Indifference Pricing for Equity-Linked Insurance & Reinsurance

The Fields Institute

Quantitative Finance Seminar Series

April 26th, 2006

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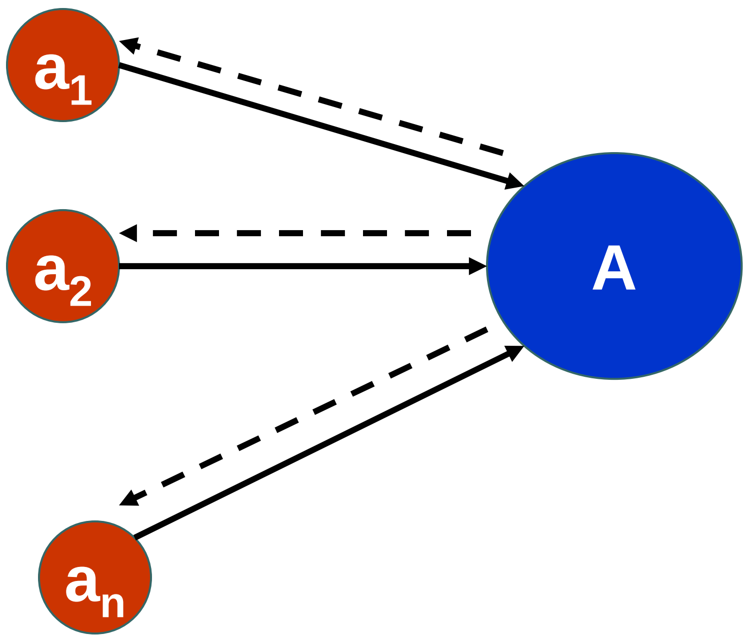
Outline

- Overview of Equity-Linked Insurance and Reinsurance Products
- Utility Indifference Pricing Principle
- Insurance Premium Valuation
- Reinsurance Price Valuation
- Counterparty Risk
- Conclusions

