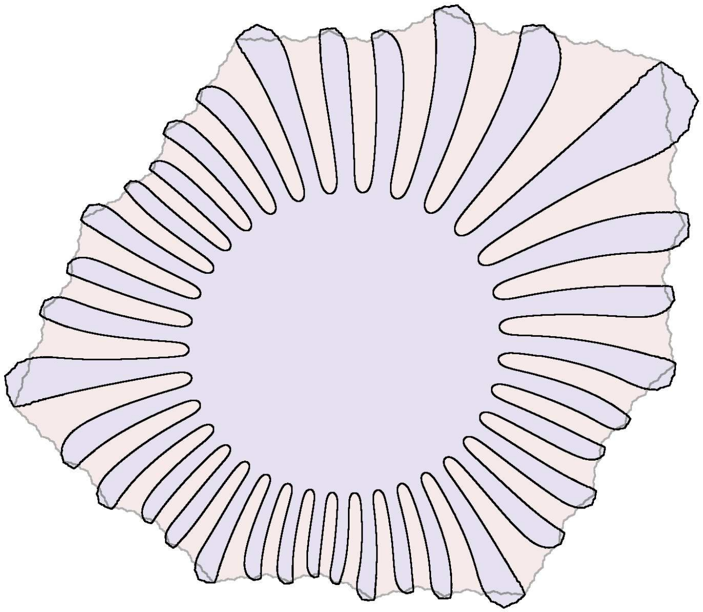
Introverted Siegel disks



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The theorem

For $\theta \in \mathbb{R}$ let

$$P_\theta(z) = e^{2i\pi\theta} z + z^2.$$

We note $\mathcal{B}$ the set of Brjuno numbers. Brjuno and Yoccoz proved that P_θ is linearizable at 0 iff $\theta \in \mathcal{B}$. In that case, we note Δ_θ its Siegel disk. It is foliated by P_θ-invariant loops.

Theorem: *$\forall \theta \in \mathcal{B}$ and for all invariant loop $\mathcal{C}$ of Δ_θ, there exists a sequence $\theta_n \in \mathcal{B}$ such that*

- $\theta_n \longrightarrow \theta$
- *P_{θ_n} has a cycle (of period $q_n \longrightarrow +\infty$) that tends to $\mathcal{C}$*
- *for all open subset U of Δ_θ,*

$$\liminf_{n \to +\infty} \frac{\operatorname{area}(U \cap \Delta_{\theta_n})}{\operatorname{area} U} \geq \frac{1}{2}$$

Special perturbations

Given $\theta \in \mathbb{R} \setminus \mathbb{Q}$, let us develop:

$$\theta = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots}} = [a_0, a_1, \ldots]$$

Let $p_n/q_n = [a_0, \ldots, a_n]$ be the n-th convergent of θ. We choose $A \in (1, +\infty)$, $N \in \mathbb{N}^*$, and set

$$\theta_n = [a_0, \ldots, a_n, \lfloor A^{q_n} \rfloor, N, N, N, \ldots].$$

This is tailored so that

$$\sqrt[q_n]{\left|\theta_n - \frac{p_n}{q_n}\right|} \underset{n \to +\infty}{\longrightarrow} \frac{1}{A}.$$

Parabolic explosion

Let $\mathcal{P}_q$ be the set of $\theta \in \mathbb{C}$ such that P_θ^q has a multiple fixed point. Let $d = d(p/q) = \mathrm{dist}(p/q, \mathcal{P}_q \setminus \{p/q\})$.

Theorem: (easy) $\exists \chi : D(0, d^{1/q}) \to \mathbb{C}$ *analytic, such that*

- $\forall \theta \in D(\frac{p}{q}, d) \setminus \{\frac{p}{q}\}$, P_θ *has a cycle of length* q *given by* $\chi(\sqrt[q]{\theta - \frac{p}{q}})$, *where* $\sqrt[q]{}$ *denotes the set of all* q*-th roots*
- $\chi(0) = 0$, $\chi'(0) \neq 0$

Theorem: (Douady) $\exists K > 0$ *such that* $\forall p/q$,

$$d(p/q) \geq K/q^3.$$

This is a corollary of the Yoccoz inequality, combinatorics of degree 2 polynomials, and Pythagora's theorem.

