currents of bidimension (k, k) act non trivially only on forms of bidegree (k, k)
- Natural notion of positivity, invariant under holomorphic mappings.
- Topology considered: weak topology.

Positive closed currents

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- Currents of dimension k are dual to differential forms of degree k:
$$\varphi \mapsto \langle T, \varphi \rangle.$$
- Example 1: if u is a function, $i\partial\bar{\partial}u$ defines a current of bidegree $(1,1)$: $\langle i\partial\bar{\partial}u, \varphi\rangle = \int u(i\partial\bar{\partial}\varphi)$.
Let $dd^c = i\partial\bar{\partial}$. u is psh $\Leftrightarrow$ $dd^c u$ is positive.
- Example 2: if V is a complex subvariety of (complex) dimension k,
$$\varphi \mapsto \int_V \varphi$$
defines a positive closed current of bidimension (k,k)

Marked critical points

Consider a family $(f_\lambda)_{\lambda\in\Lambda}$ of rational maps on $\mathbb{P}^1$, of degree d, with a holomorphically varying (marked) critical point $c(\lambda)$.

DEFINITION. $c(\lambda)$ is passive at λ_0 if $\left\{f_\lambda^n(c(\lambda))\right\}$ is normal near λ_0. Otherwise it is active.

Assume the f_λ are polynomials. Then

$$T = dd^c G_{f_\lambda}(c(\lambda))$$

is a positive closed current on Λ associated to c.

In the case of the space of quadratic polynomials this is the harmonic measure of the Mandelbrot set.

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In general, following DeMarco, $(f_\lambda, c(\lambda))$ admits an associated positive closed current T.

THEOREM. $\mathrm{Supp}(T) =$ activity locus of c

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DEFINITION. $c(\lambda)$ is passive at λ_0 if $\left\{f_\lambda^n(c(\lambda))\right\}$ is normal near λ_0. Otherwise it is active.

FACT. if c is active at λ_0, there exist parameters λ arbitrary close to λ_0 where c is (pre)periodic.

(Classical consequence of Montel's theorem)

Equidistribution

Assume the parameter space Λ is quasiprojective.
For $0 \leq k(n) < n$, let

$$\mathrm{Per}(n, k(n)) = \left\{ \lambda,\ f_\lambda^n(c(\lambda)) = f^{k(n)}(c(\lambda)) \right\}.$$

THEOREM. The following convergence statements hold

$$\begin{cases} \frac{1}{d^n + d^{k(n)}} [\mathrm{Per}(n, k(n))] \to T & \text{when the } f_\lambda \text{ are rational maps} \\ \frac{1}{d^n} [\mathrm{Per}(n, k(n))] \to T & \text{when the } f_\lambda \text{ are polynomials} \end{cases}$$

EXAMPLE. if $\Lambda = \mathbb{C}$ is the space of quadratic polynomials, this is the equidistribution of centers of components (resp. Misiurewicz points) towards the harmonic measure of the Mandelbrot set.

Two marked critical points.

Consider an algebraic family $(f_\lambda, c_1(\lambda), c_2(\lambda))$, with 2 independent critical points ($\dim \Lambda \geq 2$).

QUESTION. Assume both c_1 and c_2 are active at λ_0, can we find a nearby parameter λ for which both are preperiodic?

Recall that there is no analogue of Montel's Theorem in higher dimension so the classical approach fails.

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On the other hand $T_1 \wedge T_2$ is bidegree (2,2) current such that

$$\mathrm{Supp}(T_1 \wedge T_2) \subset \mathrm{Supp}(T_1) \cap \mathrm{Supp}(T_2) = \{c_1 \text{ active}\} \cap \{c_2 \text{ active}\} .$$

PROP. Near every $\lambda_0 \in \mathrm{Supp}(T_1 \wedge T_2)$ there is a λ for which c_1 and c_2 are preperiodic.

PROOF. $T_1 \wedge T_2 = \lim\limits_{n\to\infty} \lim\limits_{m\to\infty} \dfrac{[\mathrm{Per}_1(n, k(n)) \cap \mathrm{Per}_2(m, k(m))]}{(d^n + d^{k(n)})(d^m + d^{k(m)})}$.

Cubic polynomials

For simplicity we now work with cubic polynomials with both critical points marked but the results are valid for all degrees.

Let $\Lambda = \mathbb{C}^2_{(c,v)}$ and $f_{c,v}(z) = z^3 - 3c^2 z + 2c^3 + v$ (Kiwi).
Critical points: $+c$ and $-c$.

