A Debt Strategy Simulation Framework and Interest-rate Model Risk

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Overview

- Basic problem and objectives.
- Constrained stochastic control problem.
- Simulation framework for evaluating financing strategies.
    - Stochastic model (model risk).
    - Control model.
    - Analysis of output.
- Interest-rate models.
- Simple example.
- Summary and conclusions.
- Shameless plug.

The Basic Problem

- Existing stock of accumulated debt.
    - Bonds and bills at various maturities.
- Maturing bonds need to be paid for by issuing new ones.
- What is the best way to do this?

    or

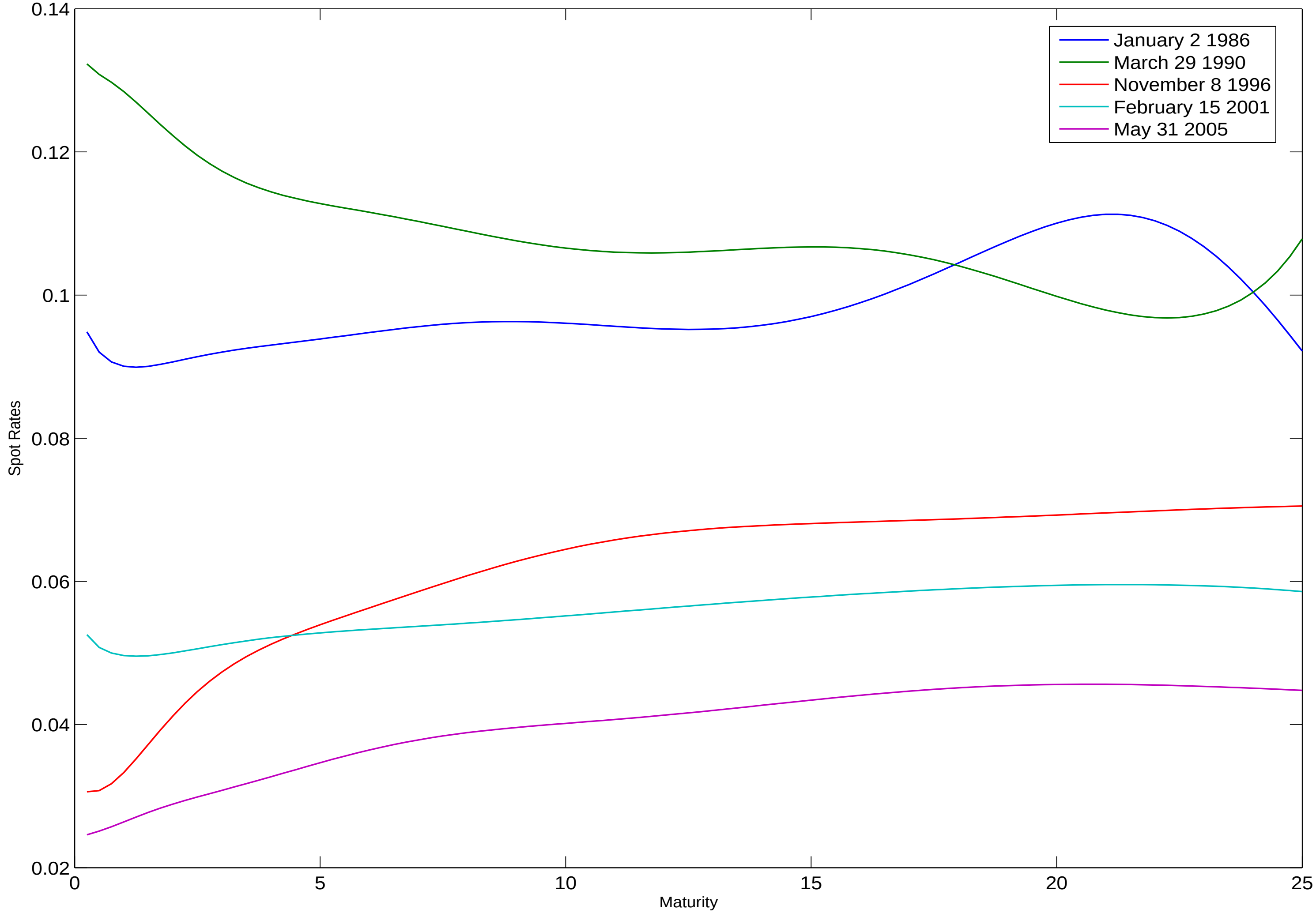
  Is there an optimal way to structure the debt portfolio?
- References
    - Bolder 2003, Adamo et al 2004, Hahm and Kim 2003, Holmlund and Lindberg 2002.
    - Discussion papers by debt managers in Sweden, U.K., Denmark, and World Bank and International Monetary Fund.
    - Missale 1994, Barro 2003.

Basic Objectives

1. Lower average debt costs.
   - Government debt portfolios are typically large.
   - Small improvements in managing the portfolio lead to significant savings.
   - Example: Consider
     - $500 Billion portfolio.
     - 1 basis point improvement.
     - leads to $50 Million in annual savings.
2. Lower Debt-cost variability (risk).
   - More on this in a moment.