Parabolic explosion

Theorem (Jellouli) *Let us fix $\theta \in \mathcal{B}$ and let $\theta_n \longrightarrow \theta$. If $\theta_n - \theta = o(1/q_n)$ and $d(q_n\theta_n, \mathbb{Z}) \longrightarrow 0$ then $P_{\theta_n}^{q_n} \longrightarrow \mathrm{id}$ uniformly on compact subsets of Δ_θ.*

Theorem *For all $\theta \in \mathcal{B}$, noting p_n/q_n its convergents, χ_{p_n/q_n} tends uniformly[1] on compact subsets of $\mathbb{D}$ to the linearizing map $\phi : \mathbb{D} \to \Delta_\theta$ of P_θ.*

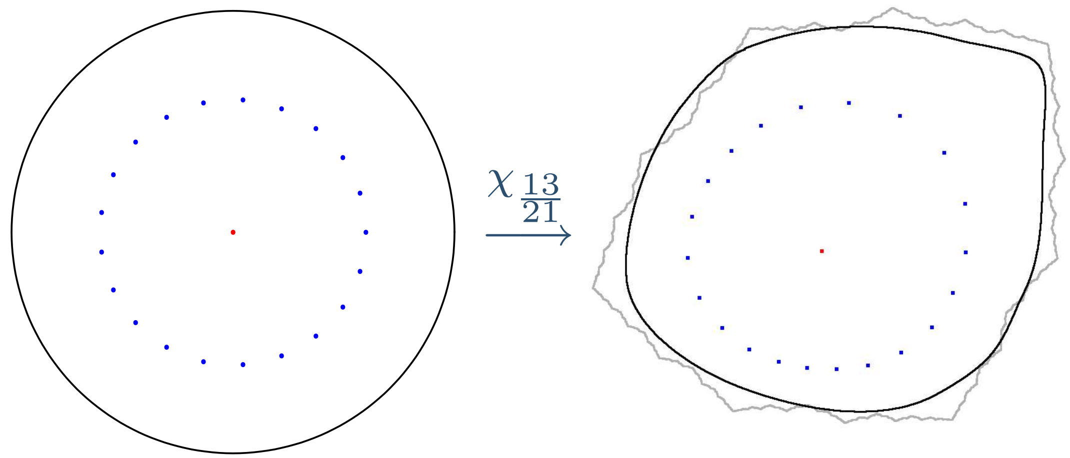
[1] up to precomposition with a rotation

We will use χ_{p_n/q_n} as a convenient change of coordinates, that tends to ϕ and on which $P_{\theta'}$ is conjugated to a map having a cycle $\sqrt[q_n]{\theta' - p_n/q_n}$ thus on a regular q_n-gon.[2]

[2] for θ' not too far from p_n/q_n.

Parabolic explosion

In particular, for our special perturbations θ_n of θ (with $|\theta_n - p_n/q_n|^{1/q_n} \longrightarrow 1/A$), P_{θ_n} gets conjugated to a map g_n defined on a disk of radius $\underset{n\to+\infty}{\longrightarrow} 1$ and having a cycle that tends to the circle $C(0, 1/A)$ for the Hausdorff topology on compact sets.



