Julia sets with
POSITIVE MEASURE

After X. Buff and A. Chéritat

by Adrien Douady
Université Paris Sud

THANKS

Thanks to Misha Lyubich, Misha Yampolsky, John Smillie, and all those who have taken part in the organization.

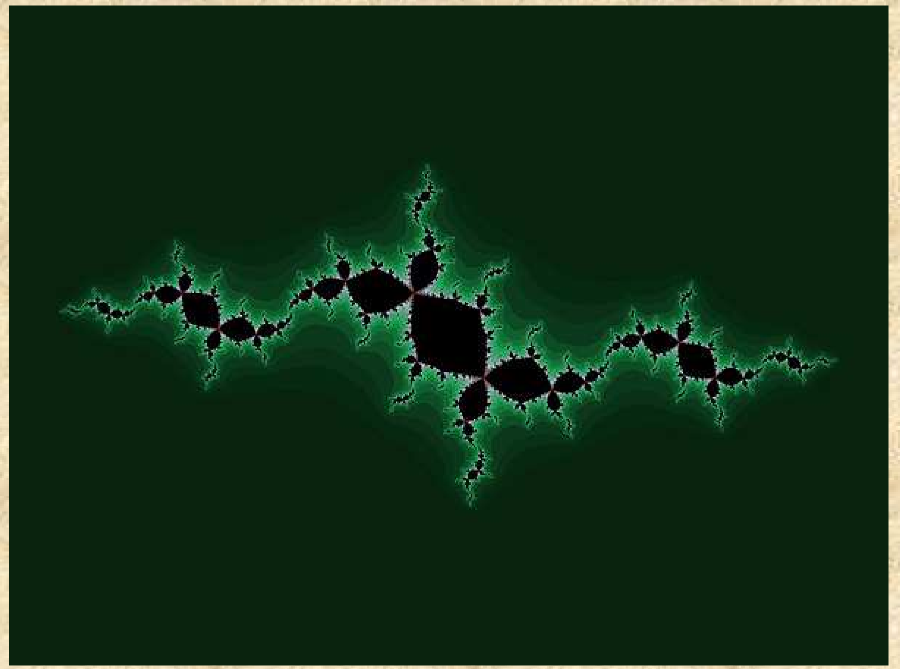
In particular the staff of Fields Institute.

I. THE CONTEXT

JULIA SETS

$f : C \mapsto C$ polynomial

Filled Julia set : $K(f) = \{ z \mid f^n(z) \qquad \}$ $\nrightarrow$



K connected
Int (K) not empty

$m(J) = 0$

Actual Julia set : $J(f)$ = boundary of $K(f)$
= closure of repelling periodic points (2)
= set of points of non-normality (3)

$f : \bar{C} \mapsto \bar{C}$ rational : definition (2) or (3)

Theorem : $J(f) =$ or $\bar{\mathbb{C}}$ $Int(J(f)) = \emptyset$

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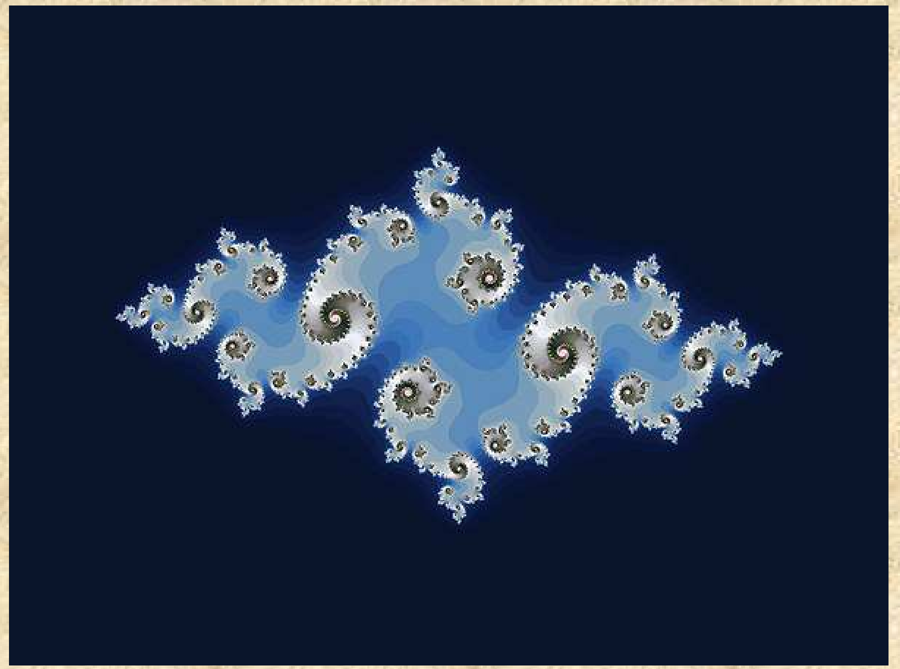

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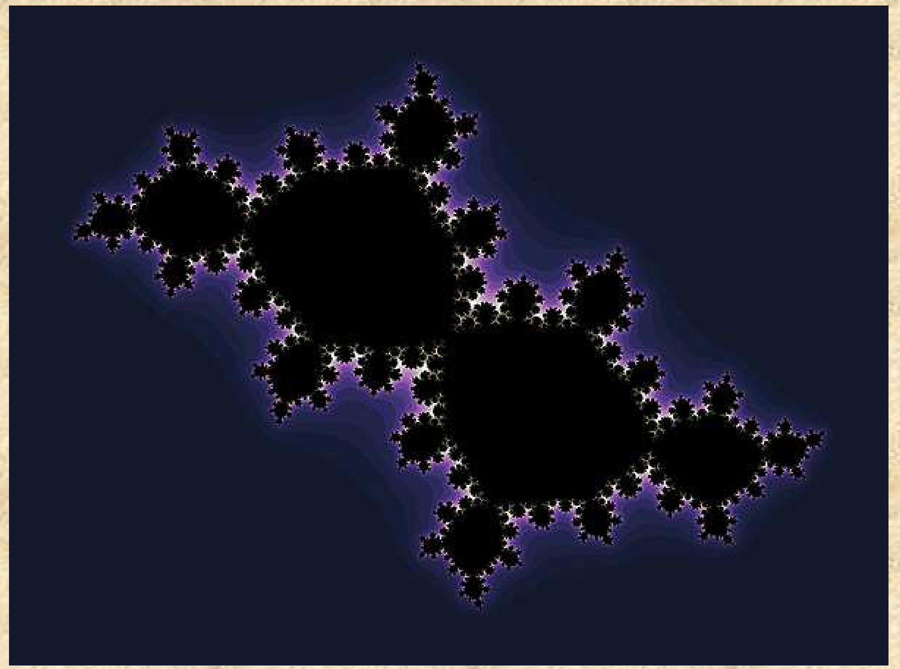

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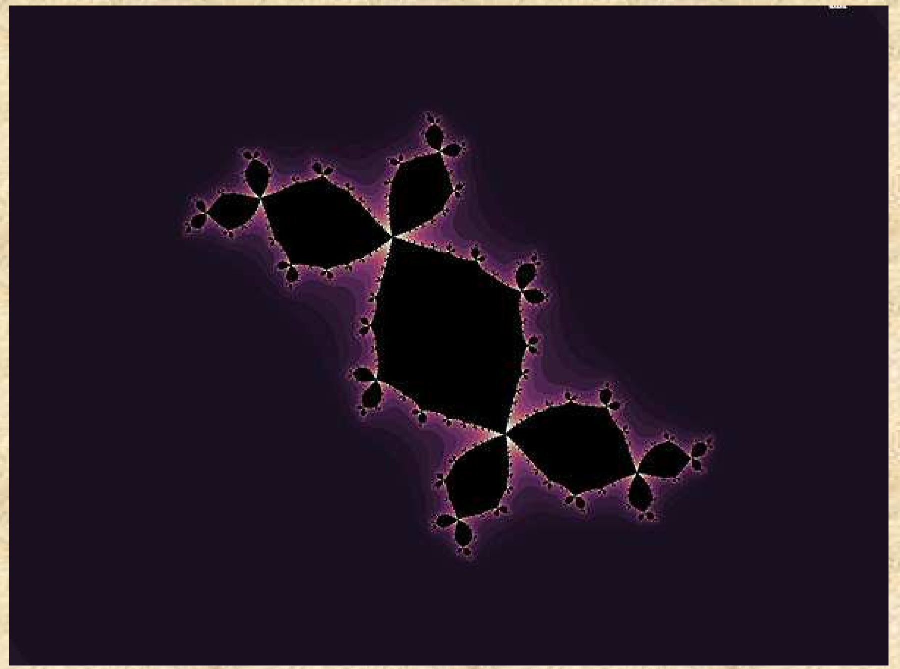