Canadian Term Structures

Data Source: Bank of Canada



Further Details and Issues

- Let
  - R_t be government revenues in period t;
  - S_t be government spending in period t; and
  - C_t be debt-service charges in period t.
- The government's primary balance in period t is

$$PB_t = R_t - S_t.$$

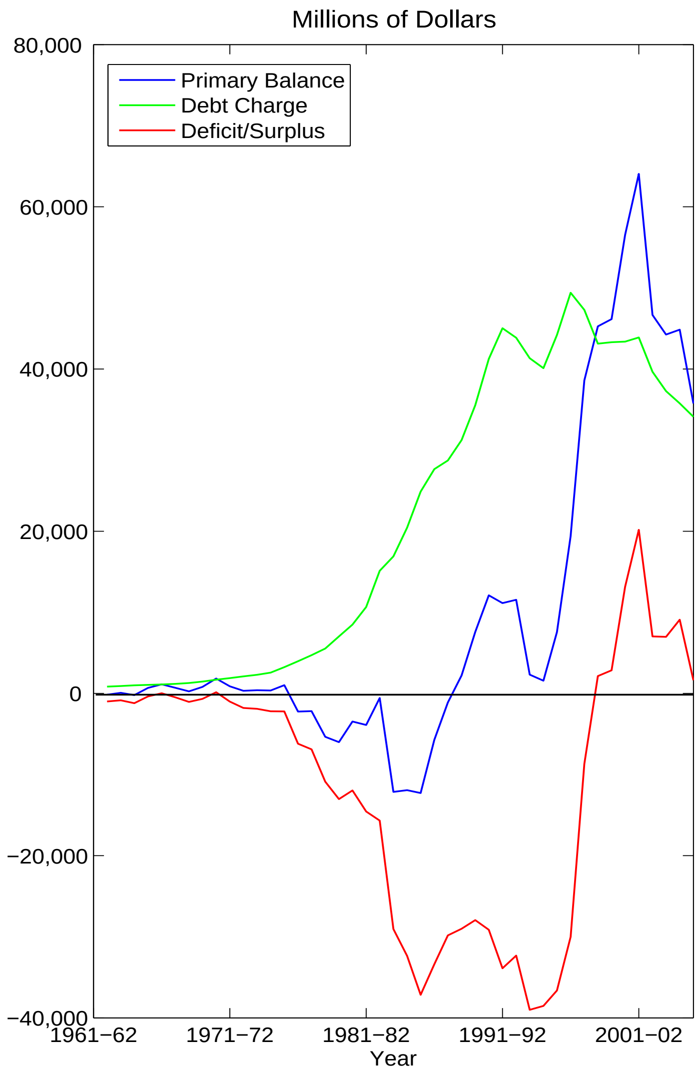
- The government's financial requirement in period t is

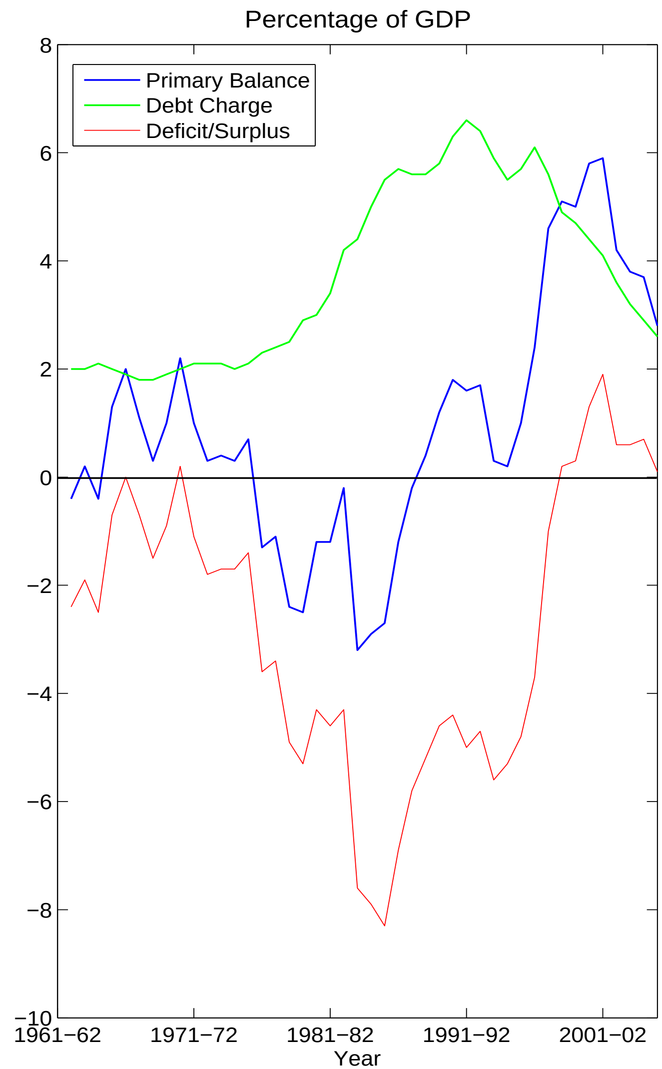
$$F_t = PB_t - C_t.$$

- Note: R_t, S_t, and C_t all depend on macroeconomic conditions.

Govt Canada Financials

Data Source: Ministry of Finance Canada





Budgetary Volatility

- The variance of the government's financial requirement is

$$Var(F_t) = Var(PB_t) + \underbrace{Var(C_t) - 2Cov(PB_t, C_t)}_{\text{Influence of Debt Charges}}.$$

- Usually $Cov(PB_t, C_t) > 0$.
- In stressed conditions (e.g., stagflation) can have

$$Cov(PB_t, C_t) < 0.$$

  which can significantly increase $Var(F_t)$.
- Under such conditions, the financial position is more volatile during a (bad) period where
  - cost of borrowing is high (C_t); and
  - may need to borrow more (low PB_t).