Vector field

By Jellouli's theorem, $g_n^{q_n}$ tends to identity uniformly on compact subsets of $\mathbb{D}$. These maps will be compared to the vector field

$$dz = 2i\pi q_n z(\varepsilon_n - z^{q_n})dt = X_n(z)dt$$

where

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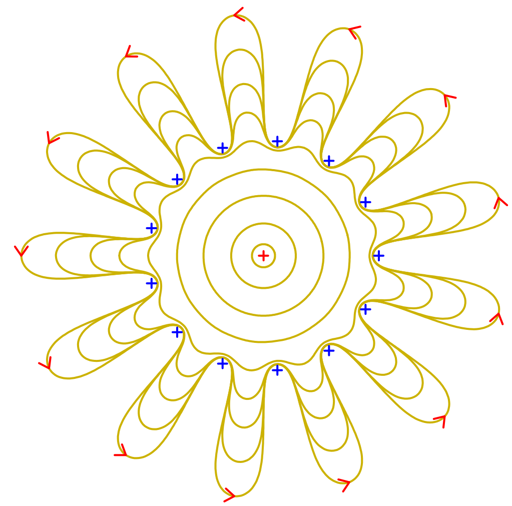

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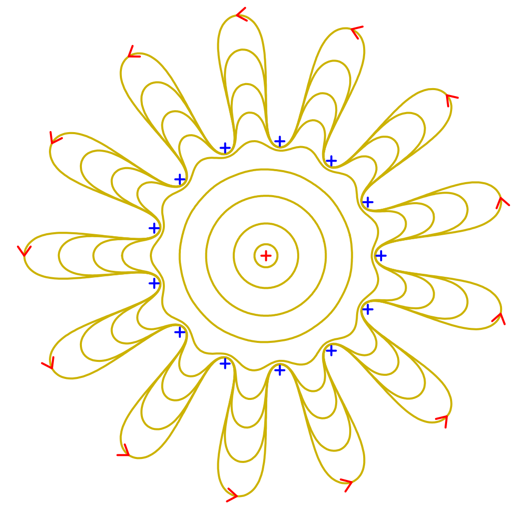

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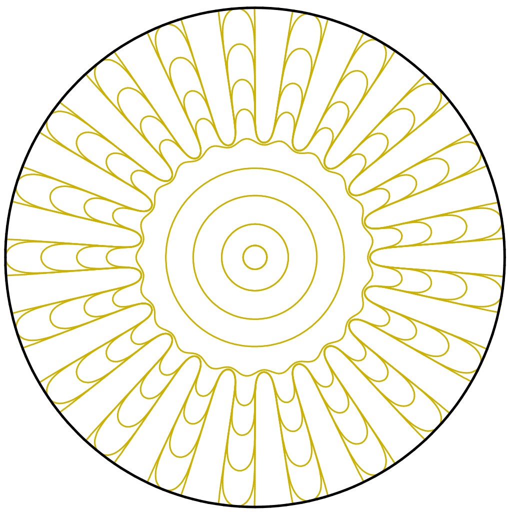
Theorem: $\dfrac{g_n^{q_n} - z}{\Phi_n(z) - z} = 1 + d_n z^{q_n} + z(z^{q_n} - \varepsilon_n)h_n(z)$ *where* $d_n \leq B_n$, $|h_n(z)| \leq B_n$ *and* $(B_n)^{1/q_n} \longrightarrow 1$.

Renormalization

The previous theorem tells us that the relative error between $g_n - z$ and X_n has order of magnitude z^{q_n}.



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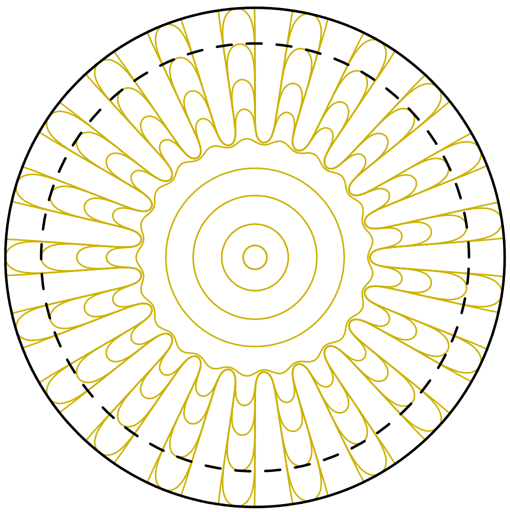