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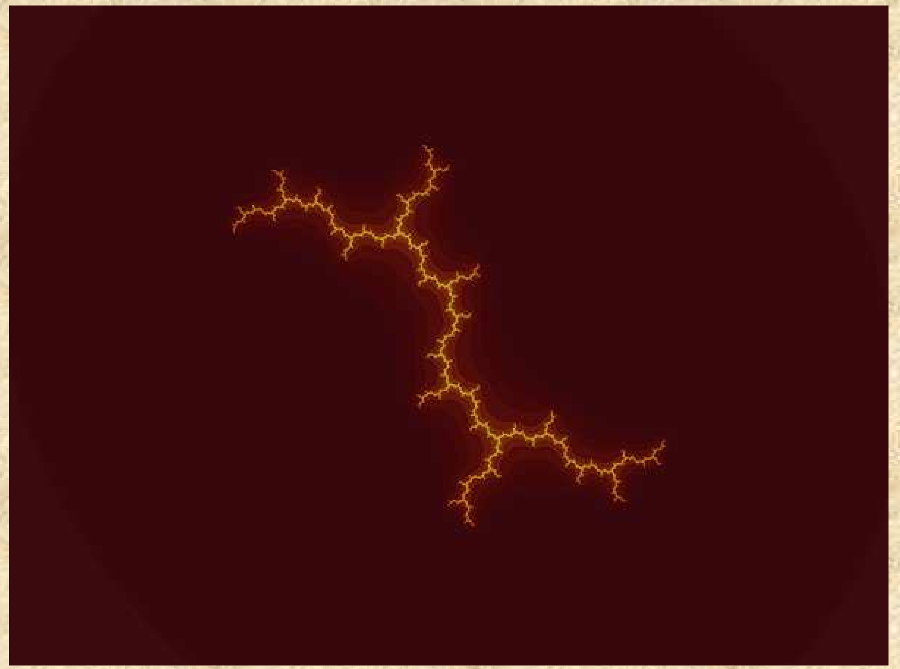

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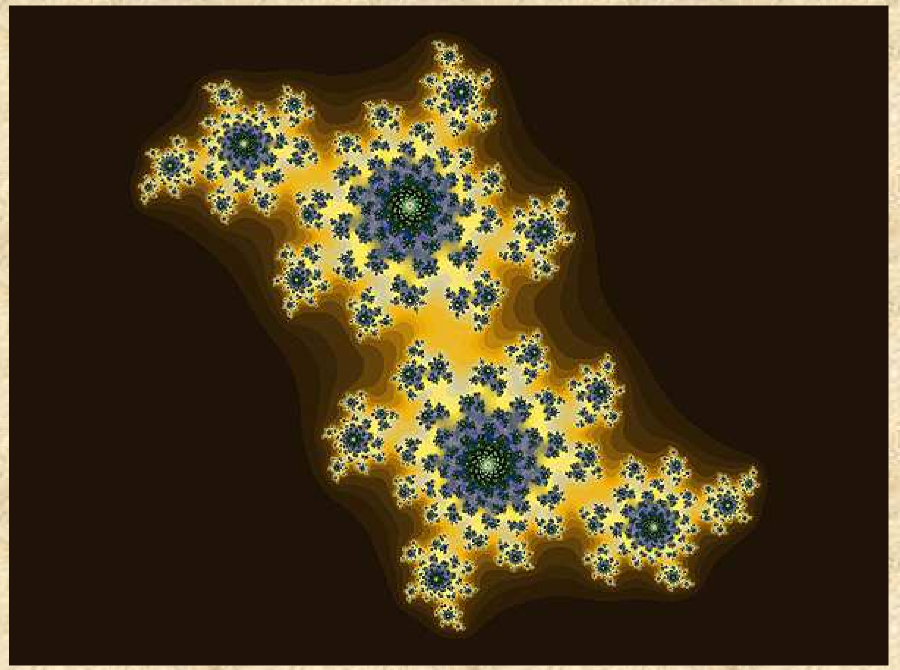

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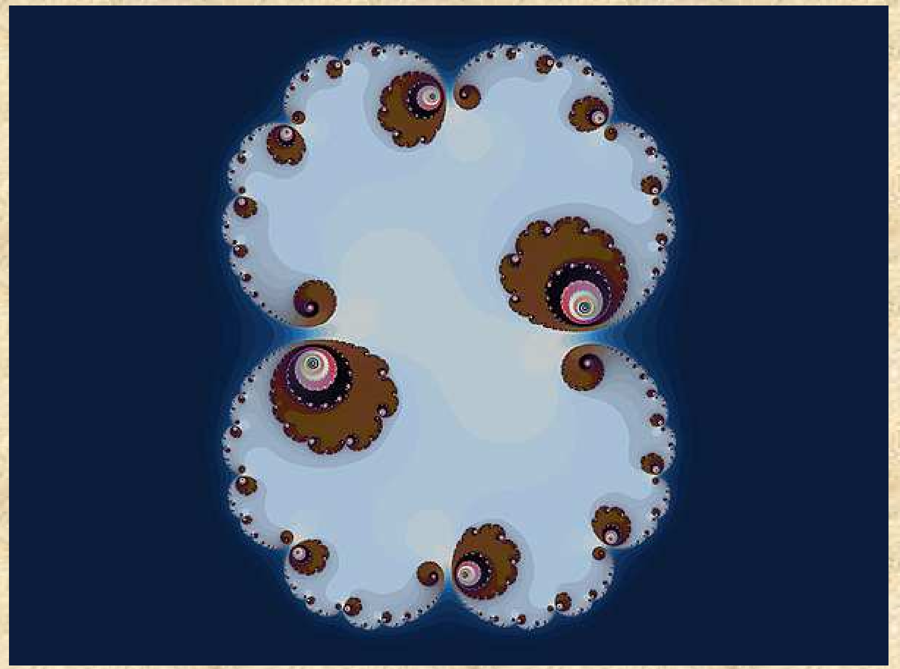

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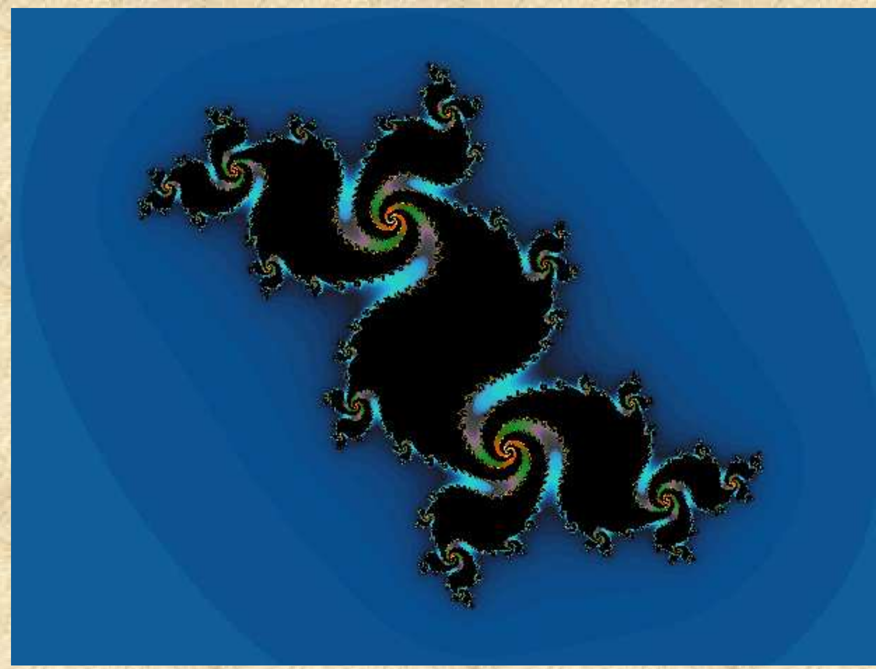

Attracting cycle :
Int (K) not empty
Cycle in Int(K)

