

Intermingled Basins

from

Elliptic Curves as Attractors in \mathbb{P}^2

Part I : Dynamics

Joint with Marius Dabija
John Milnor

SUNY Stony Brook IMS preprint series 2006/01

History

The term Riddled Basin was introduced by:

Alexander, Kan, Yorke, You in its paper
Riddled Basins (1992)

To indicate an attracting basin whose complement intersects every disk in a set of positive measure

They define two basins to be Intermingled if every disk which intersects one basin in a set of positive measure also intersects the other basin in a set of positive measure

I. Kan Open sets of diffeomorphisms having two attractors, each with an everywhere dense basin (1994)

Announced the existence of an open set of C^k -diffeomorphisms on the 3 dimensional manifold with boundary, $T^2 \times [0,1]$ with 2 intermingled basins

B. Fayad Topologically mixing flows with pure point spectrum (2003)

Gave a new simpler construction of a C^∞ diffeomorphism $F: T^3 \rightarrow T^3$ that has 2 intermingled attractors

I. Melbourne and A. Windsor A C^∞ diffeomorphism with infinitely many intermingled basins (2005)

Construct a C^∞ diffeomorphism $T: T^2 \times S^2 \rightarrow T^2 \times S^2$ with k intermingled attractors and then construct another C^∞ diffeomorphism with countably infinity intermingled minimal attractors

Let

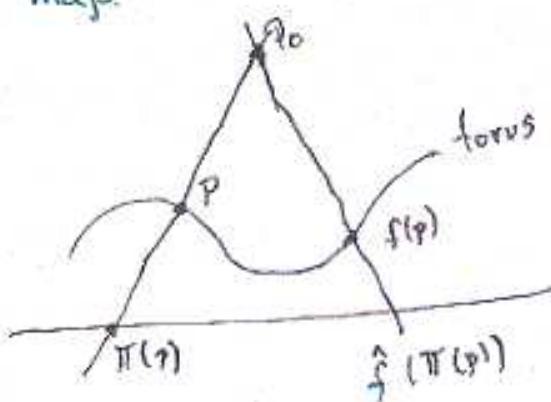
$$f(x, y, z) = (-x(x^3 + 2z^3), y(z^3 - x^3 + b\phi), z(2x^3 + z^3))$$

- f is holomorphic
- f leaves invariant the family of curves
 $\phi_k = x^3 + y^3 + z^3 - 3kxyz$
- f leaves invariant the pencil of lines through the point $(0; 1; 0) = p_0$

By setting $X = \frac{x}{z}$ and $X' = \frac{x'}{z'}$ then the correspondence

$$\hat{f} : X \mapsto X' = -X \left(\frac{x^3 + 2}{2x^3 + 1} \right)$$

is a lattice map.



Properties of \hat{f} :

- 1) \hat{f} is postcritically finite
- 2) $J(\hat{f}) = \hat{\mathbb{C}}$
- 3) \hat{f} has an ergodic invariant measure which is smooth except at its critical values, the cube roots of -1.

Thinking of $\mathbb{P}^2 \setminus \text{pol}$ as a (real or complex) line bundle over the projective line \mathbb{P}^1 with projection $\Pi(x:y:z) = (x:z)$, we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{P}^2 \setminus \text{pol} & \xrightarrow{f} & \mathbb{P}^2 \setminus \text{pol} \\ \Pi \downarrow & & \downarrow \Pi \\ \mathbb{P}^1 & \xrightarrow{\hat{f}} & \mathbb{P}^1 \end{array}$$

where f carries each fiber into a fiber by a polynomial map with coefficients which vary with the fiber

Example:

For the invariant line $x=0$, we get

$$(0:y:1) \longmapsto (0: \underbrace{by^2 + (1+b)y}_1 : 1)$$

For $z=0$

$$(-1:y:0) \longmapsto (-1: \underbrace{by^2 + (1-b)y}_0 : 0)$$

If $b \neq 0$, $(0:1:0)$ is superattracting and serves as the point at infinity for each line.

In the complex case these maps have 120° rotational symmetry and in the real case they are unimodal.