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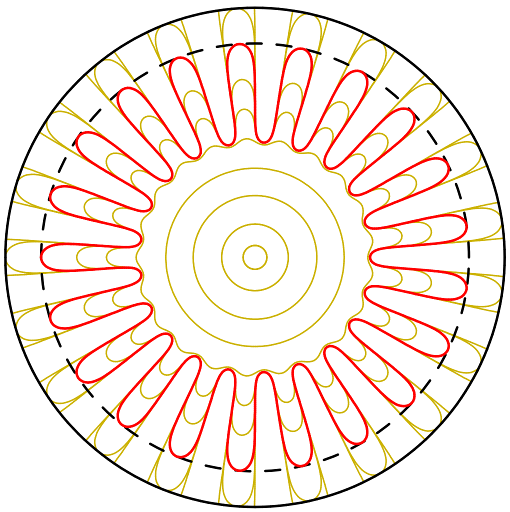

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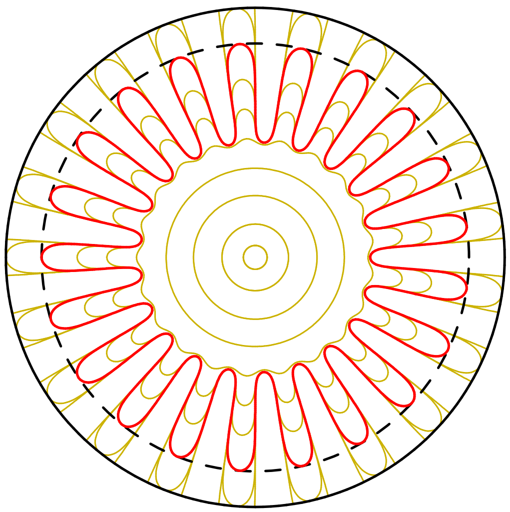

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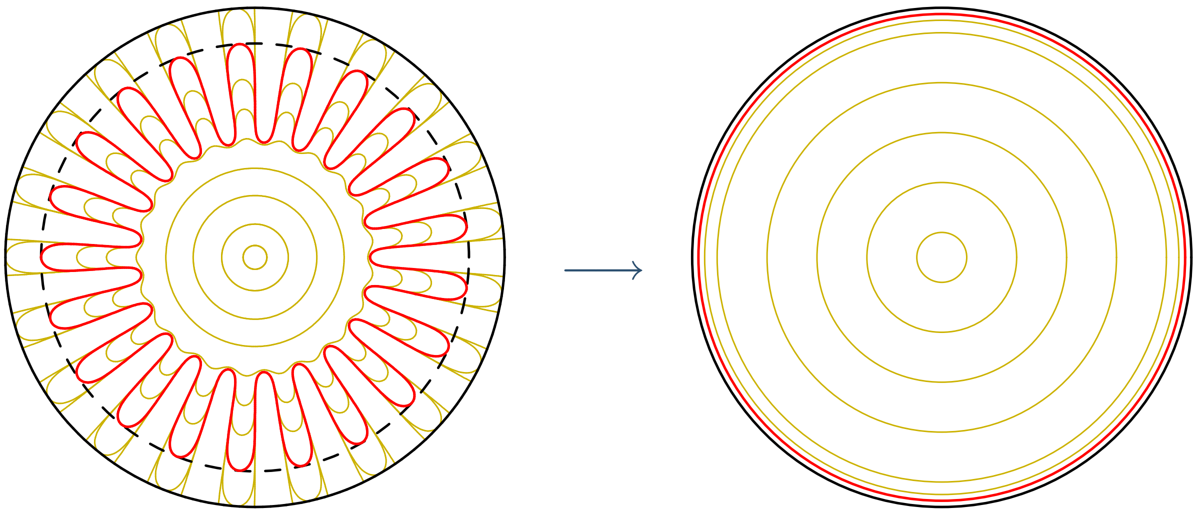

The distance between the red circle and the black circle is of order $(1/Ar)^{q_n}$.