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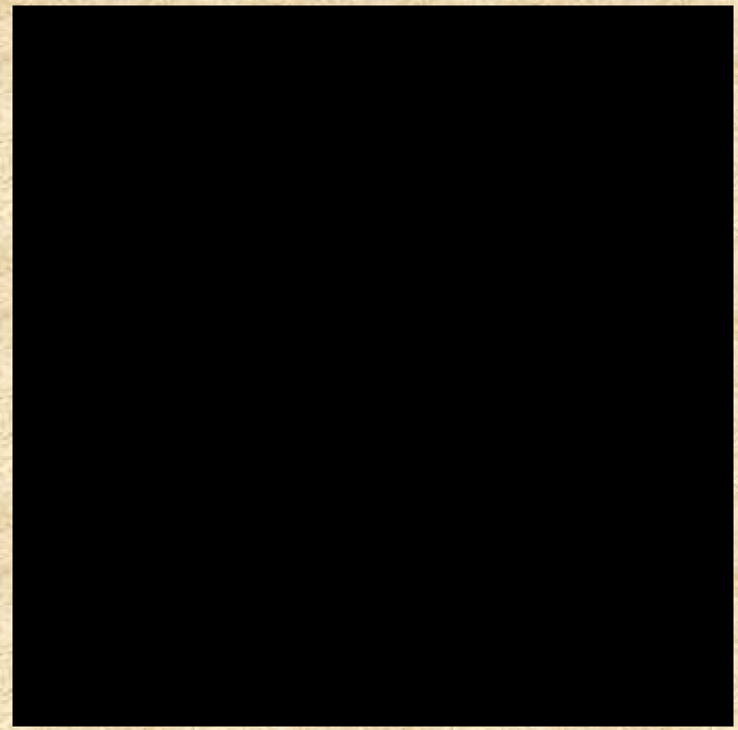
Fatou, 1919

« Ces exemples suffisent à montrer
que l'étude de l'ensemble E'
est un problème de nature arithmétique
auquel on pourrait appliquer
les méthodes de Borel-Lebesgue
pour la mesure des ensembles,
et qui appelle de nouvelles recherches… »

POSITIVE MEASURE

Compact set with empty interior, but positive measure



ZMJ Conjecture :

Impossible for Julia sets

(dominant until ~ 1990)

Attempts for a counter-example :

- *Van Strien ~ 1995 :* $z^d + c$, c real , d big
- *Jellouli (93-94) - Chéritat (98-2003) :* $\lambda z + z^2$, $\lambda = e^{2i\pi\theta}$, *Cremer*

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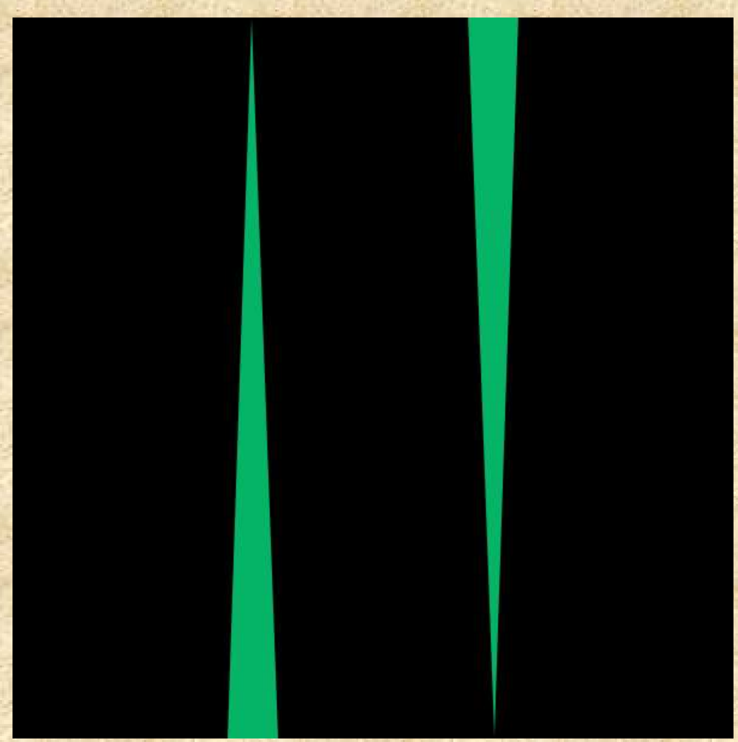

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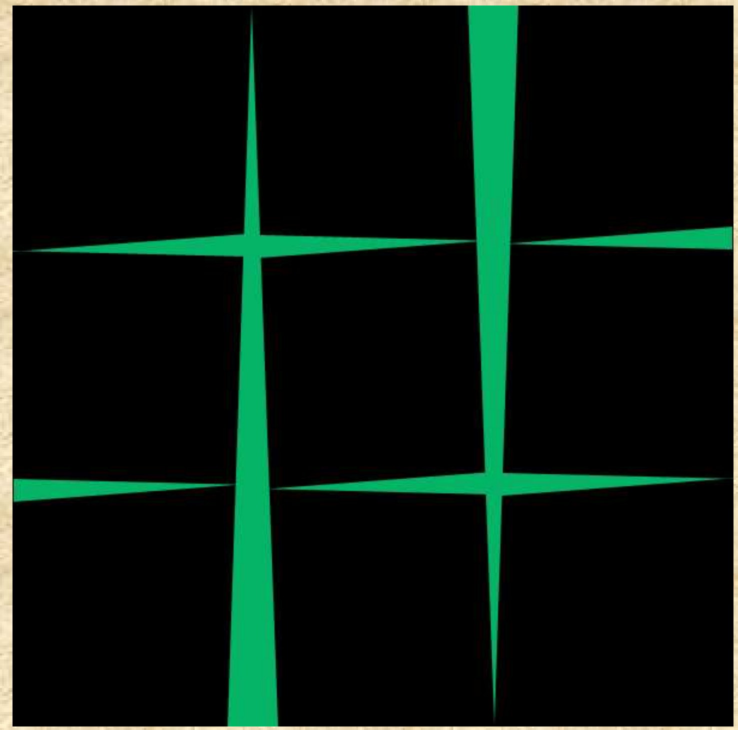

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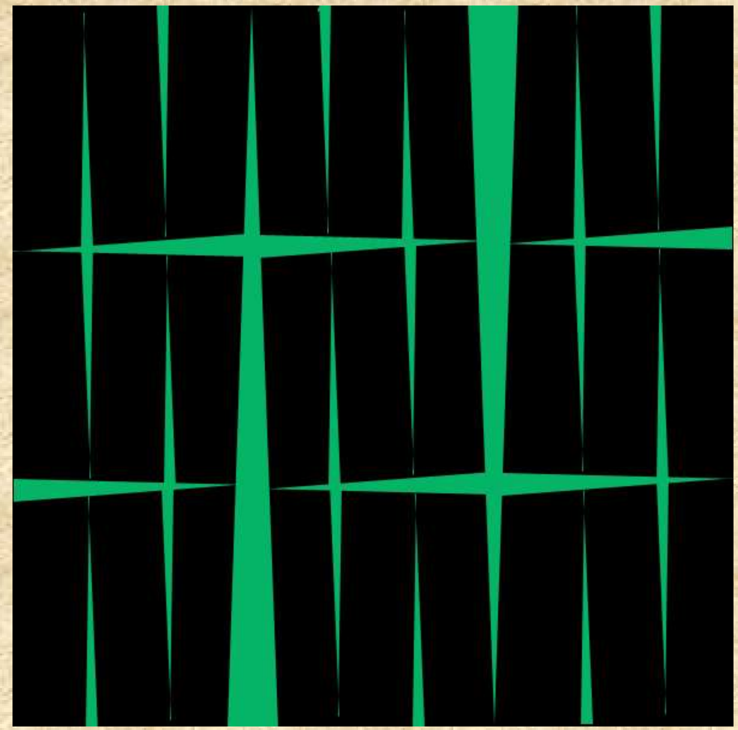

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II. THE FAMILY (f_θ)

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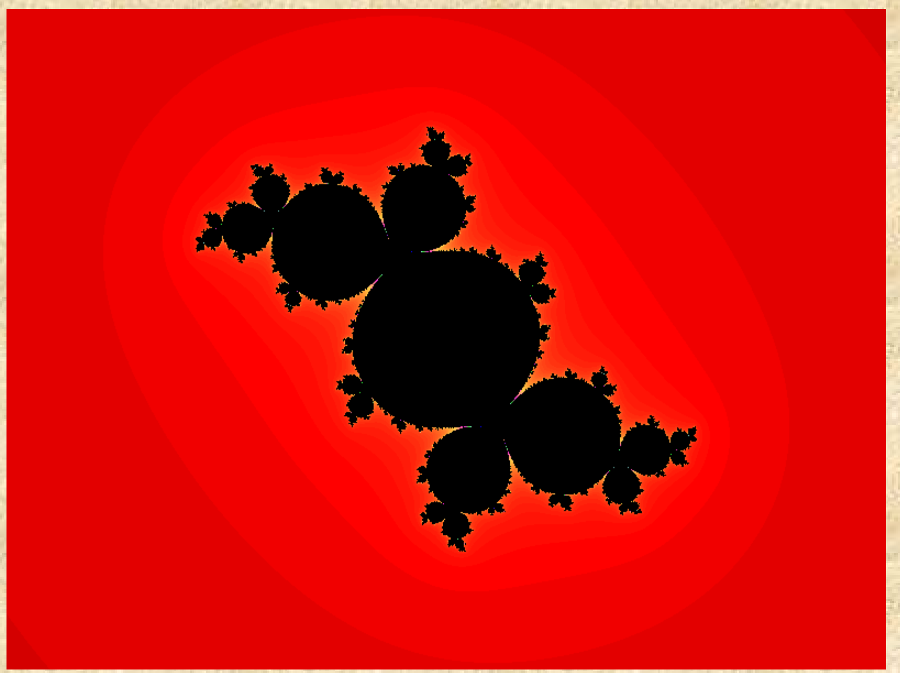
$$f_\theta : z \rightarrow \lambda z + z^2, \quad \lambda = e^{2i\pi\theta}$$