Since \hat{f} has no attracting cycles. It follows that an elementary map with invariant elliptic curve can have no other attracting cycles than its center point.

Observe that the line $\mathcal{L} = \{y=0\}$ is invariant for both f and \hat{f} . Hence it also has a canonical invariant measure and a well defined transverse Lyapunov exponent.

By definition the transverse Lyapunov exponent along the invariant curve \mathcal{L} is given by:

$$\text{Lyap}_{\mathcal{L}} = \lim_{k \rightarrow \infty} \left(\frac{1}{k} \right) \log \| (f^{\circ k})'_{\mathcal{L}} (p) \|$$

where $f'_{\mathcal{L}} (p)$ is the induced linear map

$$f'_{\mathcal{L}} (p) : T_{\mathcal{L}} (\mathcal{L}, p) \longrightarrow T_{\mathcal{L}} (\mathcal{L}, f(p))$$

and

$$T_{\mathcal{L}} (\mathcal{L}, p) = T(\mathbb{P}^2, p) / T(\mathcal{L}, p)$$

$f'_{\mathcal{L}} (p)$ has an operator norm

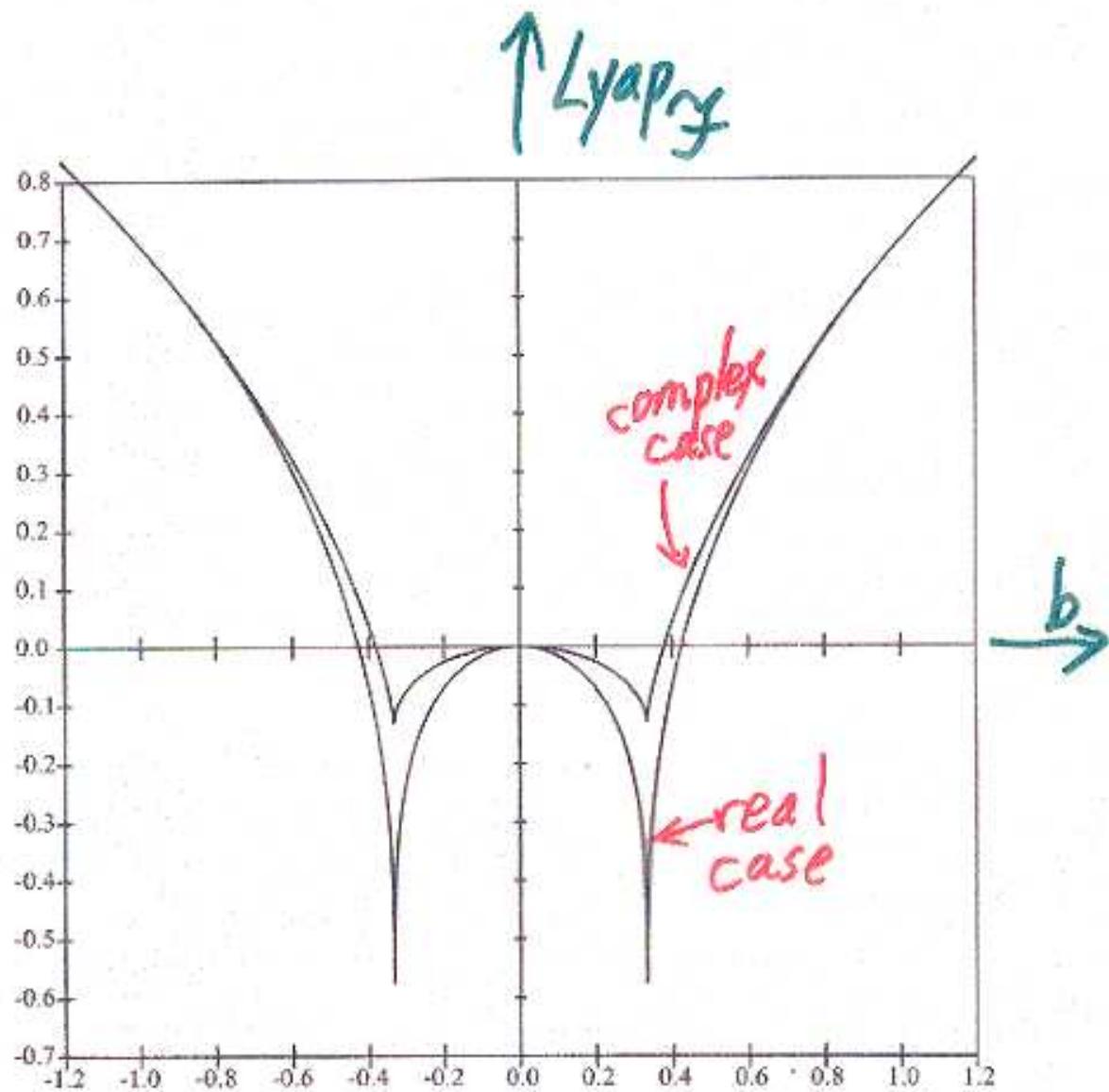
$$\| f'_{\mathcal{L}} (p) \| = \| f'_{\mathcal{L}} \vec{v} \|_h / \| \vec{v} \|_h$$