Equity-Linked Insurance
and
Reinsurance Products

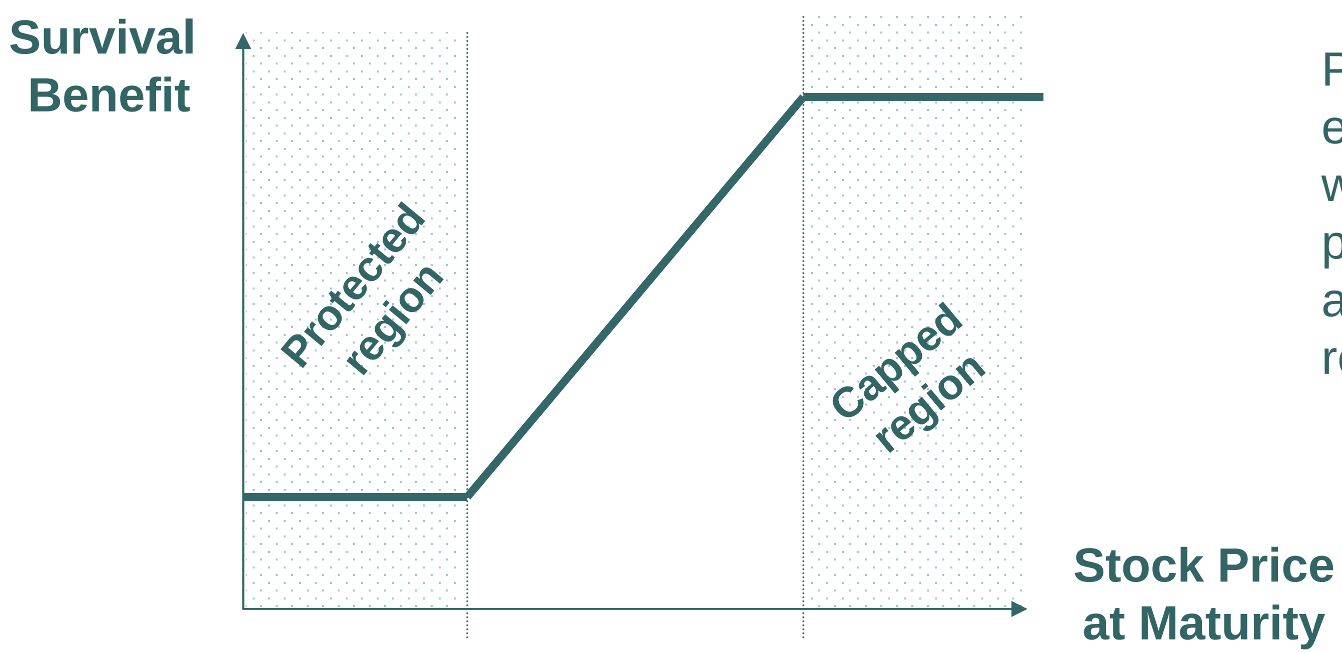
Equity Linked Insurance Products

- Insurer **A** sells insurance to individuals $\mathbf{a_1, \ldots, a_n}$
  - Individual $\mathbf{a_k}$ makes regular premium payments to **A**
  - **A** makes claim contingent payments to $\mathbf{a_k}$ at times $\mathbf{t_k^1, t_k^2, \ldots}$



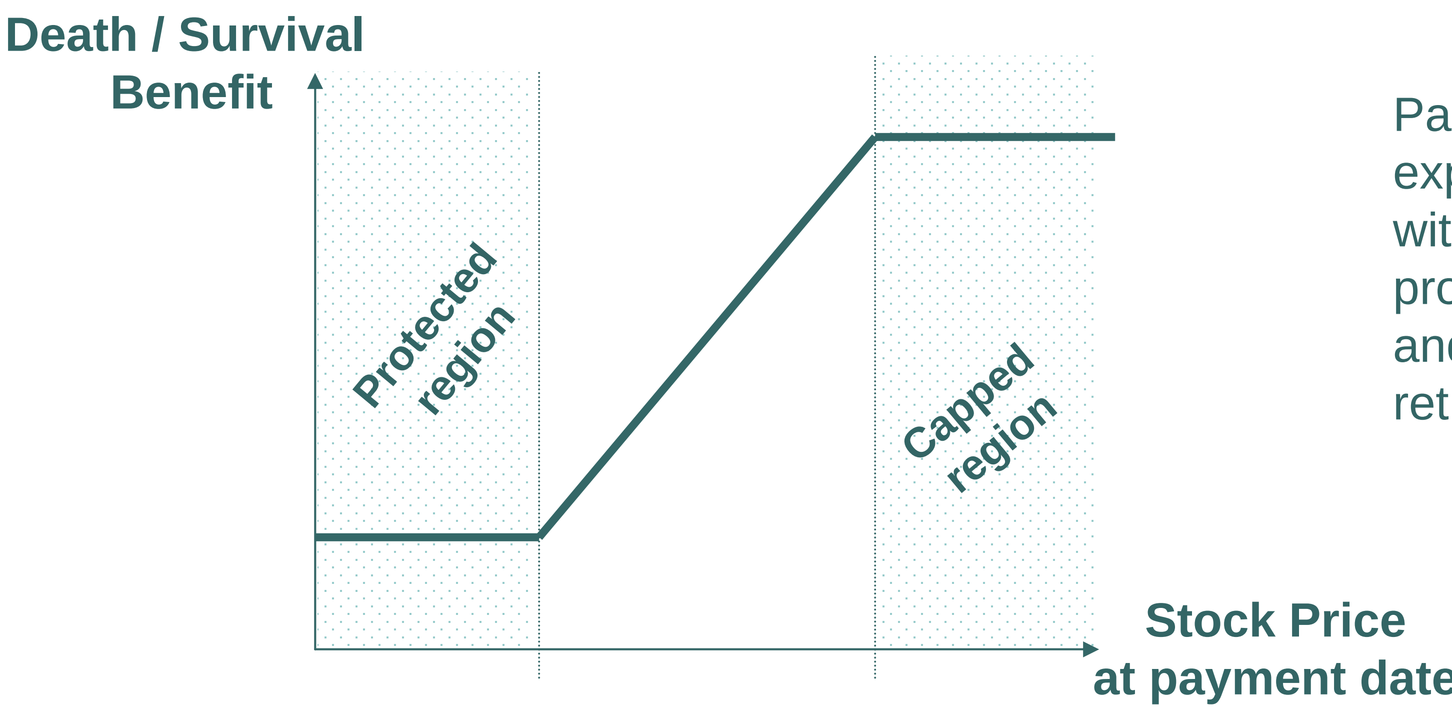
Equity Linked Insurance Products

- **Equity-Linked Pure Endowment**
  - Single benefit payment is linked to the value of an underlying asset at the maturity date – *contingent on survival*.