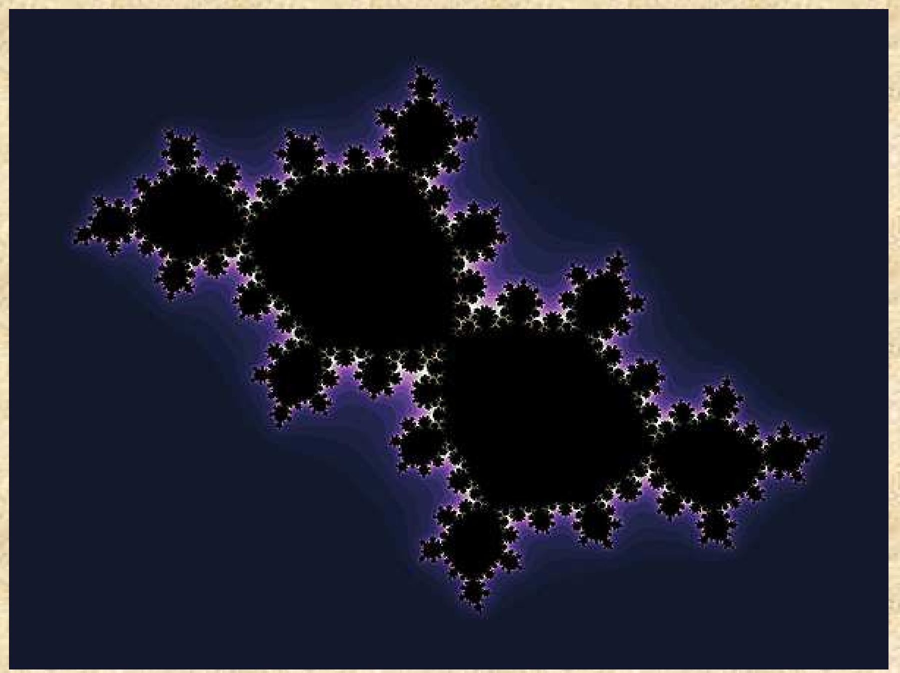
θ rational
$0 \in J$
$\omega \in Int(K)$

Parabolic case

θ rational : $Int(K) \neq \emptyset$, K locally connected
$m(J) = 0$

THE FAMILY (f_θ)

$$f_\theta : z \to \lambda z + z^2, \quad \lambda = e^{2i\pi\theta}$$



θ *bounded type*
$0 \in \; Int(\mathcal{K})$
$\omega \in \; \mathcal{J}$

θ *bounded type :* $\theta = [a_1, \ldots, a_n, \ldots]$, a_n *bounded* $\Rightarrow$

- *boundary of* Δ *quasi-circle through* ω *(Herman 1987)*
- $\mathcal{K}_\theta$ *locally connected (Petersen 1994)*
- *mes* $(\mathcal{J}_\theta) = 0$ *(Petersen - Lyubich), dim*$(\mathcal{J}_\theta) < 2$ *(Mc Mullen)*

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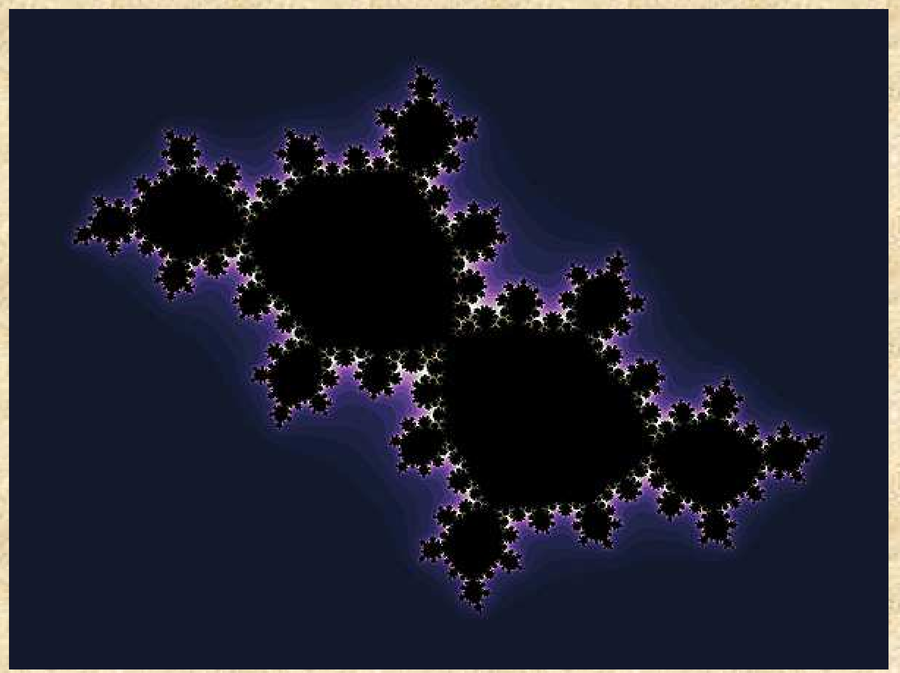


θ *bounded type*
$0 \in Int(K)$
$\omega \in J$

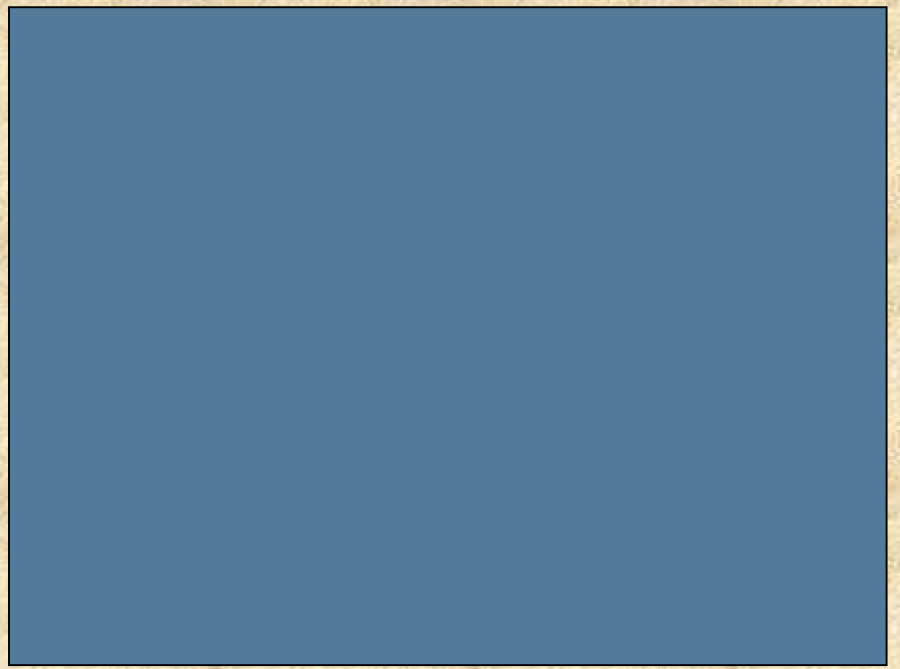
More generally
Bruno condition : $\displaystyle \sum \frac{\log q_{n+1}}{q_n} < \infty$

θ *Bruno* $\Rightarrow$ *Siegel,* $0 \in Int\,\theta(K)$, $\omega \in J$

Δ= *Siegel disc,* $f : \Delta \to \Delta$ *holomorphically conjugated to a rotation*

THE FAMILY (f_θ)

$$f_\theta : z \longrightarrow \lambda z + z^2, \quad \lambda = e^{2i\pi\theta}$$



Cremer indifferent irrational cycle:
Cycle in $\mathcal{J}(f)$

θ *irrational non Bruno* $\Rightarrow$ *Cremer* *(Yoccoz)*

$Int(\mathcal{K}) = \emptyset, \quad 0 \in \mathcal{J}, \quad \omega \in \mathcal{J}$

$\mathcal{K}$ not locally connected.