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Let $\mathcal{C}^\pm = \{(c,v),\ \pm c$ has bdd orbit$\}$.
$\mathcal{C}^\pm$ are closed and the activity locus of $\pm c$ is precisely $\partial\mathcal{C}^\pm$.

$\mathcal{C} = \mathcal{C}^+ \cap \mathcal{C}^-$ is the connecteness locus, which is compact in $\mathbb{C}^2$ (Branner-Hubbard).

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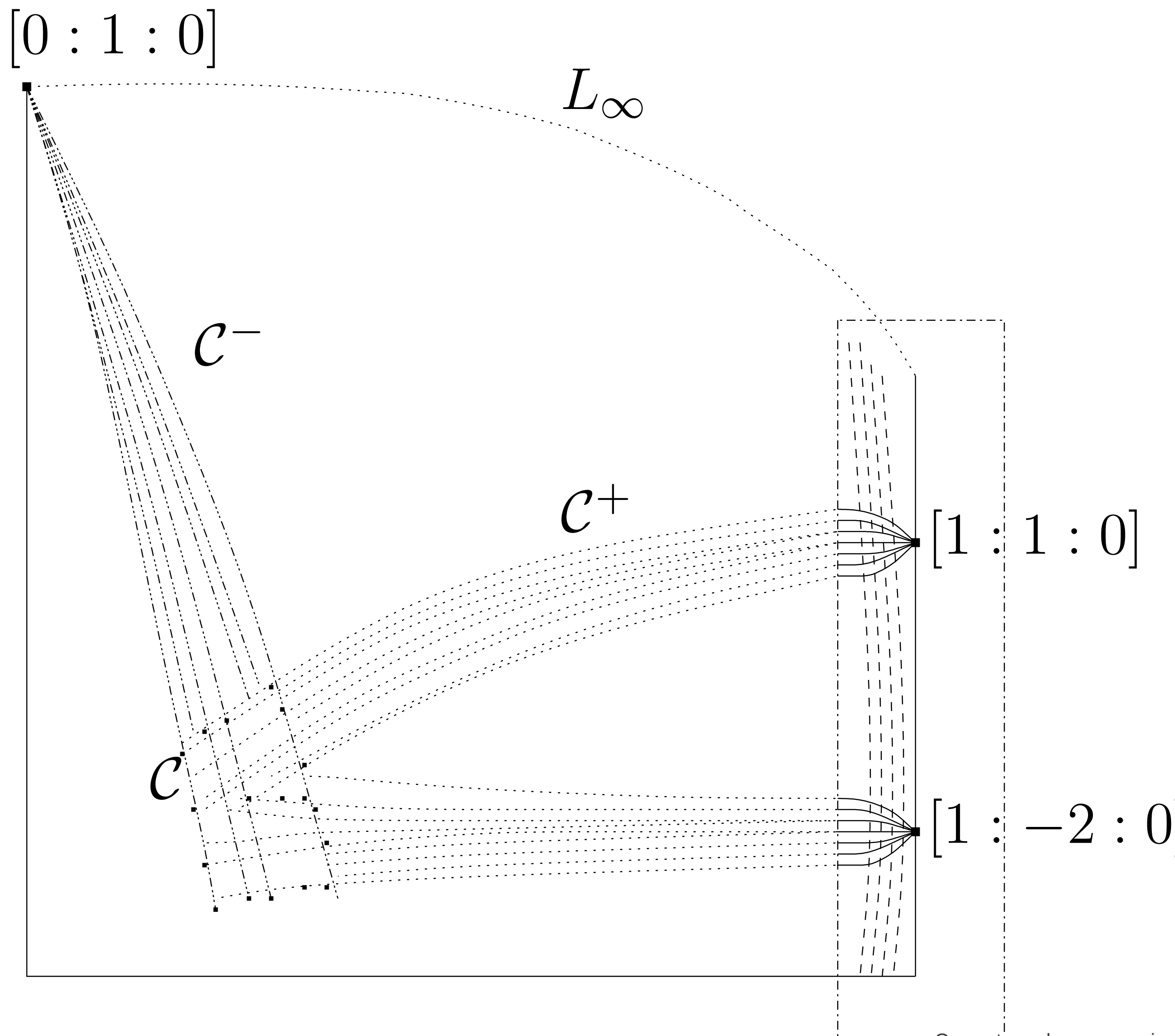
Let $G^\pm(c,v) = G_{f_{c,v}}(\pm c)$ and T^+ and T^- be the associated currents, $T^\pm = dd^c G^\pm$.