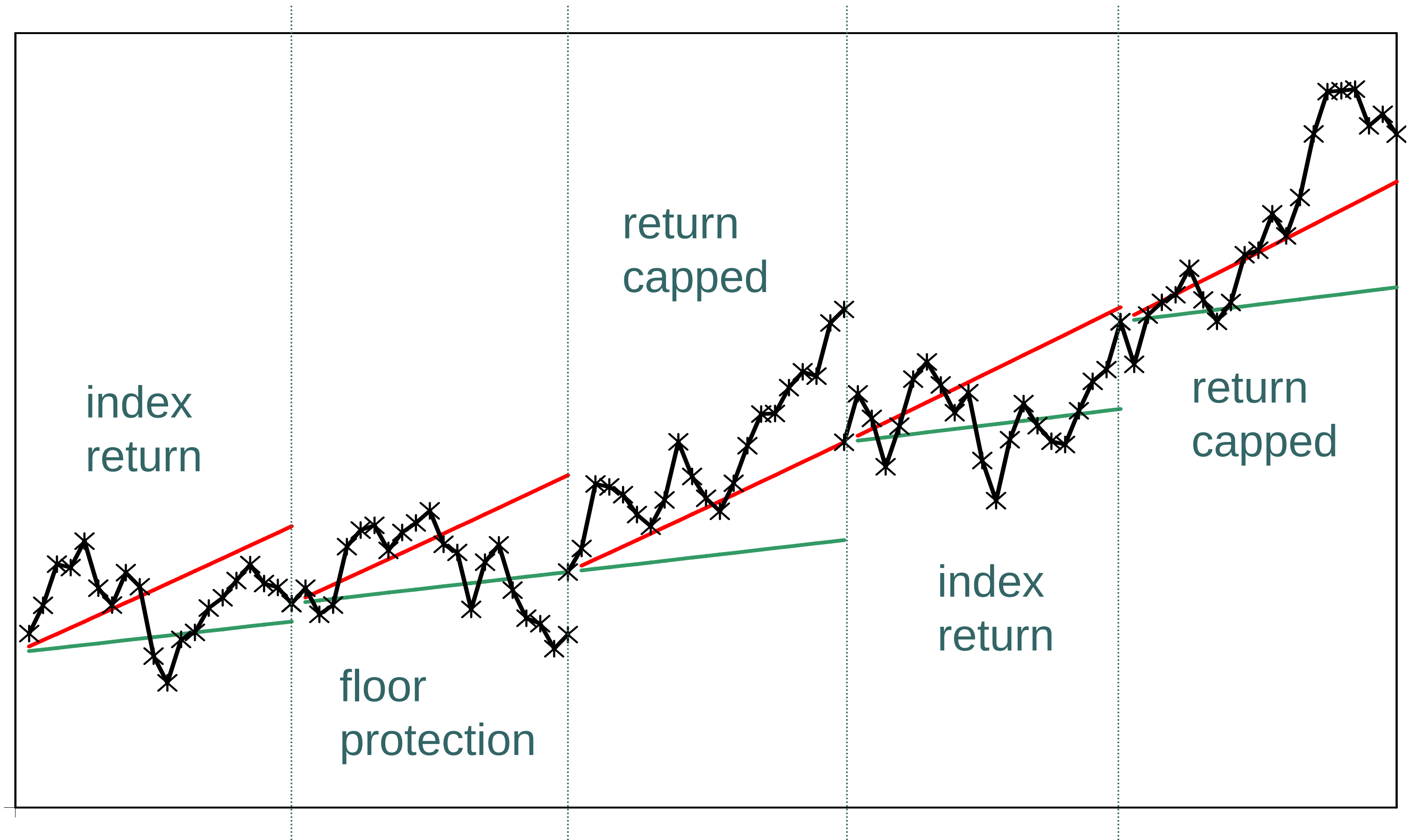
Partial Equity exposure with floor protection and capped returns