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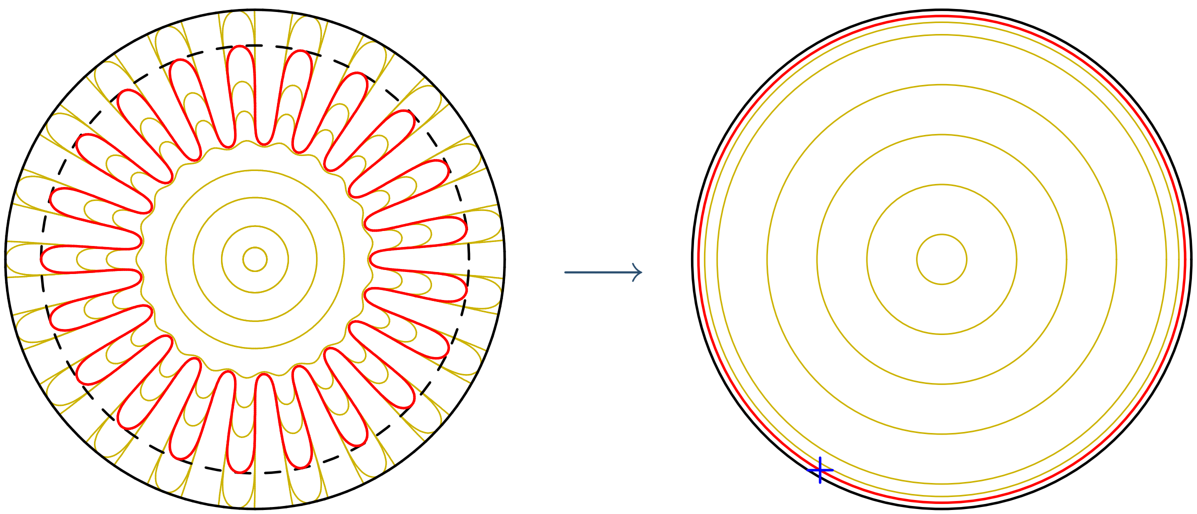

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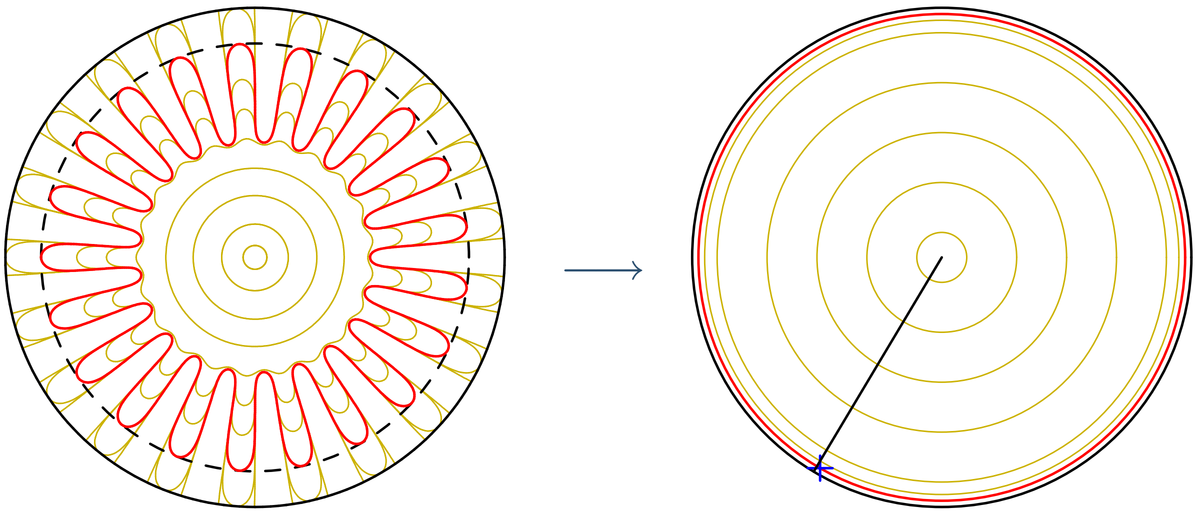