INITIAL STRATEGY

How to get a counter-example to ZMJ :

One can construct a sequence (θ_k) such that :

- mes $(\mathcal{K}_{\theta_k})$ $\theta \not\longrightarrow$
- $\theta_k \longrightarrow \theta$ irrational non Bruno

If one finds such a sequence,

- $Int(\mathcal{K}_\theta) = \varnothing$ since θ non Bruno
- mes $(\mathcal{K}_\theta) \geq$ lim sup mes $(\mathcal{K}_{\theta_k}) > 0$, since

$$\left.\begin{array}{ll} \theta & \longmapsto \mathcal{K}_\theta \\ \mathcal{K} & \text{mes}\,(\mathcal{K}) \longmapsto \end{array}\right] \quad \text{u.s.c.}$$

Initial strategy :

- θ_k rational for k odd
- θ_k irrational bounded type for k even

STEP 1 : SIEGEL $\longrightarrow$ PARABOLIC

Just indicative

$\theta = [a_1, \ldots, a_n, \ldots]$ Bruno

Δ= Siegel disc for f_α

$\tau_k = [a_1, \ldots, a_k]$ approximant

Theorem 1 (Jellouli - Chéritat)

For given θ :

$$mes\,(\Delta \setminus \mathcal{K}_{\tau_k}) \to 0 \quad \text{when} \quad k \to \infty \,.$$

Corollary

if θ bounded type :

$$mes\,(\mathcal{K}_\theta \setminus \mathcal{K}_{\tau_k}) \to 0 \quad \text{when} \quad k \to \infty \,.$$

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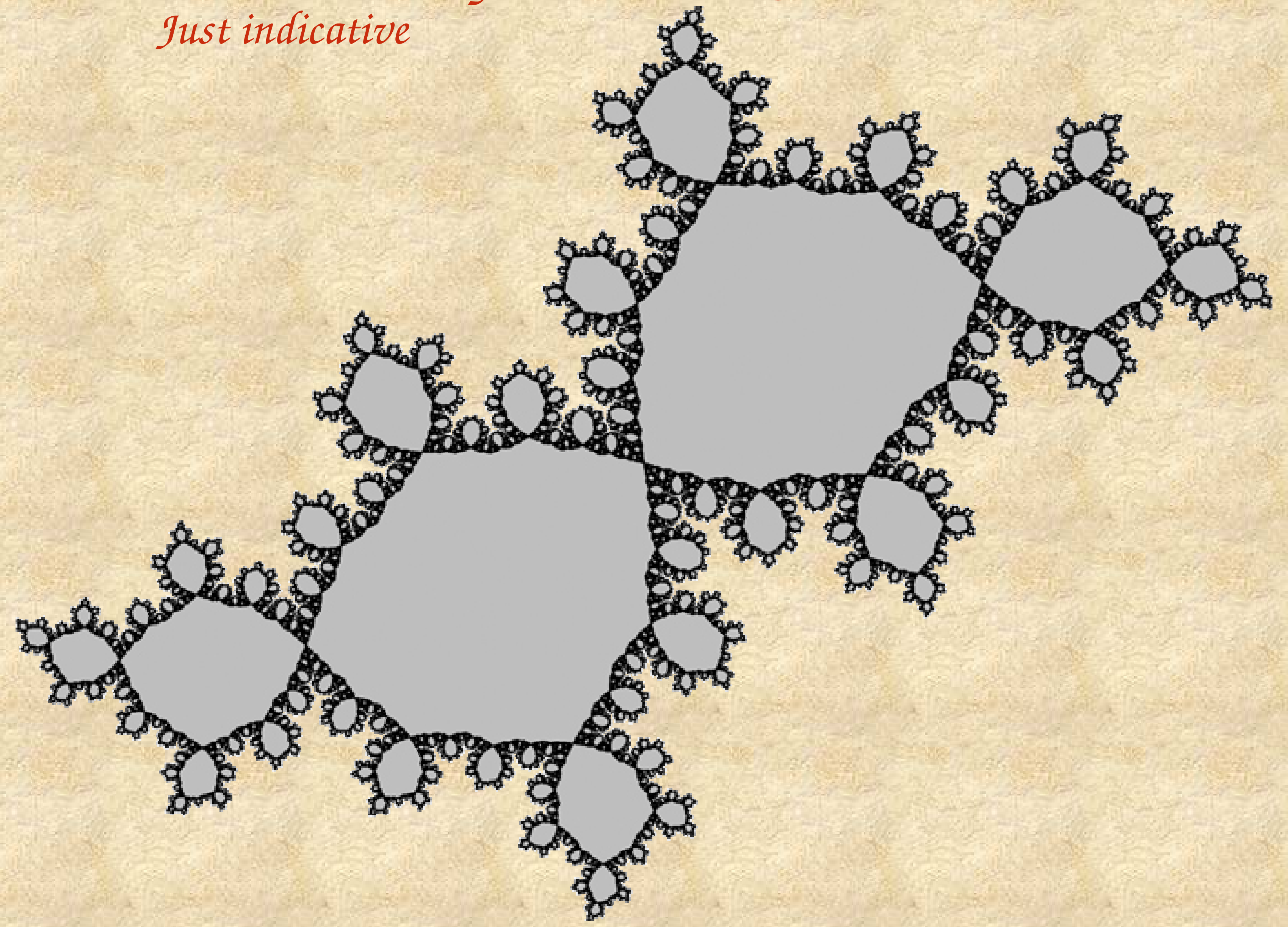

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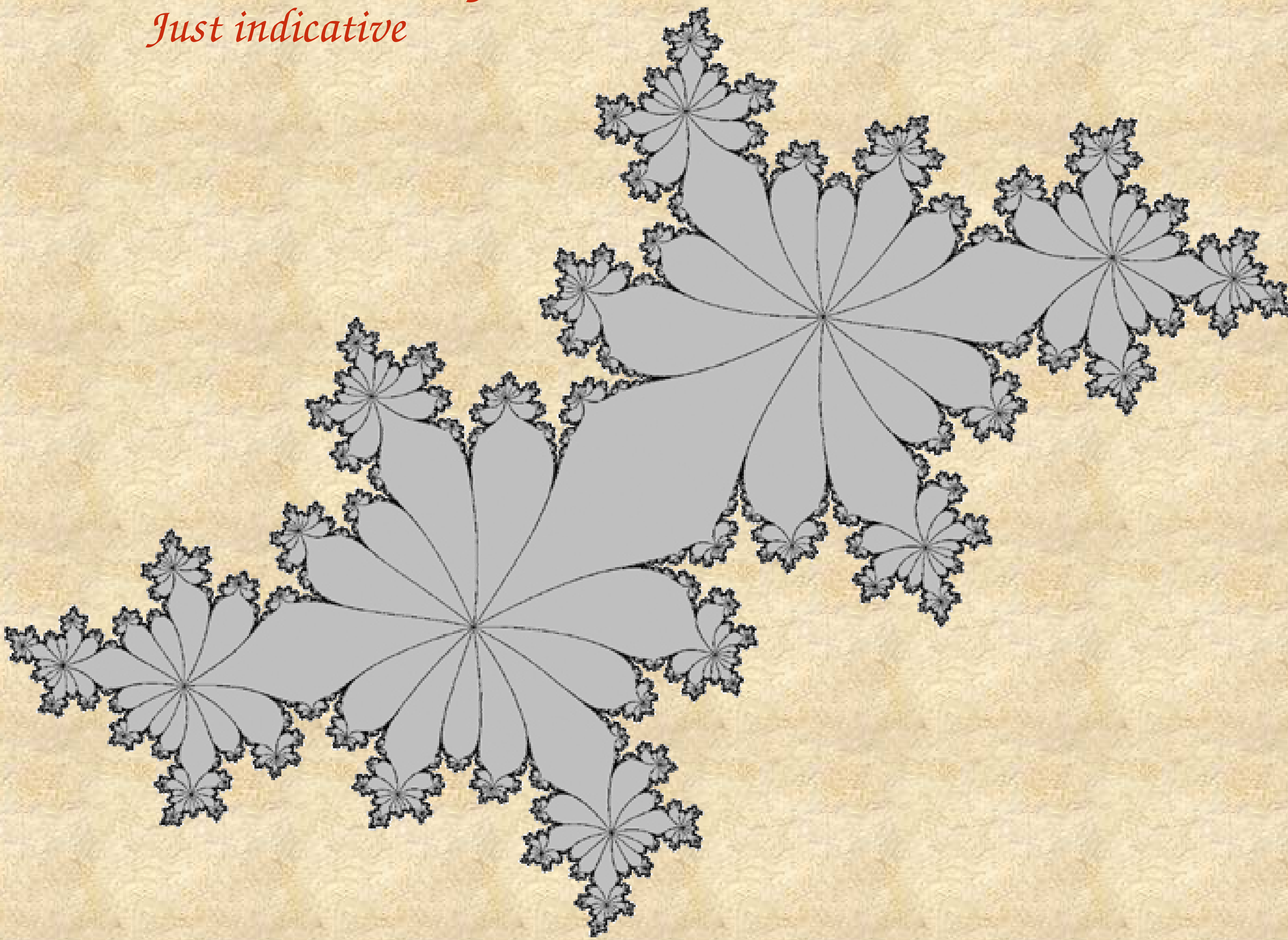