- **Equity Indexed Annuities**: **Point–to–Point design**
  - Single benefit payment linked to the value of an underlying asset at the end of the payment year (*death or survival*)



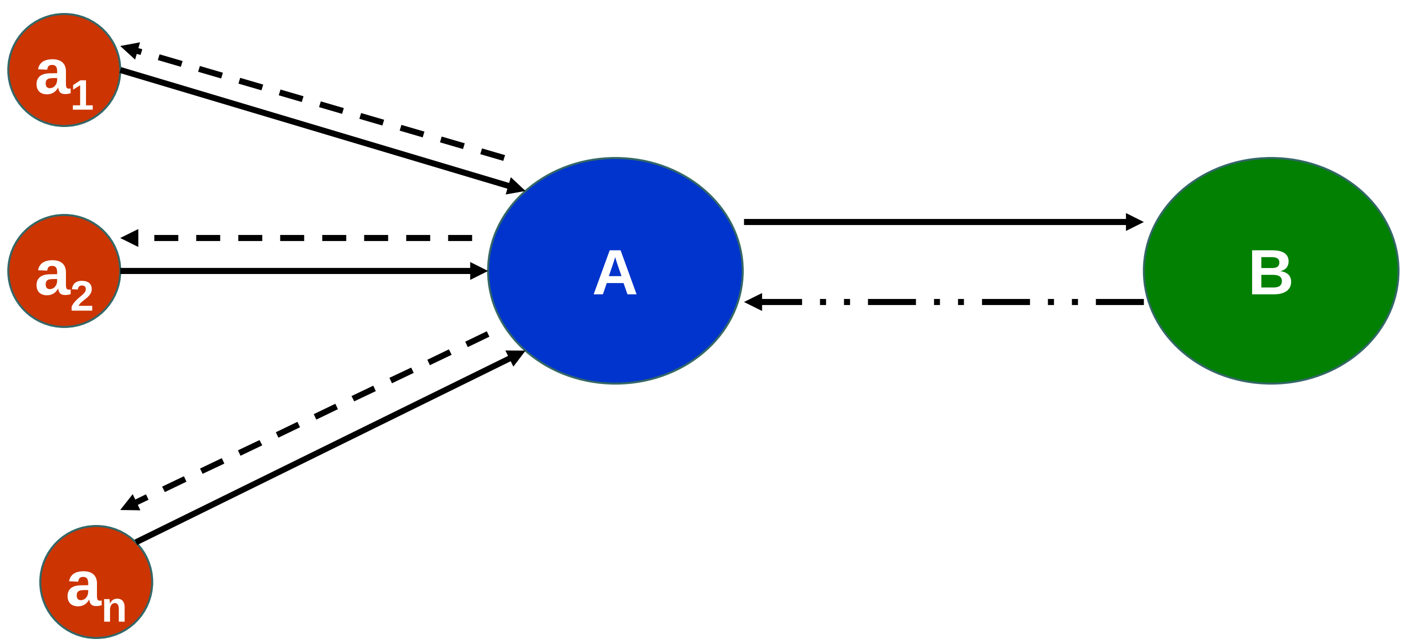
Partial Equity exposure with floor protection and capped returns

- **Equity Indexed Annuities**: **Ratchet design**



Equity Linked Reinsurance Products

- Reinsurer **B** sells reinsurance to **A**
  - **A** makes payments to **B** (can be regular or lump-sum)
  - **B** makes a contingent payment to **A** at maturity



Equity Linked Reinsurance Products

- **Risk management** traditionally falls into **two categories**:
  - **Insurance risks** (property, liability, life, etc…)
  - **Financial risks** (equity, interest rates, foreign exchange, etc…)