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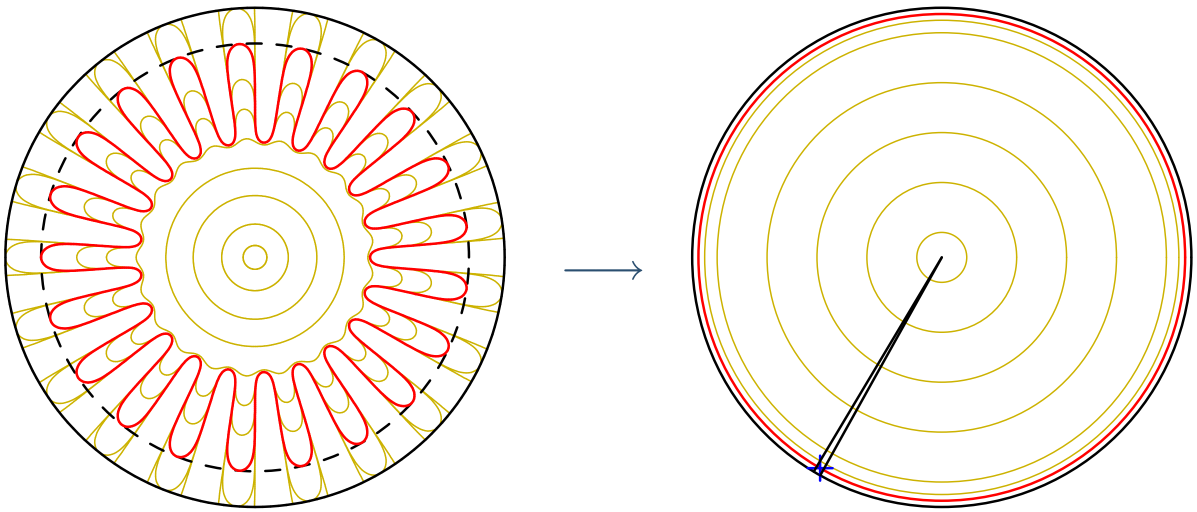
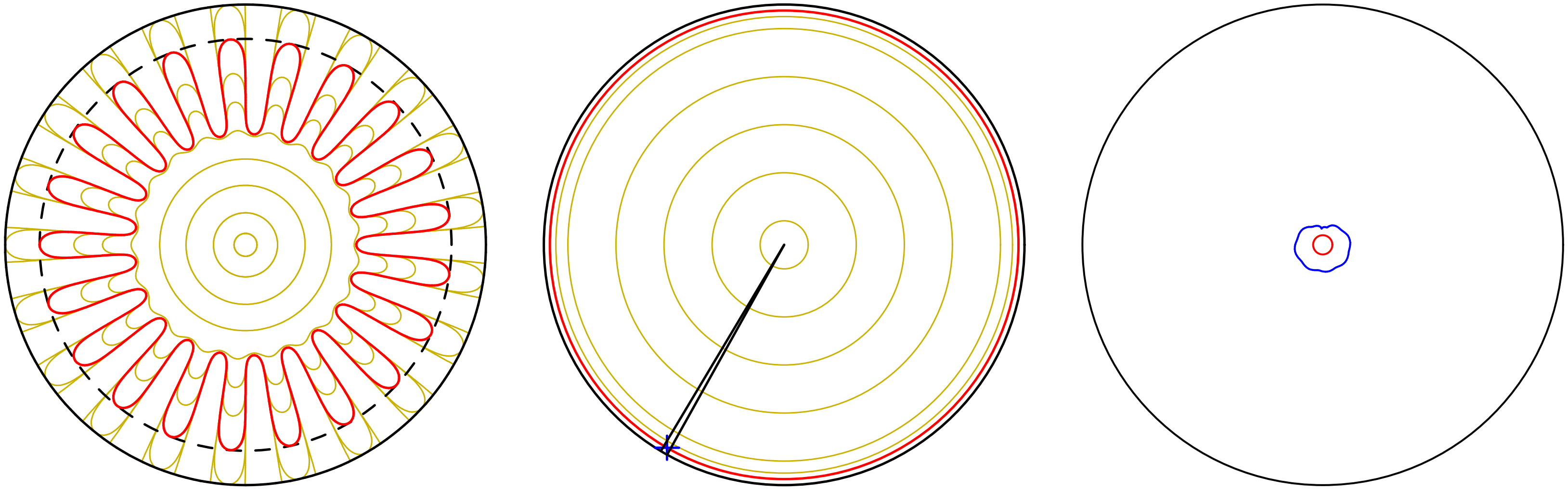