$\mathrm{Supp}(T^\pm) = \partial\mathcal{C}^\pm$

Cubic polynomials: schematic picture

p11 p12



Misiurewicz points

Let $\mu_{\rm bif} = T^+ \wedge T^-$. The bifurcation measure μ is a positive measure with support in $\partial\mathcal{C}^+ \cap \partial\mathcal{C}^-$.

DEFINITION. f is of Misiurewicz type if all critical points are strictly preperiodic.

We already know that:

$$\mathrm{Supp}(\mu_{\rm bif}) \subset \overline{\{\text{Misiurewicz pts}\}} \subset \partial\mathcal{C}^+ \cap \partial\mathcal{C}^-.$$

THEOREM. $\mathrm{Supp}(\mu_{\rm bif}) = \overline{\{\text{Misiurewicz pts}\}} \neq \partial\mathcal{C}^+ \cap \partial\mathcal{C}^-$.

Other definitions of μ_{bif}

Let $\mathrm{Lyap}(c, v)$ be the Lyapounov exponent of the maximal entropy measure of $f_{c,v}$:

$$\mathrm{Lyap}(c, v) = \log 3 + G_{c,v}(c) + G_{c,v}(-c).$$

In particular the "bifurcation current" (DeMarco) is

$$T_{\mathrm{bif}} = dd^c\mathrm{Lyap} = T^+ + T^-.$$

FACT. $(T^+)^2 = (T^-)^2 = 0$.

$\Rightarrow \mu_{\mathrm{bif}} = T^+ \wedge T^- = (dd^c\mathrm{Lyap})^2$.

(previously studied by Bassanelli-Berteloot)

Other definitions of $\mu_{\rm bif}$

PROP. $\mu_{\rm bif} = (dd^c \max\{G^+, G^-\})^2$.

COR. $\mu_{\rm bif}$ is the "pluriharmonic measure" of the connectedness locus.

(natural object from the point of view of complex analysis).

$\Rightarrow$ (Bedford-Taylor) $\mathrm{Supp}(\mu_{\rm bif})$ is the Shilov boundary of $\mathcal{C}$.

Some geometric intuition

There is a natural stratification of the space in terms of the number of critical points being active (picture):

$$\partial\mathcal{C}^+ \cap \partial\mathcal{C}^- \subset \partial\mathcal{C}^+ \cup \partial\mathcal{C}^- \subset \mathbb{C}^2.$$

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Define:

- the bifurcation locus $\mathrm{Bif}_1 = \partial\mathcal{C}^+ \cup \partial\mathcal{C}^- = \mathrm{Supp}\, T_{\mathrm{bif}}$
- the secondary bifurcation locus $\mathrm{Bif}_2 = \mathrm{Supp}\, \mu_{\mathrm{bif}}$

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Intuitively:

- a polynomial f in $\mathrm{Bif}_1 \setminus \mathrm{Bif}_2$ is unstable in $\mathbb{C}^2$ but should have a 1-parameter family of qc-deformations.
- a polynomial in Bif_2 should be "rigid".

Laminarity

(only valid for cubic polynomials)

Assume Δ is a holomorphic disk in $\partial\mathcal{C}^+ \setminus \partial\mathcal{C}^-$. Then

- $G^+ = 0$ on Δ so c is passive.
- either $-c$ escapes so it is passive, or we are in $\mathrm{Int}(\mathcal{C}^-)$ so $-c$ is again passive.

Both critical points are passive on Δ so the dynamics is J-stable.

$\Rightarrow$ all disks contained in $\mathrm{Bif}_1 \setminus \partial\mathcal{C}^+ \cap \partial\mathcal{C}^-$ are disks of stability.

Laminarity

(only valid for cubic polynomials)

THEOREM. T^+ and T^- (hence T_{bif}) are laminar outside $\partial\mathcal{C}^+ \cap \partial\mathcal{C}^-$.

So through T_{bif}-a.e. point in $\mathrm{Bif}_1 \setminus \partial\mathcal{C}^+ \cap \partial\mathcal{C}^-$ there is a disk of qc-deformation. (picture)

On the other hand:

THEOREM. For μ_{bif}-a.e. parameter λ, there is no holomorphic disk through λ and contained in $\mathcal{C}$.

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REMARK. The right definition of Bif_2 is not completely clear. For instance we could take the closure of the set of rigid parameters (contains $\mathrm{Supp}(\mu)$).