A Stochastic Control Problem

- Let
  - Θ be the set of admissible financing strategies;
  - θ denote a member of Θ; and
  - T be the time horizon of interest.
- The debt-strategy problem can be expressed as

$$\min_{\theta(t)} E\left[\int_0^T C_t dt\right]$$

subject to

$$Var\left[\int_0^T C_t dt\right] \leq \delta \quad \text{and}$$
$$\theta(t) \in \Theta,$$

where $\delta > 0$ is a risk constraint.

Constraints (Determines Θ)

- Can be imposed by
  - national rules;
  - supranational rules; and
  - market practices.
- Need to maintain a large amount of bonds in issuance at the benchmark maturities.
  - A minimum issuance amount at each benchmark maturity.
- Government is not a price-taker (typically).
  - Amount issued at a given maturity can affect the rate.
  - May not be enough demand for an extremely large issuance at a given maturity.
  - A maximum issuance amount at each benchmark maturity.

Constraints

- European Union countries must adhere to the conditions in the Growth and Stability Pact.
  "sound and disciplined public finances"
  - budget deficit $<$ 3% of gross domestic product.
  - nominal debt $<$ 60% of gross domestic product.
- Maintain a certain balance in the national treasury cash account.

Stochastic Model

- The complete stochastic environment includes models for
  - interest rates (direct influence on debt charges);
  - primary balance;
  - macroeconomic conditions; and
  - relationships between these components.
- Note: Analysis of model risk involves investigating the entire stochastic model.
  (very ambitious, also very important)
- In our current work, we are concerned only with the model risk associated with interest rates.

Control Model

- Note that $E\left[\int_0^T C_t dt\right], Var\left[\int_0^T C_t dt\right]$, and Θ depend on the composition of the initial portfolio (existing debt stock).
- Want to compare financing strategies for portfolios in equilibrium or steady-state.
- Searching for an optimal way to structure the debt portfolio
  not
  an optimal way to transition from the existing portfolio to a new one.

Control Model

- Time horizon of interest.
- Initial portfolio.
- Financing strategy (θ)
  - keep initial portfolio in steady-state.
  - deterministic.
  - stochastic.
- Feedbacks
  - Primary balance feedback.
  - Issuance feedback.

Simulation Framework

- Stochastic and Control Models are input into a debt strategy engine.
- Any feedbacks are accounted for accordingly.
- For each financing strategy, the output consists of simulated debt-charge sample paths, corresponding to the simulated term-structure paths.
- Perform an analysis of the output.

Interest-rate Model

- Development of most interest-rate models focussed on pricing and hedging relatively short-term derivatives (CIR, Vasicek, forward-rate models).
- There is a need for models to deal with long-term interest-rate derivatives and risk-management problems
  - Insurance.
  - Government debt management.

Interest-rate Model

- Desirable properties of an interest-rate model include:
  - positive rates;
  - mean reversion;
  - simple formula for bond prices and other derivatives;
  - realistic long-term behaviour of interest rates; and
  - a wide variety of term-structure shapes.
- Models used in our example
  - 2-factor Cox-Ingersoll-Ross (CIR).
  - 2-factor positive-interest model (Cairns 2004).

CIR Model

- Two independent state variables, X_1 and X_2, that satisfy

$$dX_i(t) = \alpha_i(\mu_i - X_i(t))dt + \sigma_i\sqrt{X_i(t)}dW_i(t).$$

- The condition $2\alpha_i\mu_i \geq \sigma_i^2$ keeps all rates positive.
- The short rate is defined as