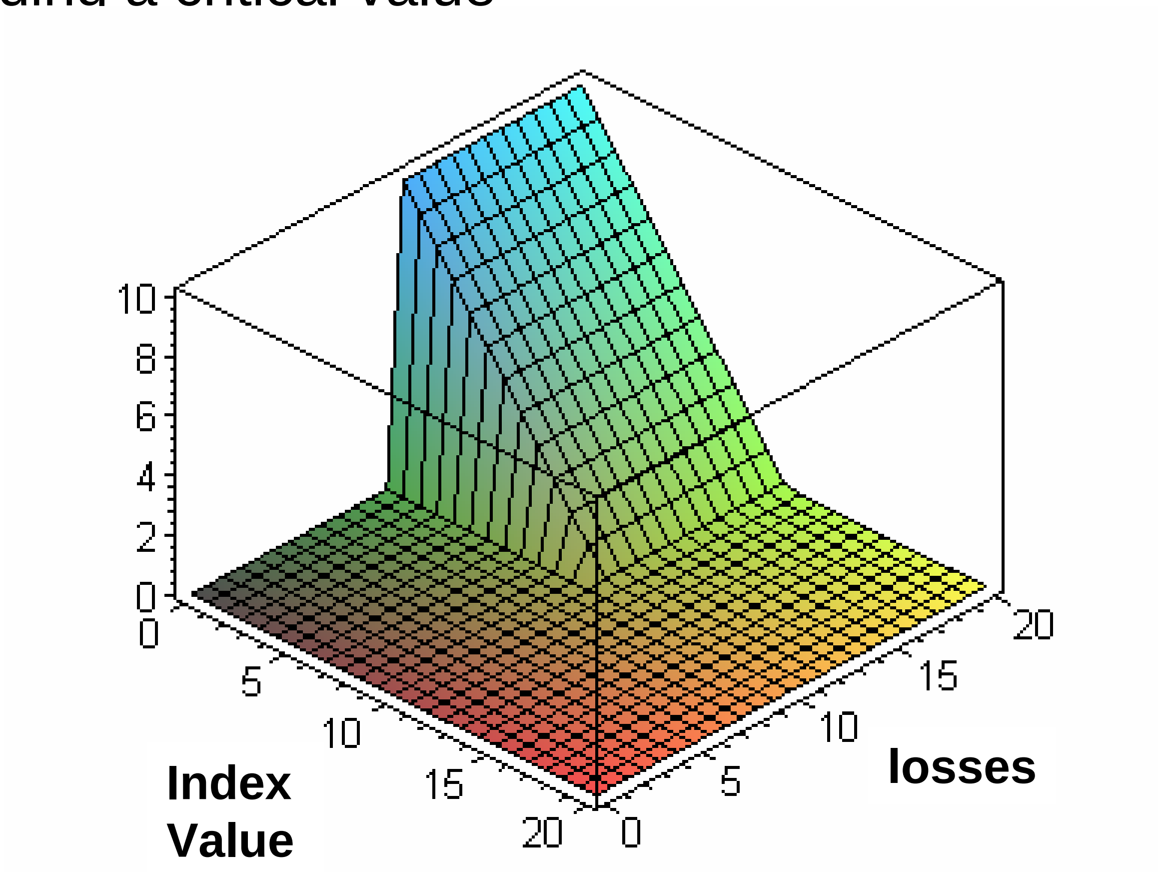
- **(Re)insurance** is purchased **yearly** to cover specific risks

- There is an **emerging market** of hybrid reinsurance products **linking insurance and financial risks**