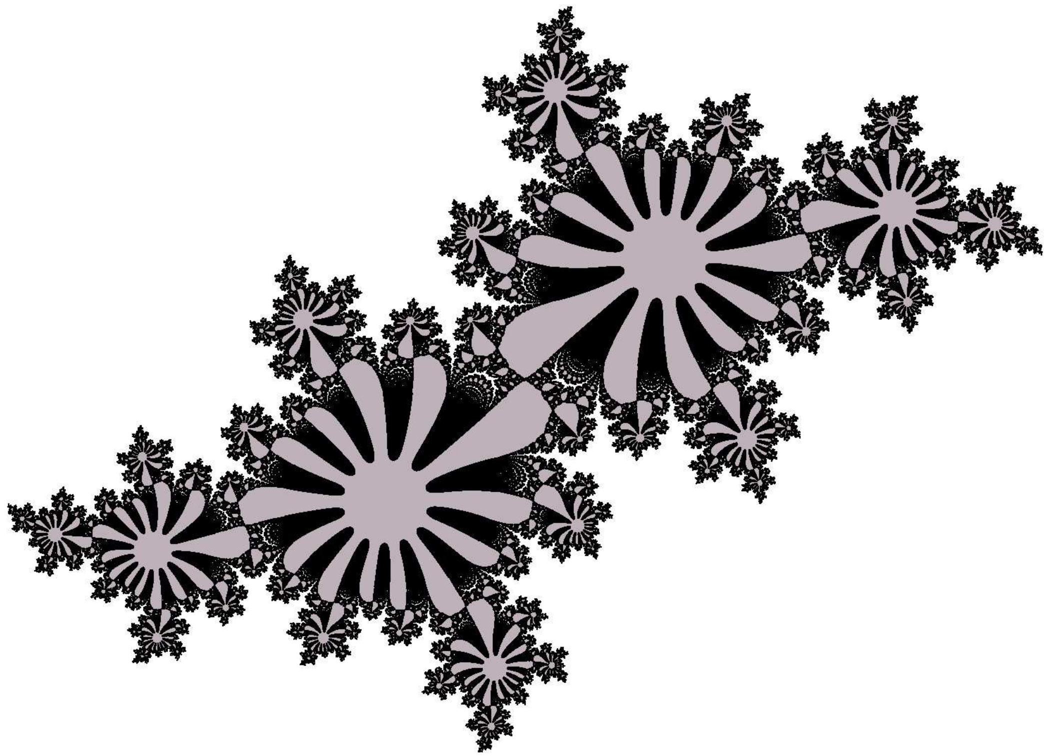
The distance between the red circle and the black circle is of order $(1/Ar)^{q_n}$. The width of the sector is of order $(1/A)^{q_n}$

Renormalization



Theorem: (Siegel) *For a fixed diophantine θ, the Siegel disk of a univalent map f on $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = e^{2i\pi\theta}$ must contain a disk of definite radius (that depends only on θ).*

Acknowledgements



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