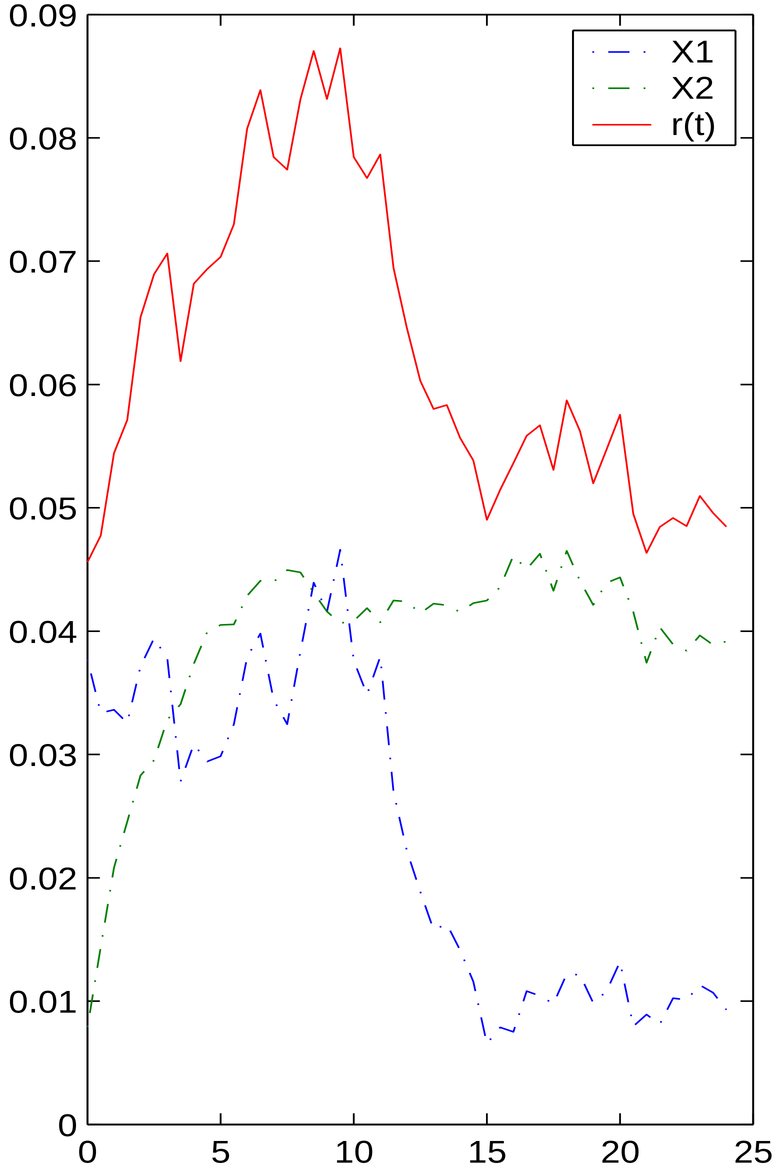
$$r(t) = X_1(t) + X_2(t).$$

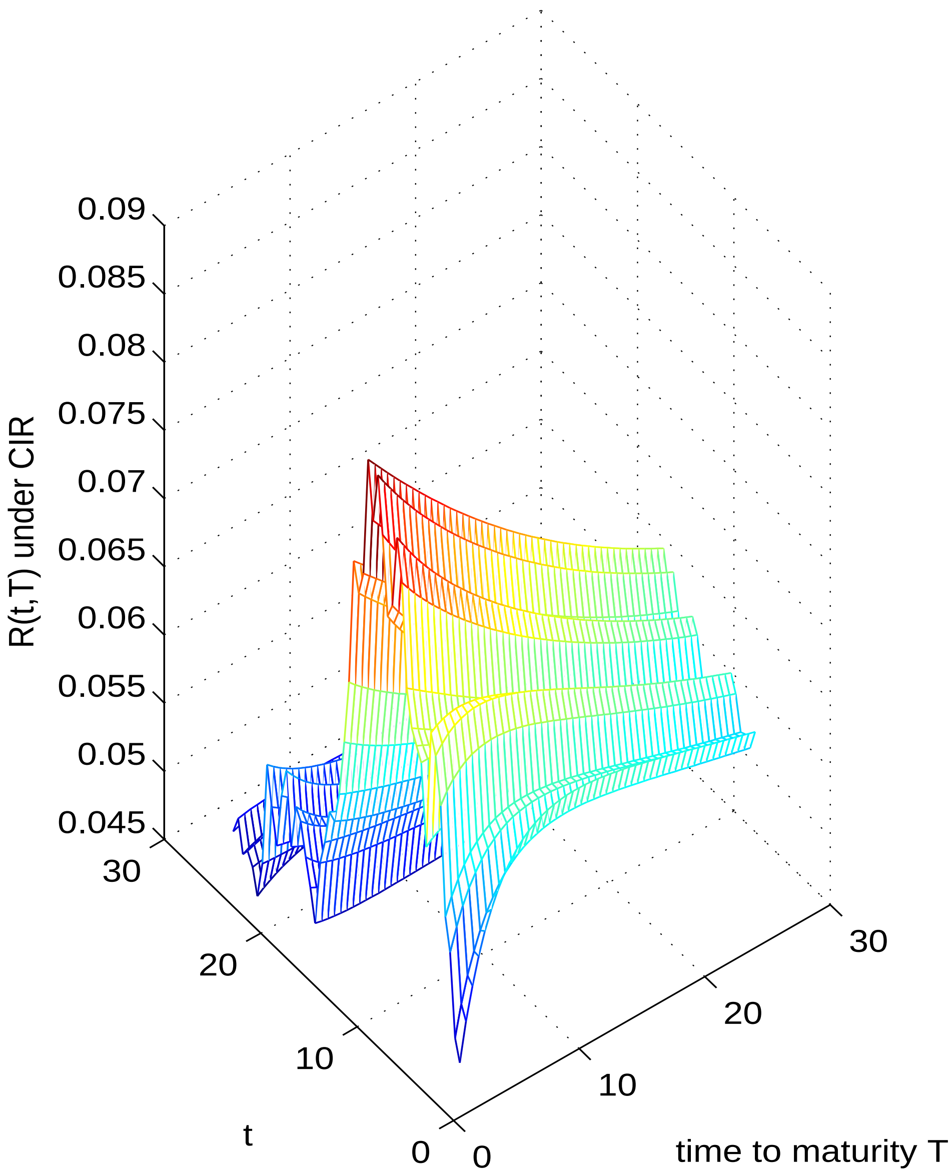
- The zero-coupon bond price is

$$P(t,T) = \exp\left[A_1(\tau) + A_2(\tau) - B_1(\tau)X_1(t) - B_2(\tau)X_2(t)\right],$$

  where $\tau = T - t$ and A_i, B_i are known functions.
- Model used in Bolder 2003.

CIR Model Sample Path





Positive-interest Model (Cairns 2004)

- Two correlated state variables, X_1 and X_2 that satisfy

$$dX_i(t) = \alpha_i(\mu_i - X_i(t))dt + \sigma_i dW_i(t).$$

- Define the function

$$H(u, x_1, x_2) = \exp\Big[-\beta u + \sigma_1 x_1 e^{-\alpha_1 u} + \sigma_2 x_2 e^{-\alpha_2 u} - \frac{1}{2}\sum_{i,j=1}^{2} \frac{\rho_{ij}\sigma_i\sigma_j}{\alpha_i + \alpha_j} e^{-(\alpha_i+\alpha_j)u}\Big].$$

Positive-interest Model (Cairns 2004)

- The zero-coupon bond price is defined by

$$P(t,T) = \frac{\int_{T-t}^{\infty} H(u, X(t))du}{\int_0^{\infty} H(u, X(t))du}$$