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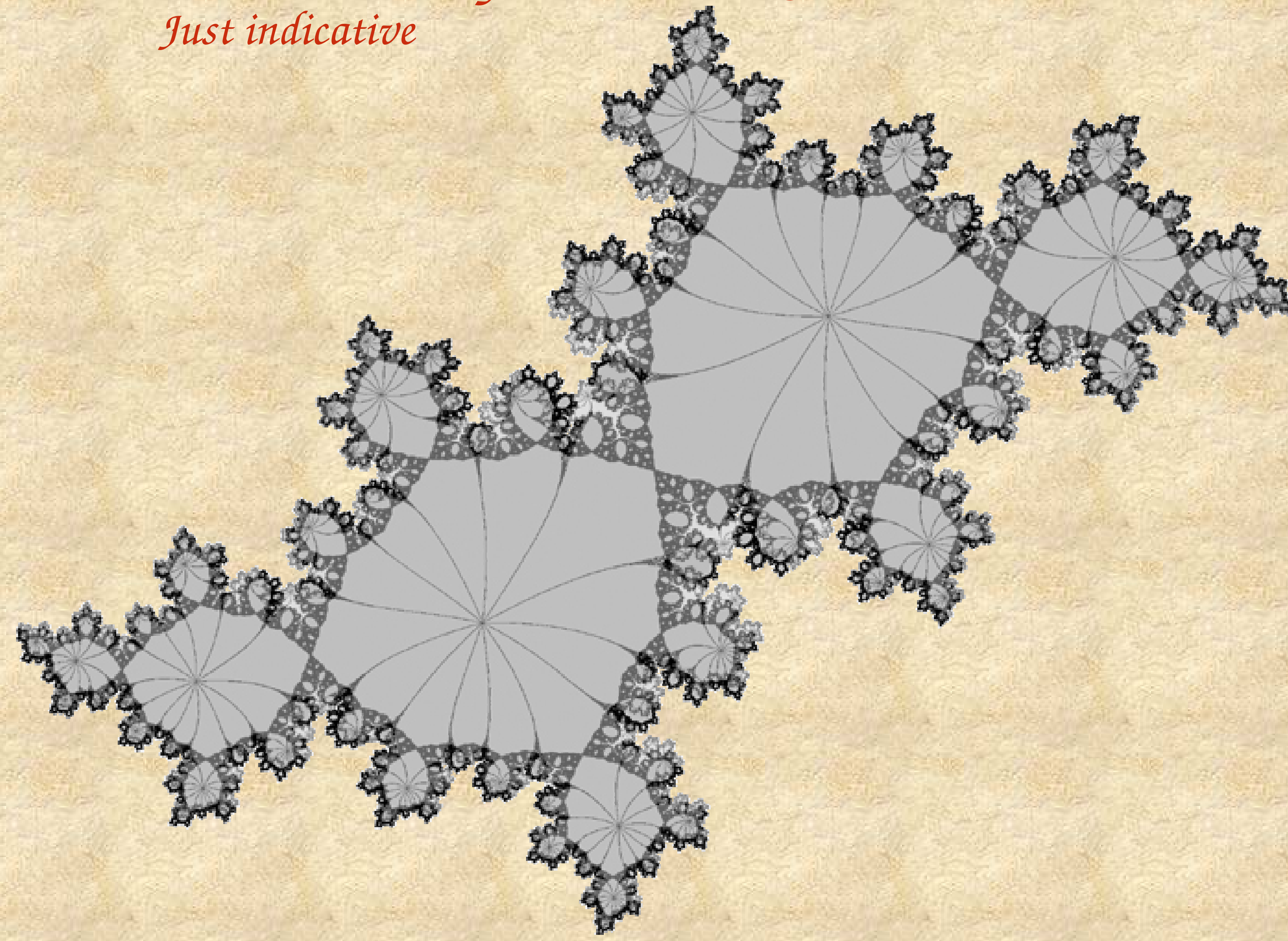

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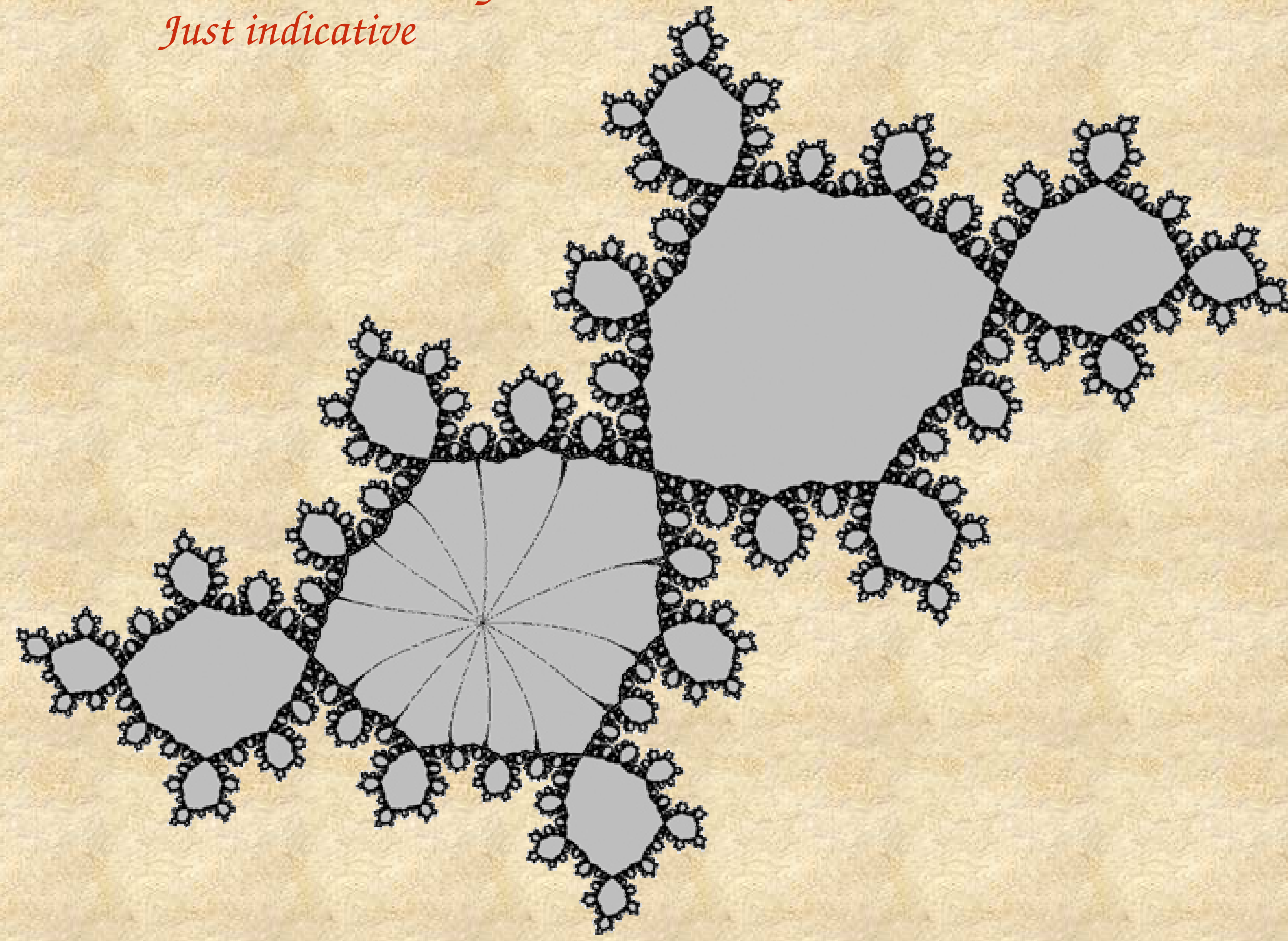

SHUNTING THE PARABOLIC

Central statement : θ *bounded type*

($\varepsilon_1, \varepsilon_2, \varepsilon_3$) (θ' bounded type)

(i) $|\theta' - \theta| < \varepsilon_1$;

(ii) $f_{\theta'}$ has a cycle in $D(0, \varepsilon_2)$;

(iii) $m(\mathcal{K}_\theta \setminus \mathcal{K}_{\theta'}) < \varepsilon_3$.