where $\| \vec{v} \|_h$ is the norm on each $T_{\mathcal{L}} \mathcal{L}$

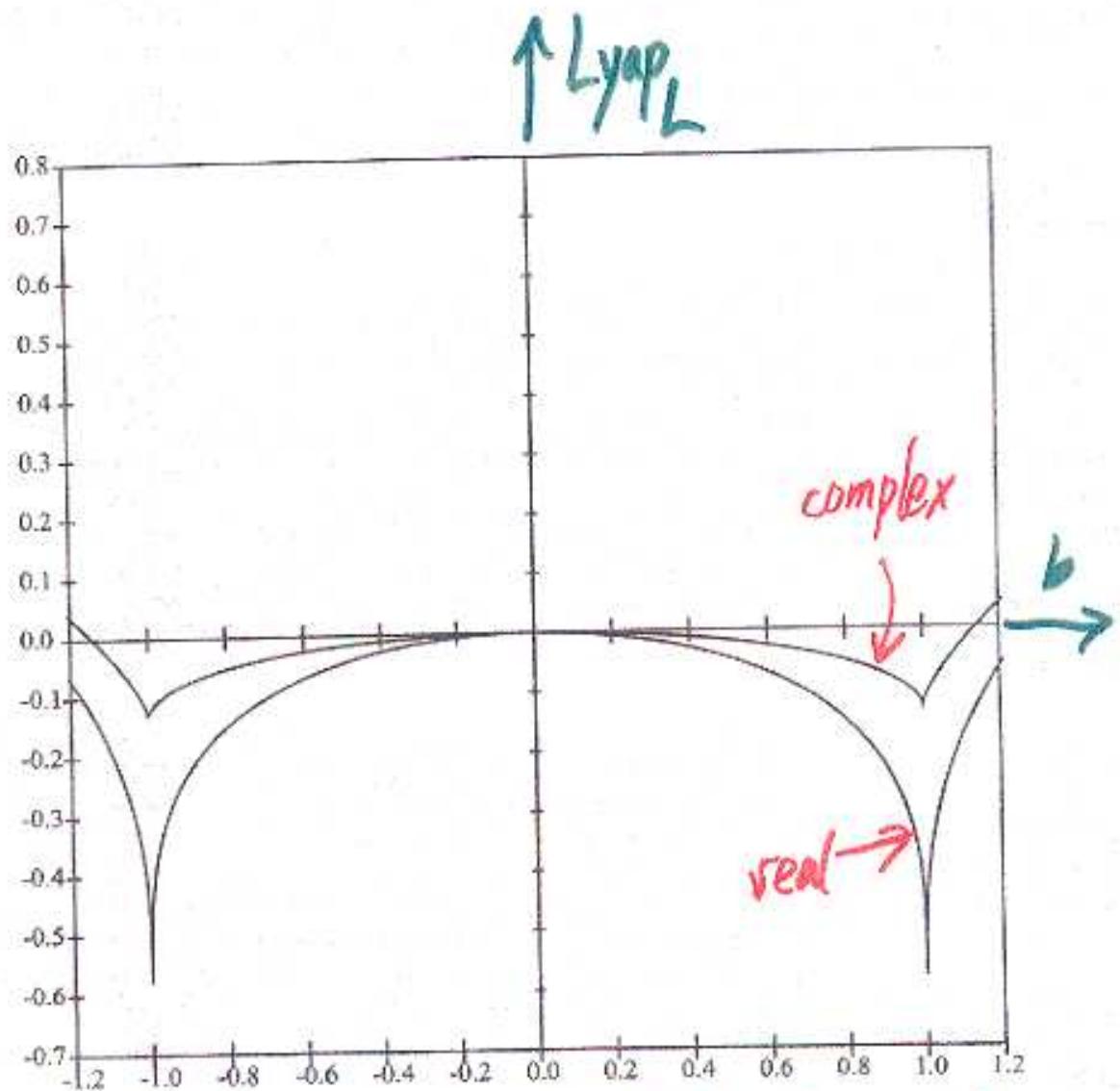
By the Birkhoff ergodic theorem

$$\text{Lyap}_G(f) = \int_G \log \|f'_A(p)\| d\pi(p)$$

independent of the choice of the metric



transverse exponent for the
Fermat curve \mathcal{F} ($x^3+y^3+z^3=0$)
for $f_{-1,b,1}$



transverse exponent for the
Line L ($y=0$) for
 $f_{-1,b,1}$

Theorem:

Let f be a real or complex elementary map with an invariant elliptic curve \mathcal{C} .

If $\text{Lyap}_f < 0$, the attracting basin $B(\mathcal{C})$, consisting of points which orbits converge to \mathcal{C} has strictly positive measure.

In fact any nbhd of a point of \mathcal{C} intersects $B(\mathcal{C})$ in a set of positive measure.

Similarly if such an f has an invariant line L with strictly negative transverse exponent then the attracting basin for this line has positive measure and intersects any nbhd of a point of this line in a set of positive measure.

Proof:

Assume that the center of the real or complex elementary map is the point $(0:1:0)$

An invariant line for this map with negative transverse exponent can not pass through this center, so let us choose $L = \{y=0\}$