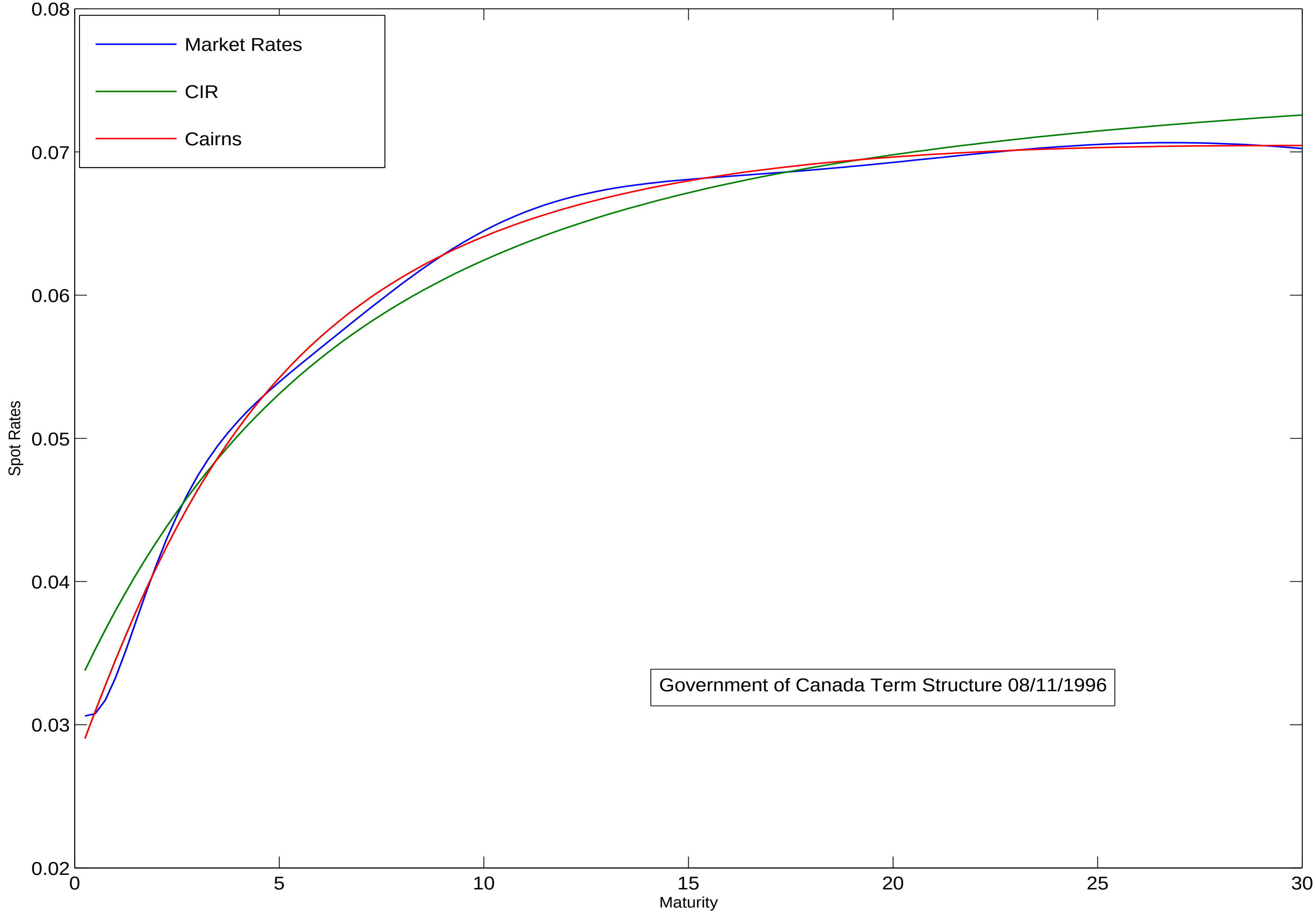
- The short rate is

$$r(t) = \frac{H(0, X(t))}{\int_0^{\infty} H(u, X(t))du}$$

- Flesaker and Hughston 1996, Rutkowski 1997, Rogers 1997

Fit Canadian Term Structure

Data Source: Bank of Canada



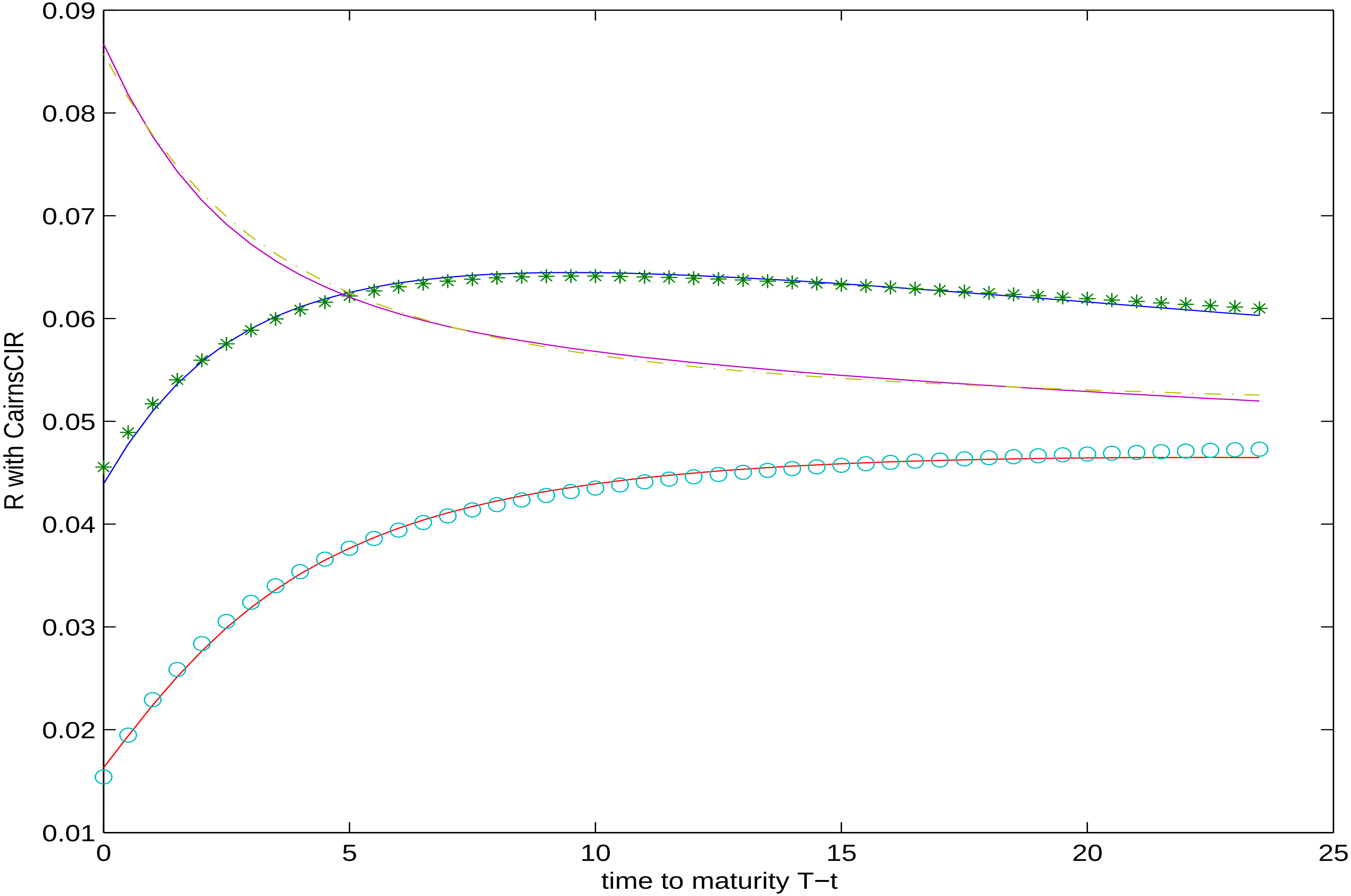
A Simple Example

- Assess the potential interest-rate model risk associated with maintaining a portfolio of zero-coupon bonds.
- Debt charges are the interest expenses.
- Stochastic model
  - 2-factor CIR and positive interest-rate models.
    - Calibration of the two interest-rate models.
  - Government's primary balance and the macroeconomic conditions are not modelled.
- Black and Telmer 1999.

A Simple Example

- Control model
  - Time horizon of 20 years.
  - Initial portfolio balance of \$150, with equal amounts at the 6-month, 5-year, and 10-year maturities.
  - Financing strategy — issue new debt to pay off maturing debt such that
    - the initial balance is maintained.
    - the duration and convexity of the portfolio remain constant.
    - issue only at the 6-month, 5-year, and 10-year maturities.
    - no other constraints.
  - Feedbacks
    - Issuance feedback only.
- Distributional Analysis
  - Based on 1000 sample paths.