Central statement $\Rightarrow$ *(* θ Cremer) $m(K_\theta) > 0$

Remarks:

1) Condition (iii) can be replaced by

(iii') $m(\Delta \setminus \mathcal{K}_{\theta'}) < \varepsilon_3$.

2) What is proved is a variant
(restriction on θ and the same on θ')

III. INGREDIENTS

INGREDIENT 1

QuickTime™ et un décompresseur
Sorenson Video sont requis pour visualiser
cette image.

Theorem (Mc Mullen) : For θ of *bounded type*

any point in the boundary of Δ

is a density point *of* $\mathcal{K}_\theta$.

INGREDIENT 2

QuickTime™ et un
d□compresseur H.264
sont requis pour visionner cette image.

Theorem (Buff-Chéritat) : θ *bounded type*

$(\ \varepsilon, \varepsilon')$ $(\ \mathcal{U}\ \square_{\theta})$ $(\ \theta'$ bounded type)

(i) $f_{\theta'}$ has a cycle in $D(0, \varepsilon)$

(ii) $m(\square_{\theta'} \square U) / m(U) > 1/2 - \varepsilon'$.

For $\theta = [a_1,\dots,a_n,\dots]$, take $\theta' = [a_1,\dots,a_k,N,s,s,s,\dots]$.

A STILL OPEN QUESTION

QuickTime™ et un
décompresseur DV - PAL
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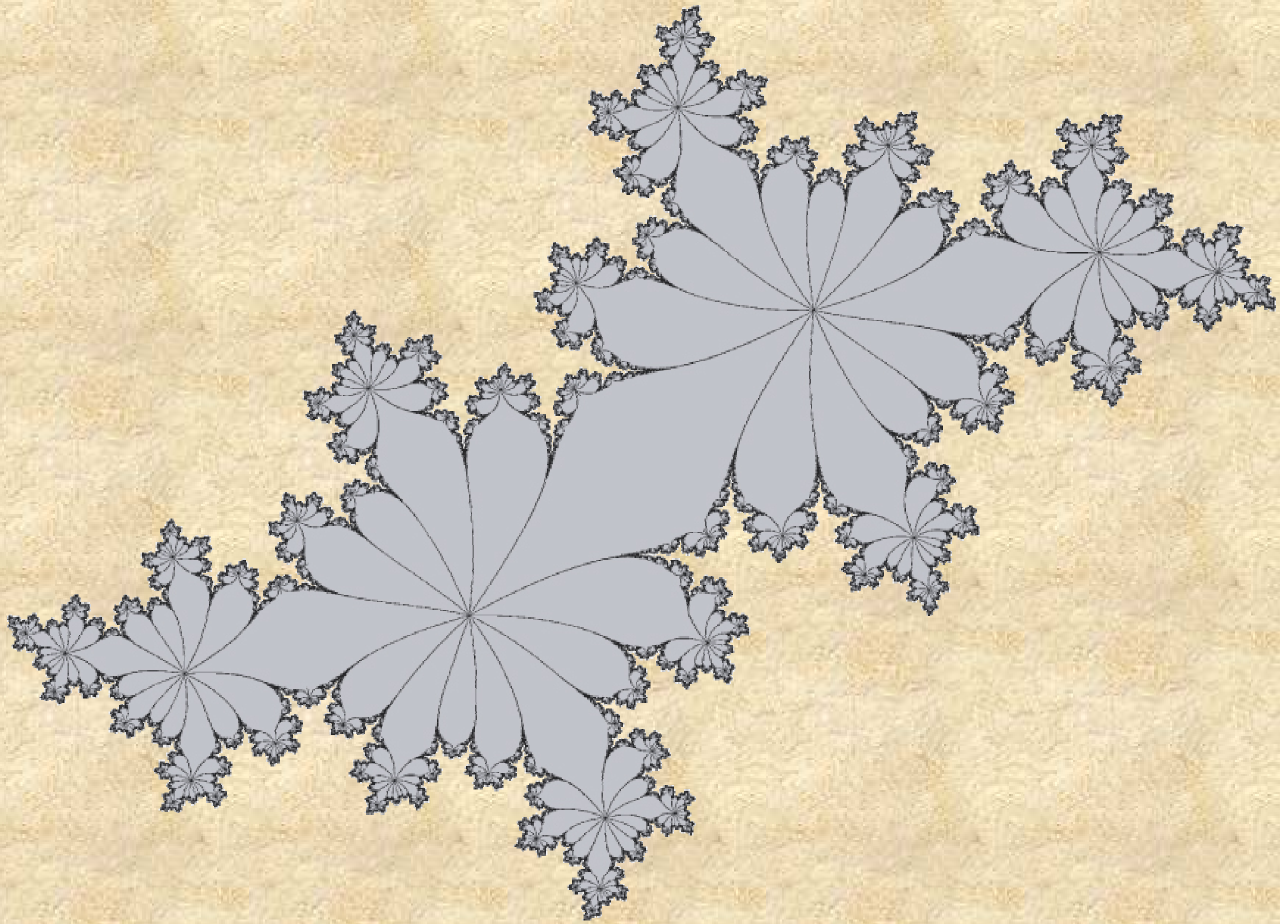
Conjecture: $\partial(\Delta_{\theta'}, \Delta_\theta)$ *can be made arbitrarily small,*

∂ *Hausdorff semi-distance:* $\partial(X,Y) = \sup_{x \in X} d(x,Y)$

A STILL OPEN QUESTION

θ *bounded type,* $\tau_n = p_n/q_n$ ctd. frac. approximants



Conjecture: $\partial(F_{\tau n}, \Delta_\theta) \to 0$,

$F_{\tau n}$: flower of half components

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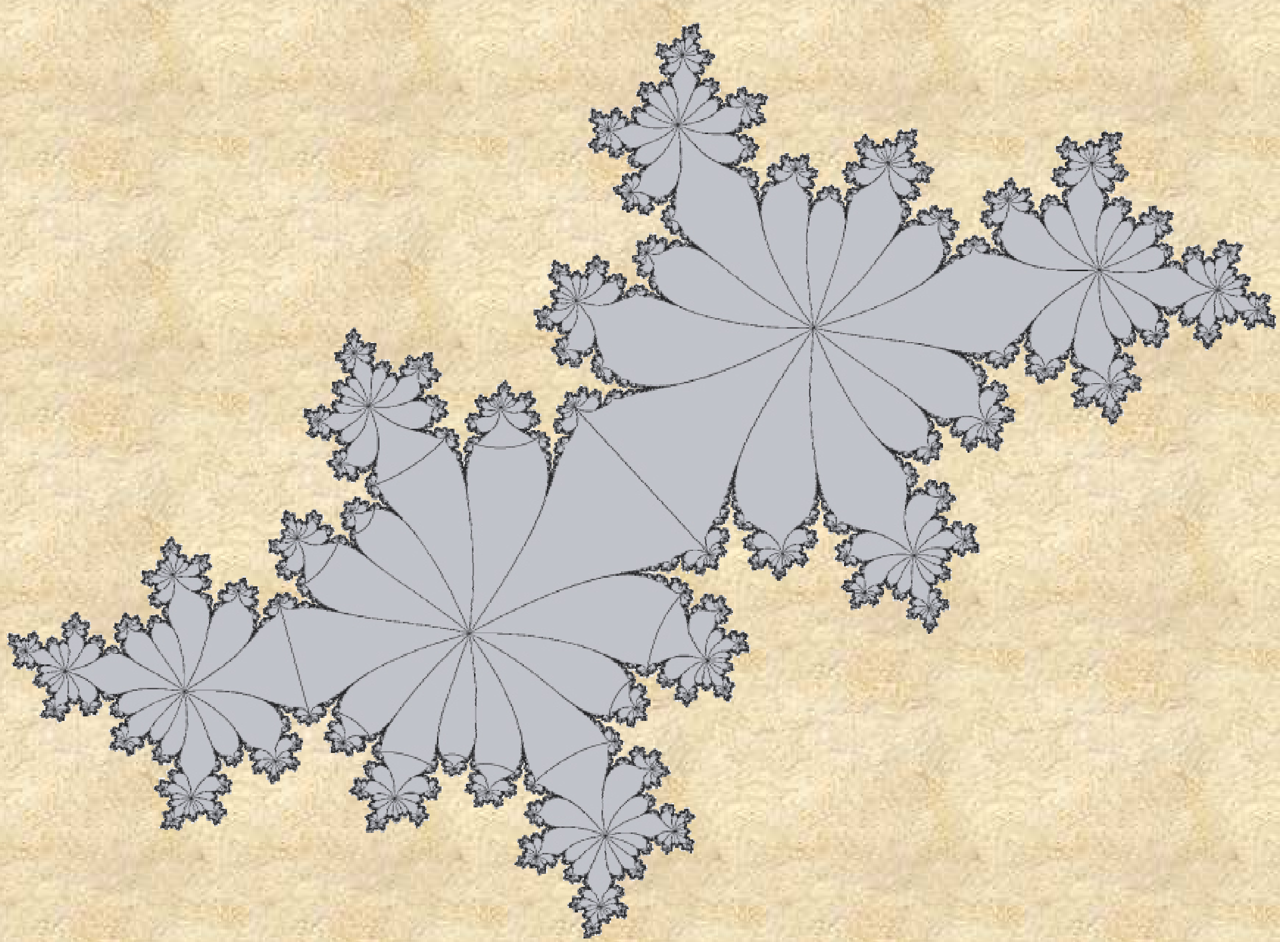