Since we are assuming that there is an invariant elliptic curve \mathcal{C} , any invariant line not passing through the center is mapped to itself by the Lattès map with absolutely continuous invariant measure, so the transverse exponent is well defined

Each fiber $(x:z) = \text{constant}$ of the fibration $\pi(x:y:z) = (x:z)$ has a canonical flat metric

$$\frac{|dy|}{\sqrt{|dx|^2 + |dz|^2}}$$

which gives rise to the norm $\|\vec{v}\|_t$ for vectors tangent to the fiber

Let

$$\|f'_t(p)\| = \|\delta' \vec{v}\|_t / \|\vec{v}\|_t$$

At points of \mathcal{C} we want to compare $\|\vec{v}\|_t$ with $\|\vec{v}\|_A$

Since most points intersect the degree 3 curve \mathcal{C} transversally in 3 distinct points and there is only a finite number of points which intersect tangentially

$$\|\vec{v}\|_A / \|\vec{v}\|_t \geq 0$$

is a continuous function
on \mathcal{C} which vanishes
only at the points
of tangency

The

$$\log \left(\|\vec{v}\|_t / \|\vec{v}\|_0 \right) = \log(p)$$

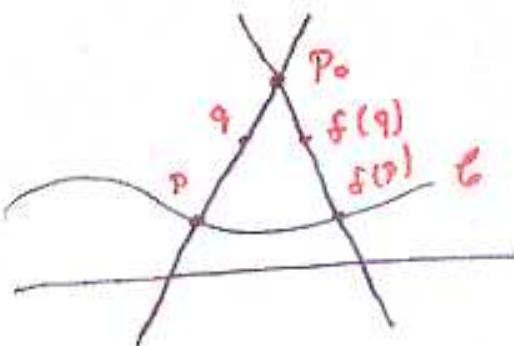
has only logarithmic singularities and hence
is an integrable function on \mathcal{C} .