Interest-rate Model Calibration

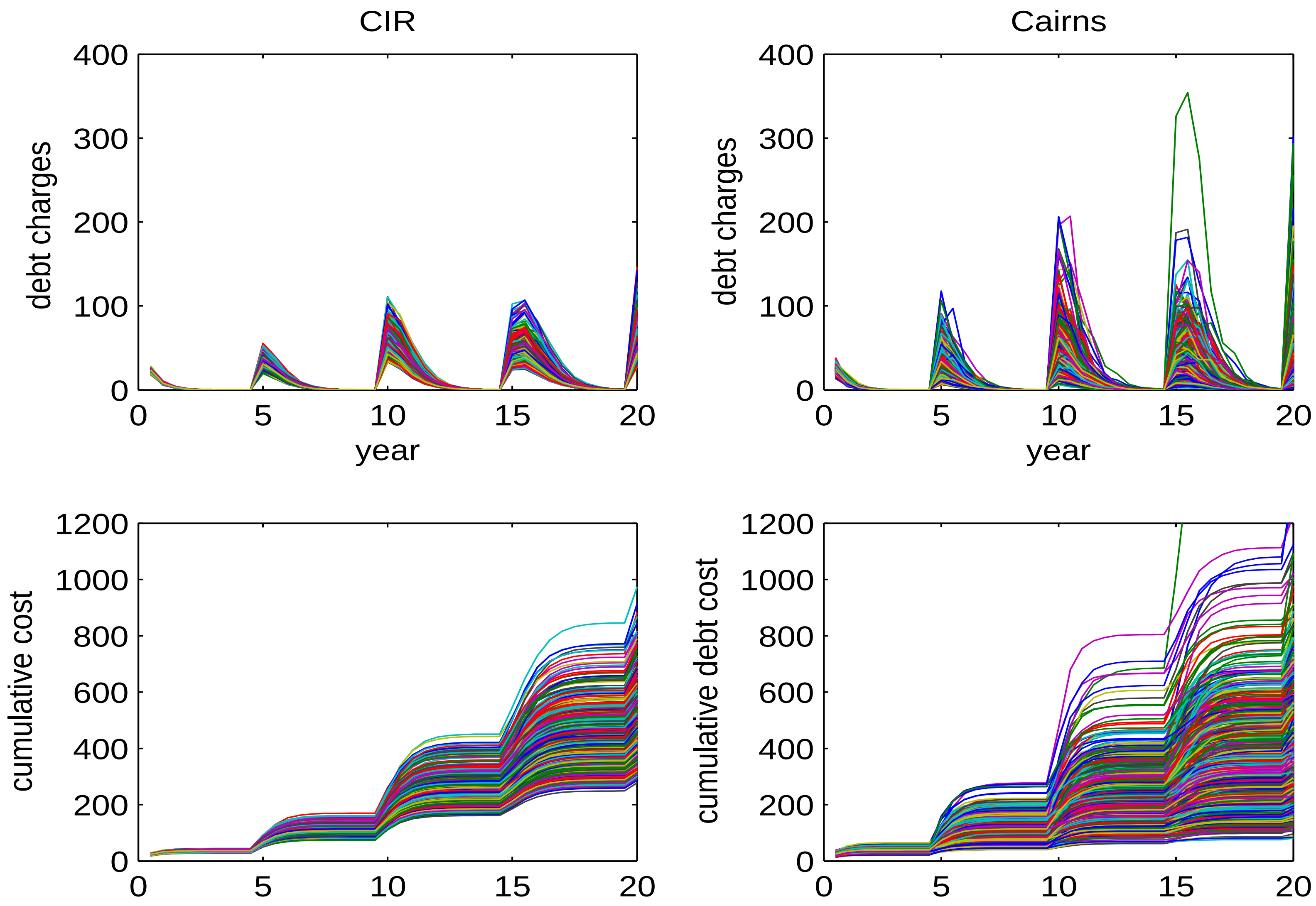


Interest-rate Model Calibration

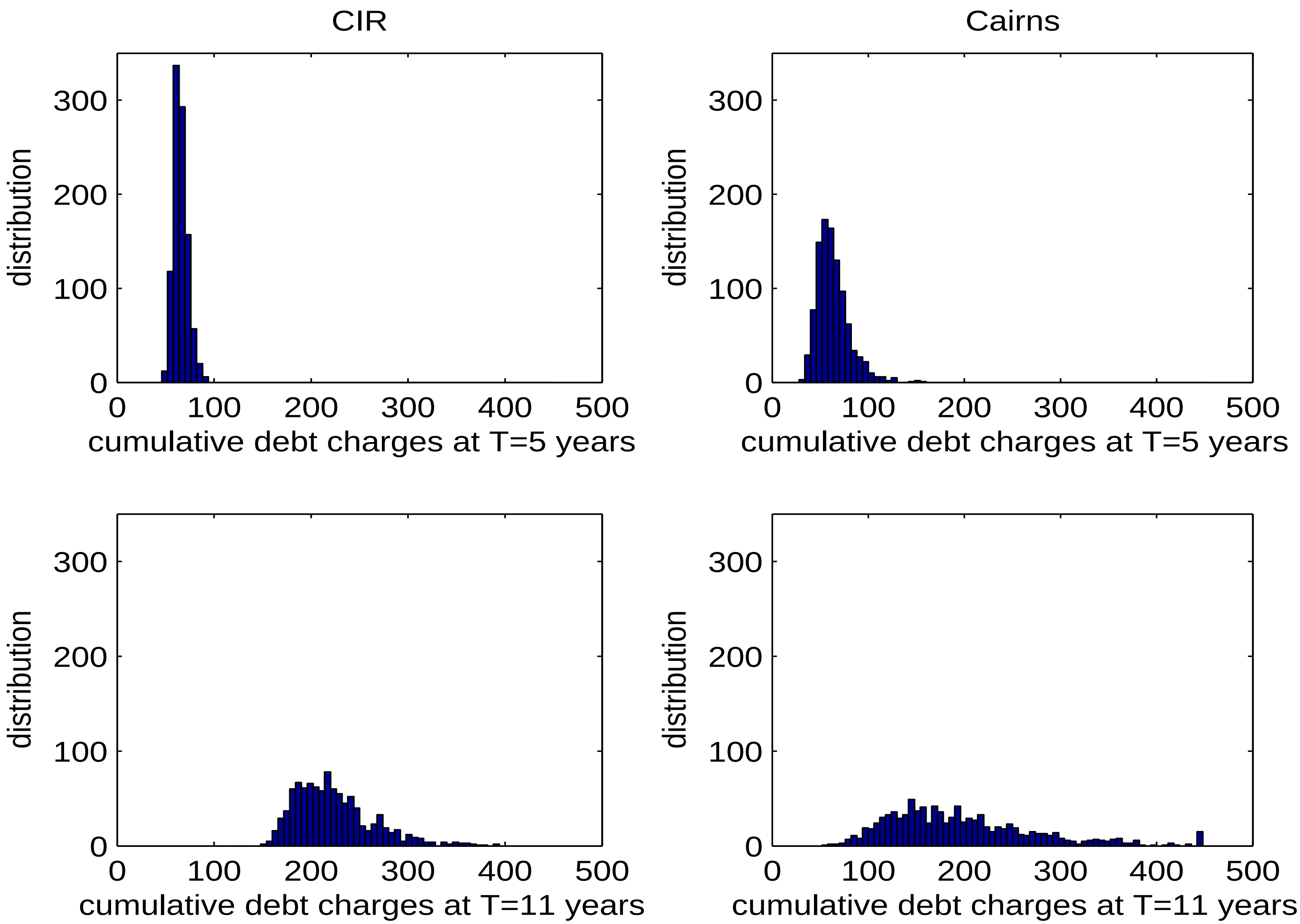
- Parameters for the CIR model are risk-neutral-measure (Q) parameters.
- Parameters for the Cairns model are pricing-measure ($\hat{P}$) parameters.
- For risk management, need real-world measure (P) dynamics.