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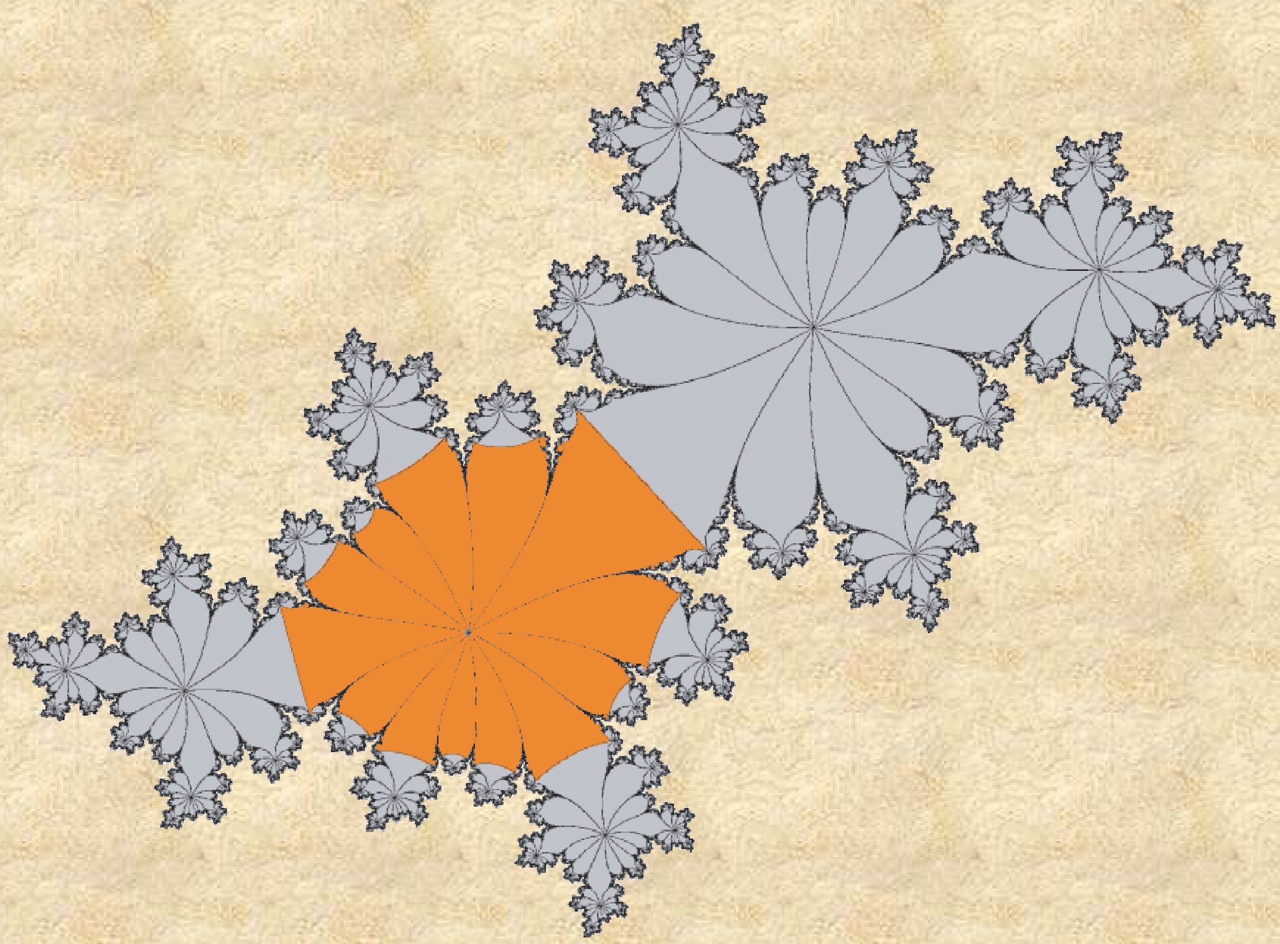

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INGREDIENT 3

S the Inou-Shishikura bound

$\Theta = \{\theta = [a_1,\ldots,a_n,\ldots] \mid (\forall n)\ \ a_n$ [illegible] $s\}$

Theorem (Buff-Chéritat / Inou-Shishikura) :

$\theta \in \Theta$ *and bounded* type

$\theta_n \to \theta$ in Θ

$$\Downarrow$$

$\partial(P(\theta_n), \Delta_\theta) \to 0$,

$P(\theta)$ the postcritical set of f_θ.

Relies on some renormalization property …

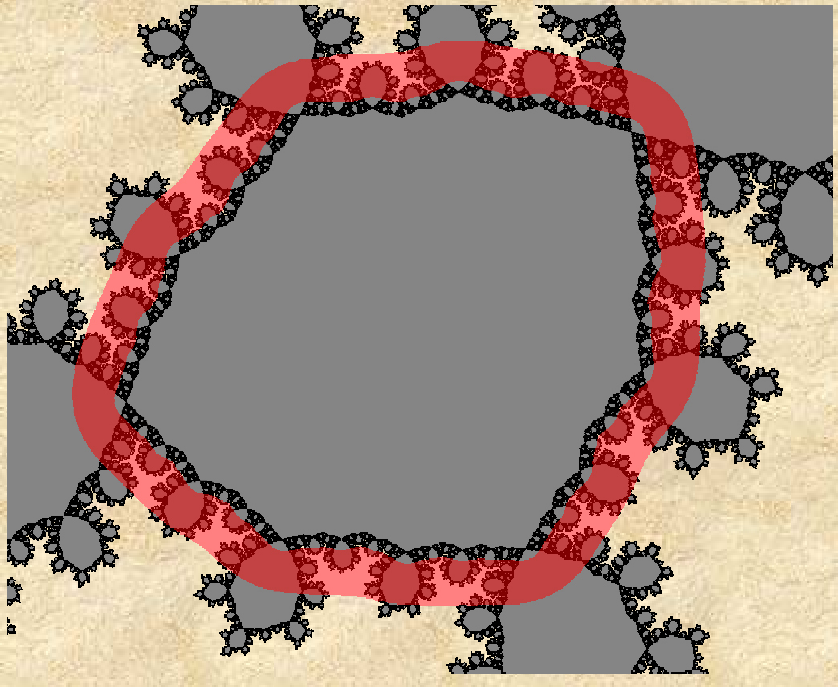
IV. SKETCH OF PROOF

TOLL RING

$\theta \in \Theta$ *and bounded* type

δ Chosen small

$$W = \{z \in C \mid 2\delta < d(z, \Delta) < 9\delta\}.$$



A point $z \in \Delta$ cannot escape to infinity under $f_{\theta'}$,

θ' close to θ , without stepping in W .

For $z \in W$ the density of C-K_{θ} in $D(z,\delta)$ is $< \eta_W$,

$\eta_W = 100.\eta_{McM}(10\,\delta)$.

THE BOUNCE

$z \in \Delta$ *bounces p* times $\Leftrightarrow (\exists\, N_0=0 < N_1 < \dots < N_p)$

$f_{\theta'}^{N_k}(z) \in \Delta$ for k even, $\in W$ for k odd .

$X_p = \{z \in \Delta \mid z$ bounces p times $\}$,

$X_p^* = X_p - X_{p+1}$.

- $m(X_{2p+1}^*) < \eta . m\,(X_{2p+1})$, $\eta = \Lambda \eta_W$,

  Λ an absolute constant involving Kœbe, Vitali.

- $m(X_{2p}^*) > (1/2 - \varepsilon) . m(X_{2p})$

Finally $m(\Delta \setminus K_{\theta'}) = \sum m(X_{2p+1}^*)$ can be made arbitrarily small.

qed.

OTHER RESULTS

- There exists θ such that f_θ has a Siegel disk Δ whose boundary does not contain the critical point and $m(J(f_\theta))>0$

- There exists an infinitely renormalizable z^2+c with a Julia set of positive measure

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