Since $d\pi$ is f -invariant

$$\int_{\mathcal{C}} \log \|f'_n\| d\pi - \int_{\mathcal{C}} \log \|f'_t\| d\pi = \\ = \int_{\mathcal{C}} l \circ f d\pi - \int_{\mathcal{C}} l d\pi = 0$$

$$\therefore \text{Lyap}_t(f) = \int_{\mathcal{C}} \log \|f'_n\| d\pi = \int_{\mathcal{C}} \log \|f'_t\| d\pi$$

For any p, q in the same fiber



Let $\delta(p, q) > 0$ be the distance from p to q .

Then $\delta(f_t(p), f_t(q)) = \|f'_t(p)\| \delta(p, q) + o(\delta(p, q))$

uniformly throughout a nbhd of ℓ

Hence given $\epsilon > 0$ we can choose s_0 so that

$$\delta(f_t(p), f_t(q)) \leq (\|f'_t(p)\| + \epsilon) \delta(p, q)$$

when $p \in \ell$, and $\delta(p, q) < s_0$

Choose ϵ small enough so that

$$\int_{\ell} \log (\|f'_t(p)\| + \epsilon) d\pi(p) < 0 \quad (*)$$

Let $p_0 \mapsto p_1 \mapsto \dots$ be the orbit of an arbitrary initial point $p_0 \in \mathcal{G}$ under f .

By the Birkhoff Ergodic Theorem
the averages

$$\frac{1}{n} \left(\underbrace{\log \|f'_t(p_0)\| + \varepsilon}_{K_0} + \underbrace{\log \|f'_t(p_1)\| + \varepsilon}_{K_1} + \dots + \underbrace{\log \|f'_t(p_{n-1})\| + \varepsilon}_{K_{n-1}} \right)$$

converges to $(*)$ for almost all $p_0 \in \mathcal{G}$

In particular

$\log(K_0 K_1 \dots K_{n-1})$ is negative for large n

Hence

$$M(p_0) = \max(1, K_0, K_0 K_1, \dots, K_0 K_1 \dots K_{n-1}) \geq 1$$

is well defined measurable and finite almost everywhere

$$\text{If } \delta(p, q) \leq \delta_0 / M(p_0), \quad \delta(f^n(q), f^n(p)) \leq \delta_0$$

$\forall n$ and it goes to zero as $n \rightarrow \infty$

Let $S = \{q : \pi(p) = \pi(q) \text{ and } \delta(p, q) \leq \delta_0/M(p)\}$
 S has positive measure

For all $q \in S$, the orbit of q converges to b
 $\therefore S$ intersects every nbhd of a point
of b in a set of positive measure #

Thm (—, Dabija)

If p is any point of an invariant
elliptic curve then the iterated preimages of
 p are dense in the Julia set.

Proposition

If f is a complex elementary map with
smooth invariant elliptic curve and if the center
of p_0 is not a point of indeterminacy then p_0
is superattracting fixed point whose basin coincides
with the Fatou set. This basin is connected and
everywhere dense in \mathbb{P}^2 . If U is a small nbhd
of a point in the Julia set, the union of forward images

It follows that:

- The attracting basin for the elliptic curve has no interior points
- The attracting basin of L can not have any interior point
- f is topologically transitive on the Julia set

Corollary (Intermingled Basins)