Conjecture: Suppose that the CIR and Cairns models produce similar term-structures with $Q-$ and $\hat{P}-$parameters, respectively. Then converting each of these models to their real-world counterparts requires adjusting each in a similar manner.

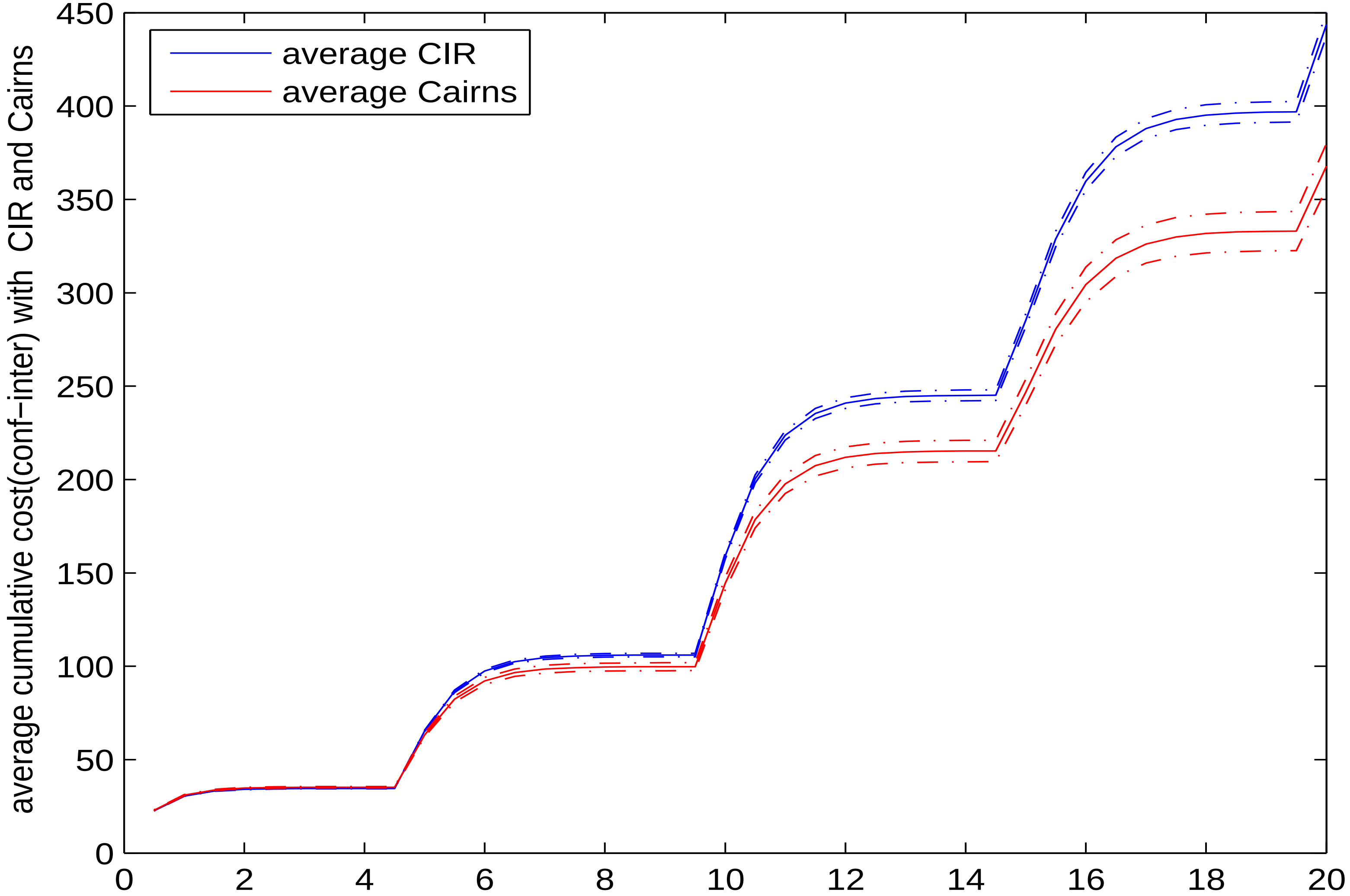
Simulation Output



Cumulative Debt Charges



Average Cumulative Debt Charges



Summary and Conclusions

- Debt-strategy problem is a stochastic optimal control problem.
- Simulation framework is valuable for analysing various debt strategies or portfolio structures.
- Example investigating interest-rate model risk.
- Implication for debt-strategy analysis — use a variety of interest-rate models. Why?
  - Strategies are evaluated on a cost-risk basis.
  - Measures of cost and risk are features of the debt-charge distribution.
  - Interest-rate model plays a significant role in determining the debt-charge distribution.

Current and Future Work

- Better calibration of the two models.
- Estimation of the models using historical data.
- Look at other interest-rate models.
- Expand the example to be more realistic.
  - Wider range of financing strategies.
  - Coupon bonds.
  - Issuance constraints.
  - Models for the primary balance and macroeconomy.

Shameless Plug

- Quantitative Finance Conference on Credit Risk
  Saturday November 5, 2005
  University of Western Ontario
- www.fields.utoronto.ca/programs/cim/05-06/credit-risk/
- Sponsored by Fields, MITACS, and SHARCNet.
- Speakers
  - Kay Giescke, Stanford University.
  - Michael Gordy, US Federal Reserve.
  - Greg Gupton, Moodys/KMV.
  - Tom Hurd, McMaster University.
  - Alex Kreinin, Algorithmics.
  - Niall Whelan, Scotia Capital.
  - Weidong Tian, University of Waterloo.
  - Michael Walker, University of Toronto.