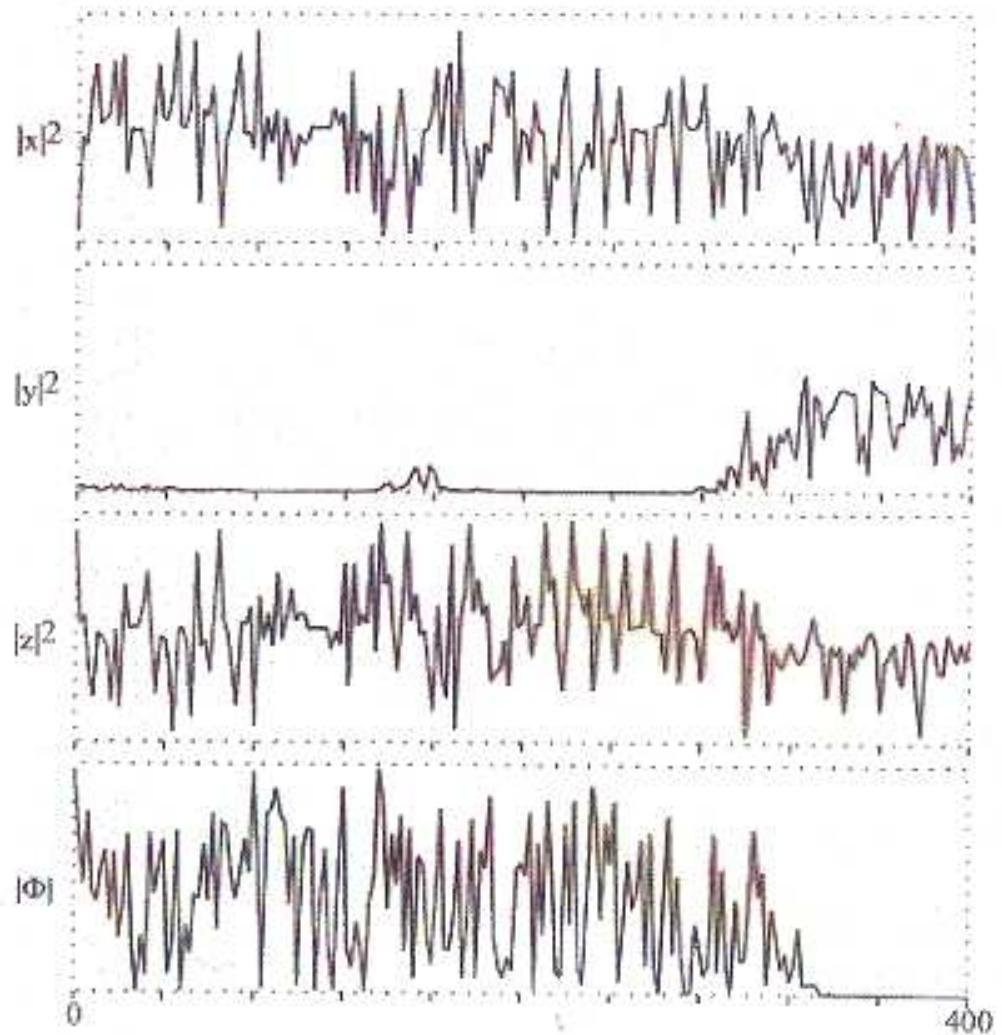
If a complex elementary map has both: an invariant line that doesn't pass through the center and an invariant elliptic curve then the two topological closures $\overline{B(G)}$ and $\overline{B(L)}$ are both precisely equal to the Julia set

Furthermore if the transverse Lyapunov exponent for G is negative then every nbhd of a point of the Julia set intersects $B(G)$ in a set of positive Lebesgue measure



$$(a, b, c) = (\pi i, \frac{1}{3}, 1)$$

	Real	Complex
Basin of Fermat in red	17%	6%
Basin of $ y =0$ in blue	17%	13%
Basin of $(0:1:0)$ gray	66%	81%



Proof:-

The basins of \mathcal{L} and \mathcal{R} are contained in the Julia set. If $p \in \mathcal{L} \cap \mathcal{R}$ then the iterated preimages of p are contained in both basins and are dense in J . Therefore the closure of either basin is equal to J .

Now if the open set U intersects the Julia set then it contains an iterated preimage of p . Since f is an open mapping $f^{-n}(U)$ is an open nbhd of p . If $(\mathcal{L} \text{ or } \mathcal{R})$ has negative transverse exponent then $f^{-n}(U)$ intersects the corresponding basin in a set of positive measure.

Choosing a regular value of f^n which is a point of density for this intersection and choosing a point $q \in U$ which maps to this

regular value. It follows that any nbhd of q intersects the corresponding basin in a set of positive measure $\#$