- Particularly relevant for **equity-linked liabilities**

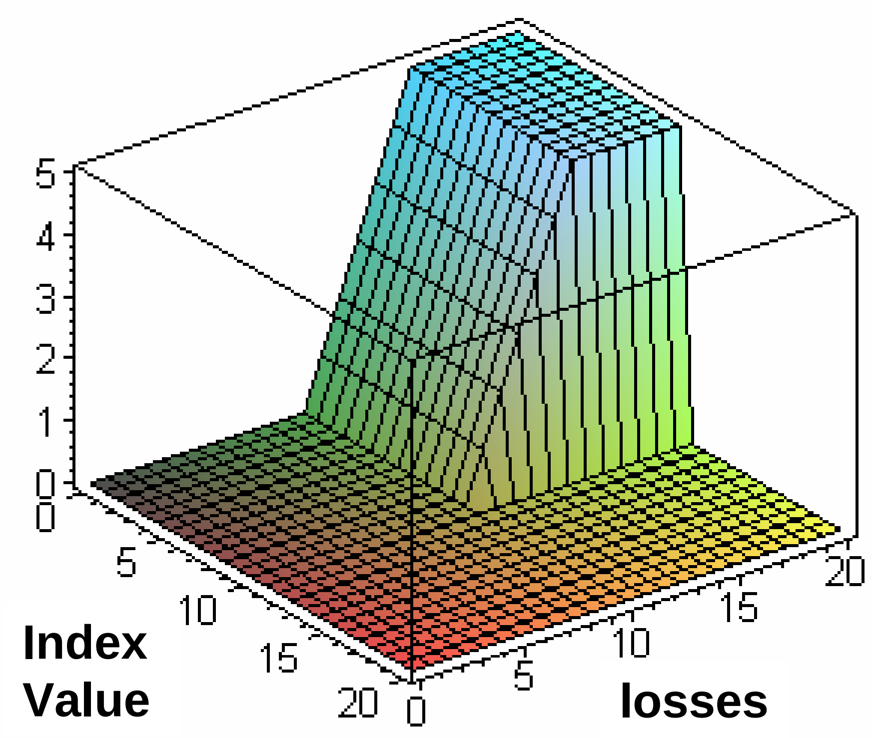
Equity Linked Reinsurance Products

- **[Catastrophe Equity Put Option]** (CatEPut)
  Protects the insured from losses contingent on the losses exceeding a critical value



Equity Linked Reinsurance Products

- **Double Trigger Stop-Loss Options**
  Protects the insured from losses contingent on the share value of the insured dropping below some critical level.



Insurer’s Problem :
Insurance Risk

The Insurer's Problem

- Insurer trades in:
  1. **Risky asset** (index) with price process **S(t)**

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma\, dX(t) \quad \text{where } X(t) \text{ is a } \mathbb{P} - \text{Wiener process}$$

  2. **Riskless asset** which grows at a rate of **r**

- **Claims** arrive according to a **Poisson process N(t)** with activity rate **λ(t)**

- The **claim sizes** are a **function of S(t)** at claim arrival times

$$L(t) = \int_0^t \int_{-\infty}^{\infty} y\, l(dy, dt) = \sum_{n=1}^{N(t)} g(S(t_n), t_n)$$

The Insurer's Wealth Process (without risk)

- The insurer invests $\pi(t)$ in **S(t)**
- The insurer invests $\pi_0(t)$ in the **risk-free money-market**
- The insurer has wealth **w** at time **t**
- The wealth process of the insurer is $W(t) = \pi(t) + \pi_0(t)$
- For self-financing strategies, **W(t)** satisfies:

$$\begin{cases} dW(u) &= [r\,W(u) + (\mu - r)\,\pi(u)]\,du + \sigma\,\pi(u)\,dX(u) \\ W(t) &= w\,. \end{cases}$$

The Insurer's Wealth Process (with risk)

- Insurer is **exposed to losses L(t)**
- Insurer receives **premium payments of q**
- The wealth process of the insurer is

$$\mathbf{W^L(t) = \pi(t) + \pi_0(t) + q\, t - L(t)}$$

- For "self-financing" strategies, $\mathbf{W^L(t)}$ satisfies:

$$\begin{cases} dW^L(u) & = \left[r\, W^L(u_-) + (\mu - r)\, \pi(u_-) + \mathbf{q}\right] du + \sigma\, \pi(u_-)\, dX(u) - \mathbf{dL(u)} \\ W^L(t) & = w\,. \end{cases}$$

Insurer's Problem :
Valuing the Insurance Risk

The Insurer's Indifference Premium

- The insurer wishes to **maximize expected utility of terminal wealth without risk**:

$$V(w,t) \quad = \quad \sup_{\{\pi(s)\}\in\mathcal{S}} \mathbb{E}_t^{\mathbb{P}}\left[u(W(T))\right]$$

- Then separately wishes to **maximize expected utility of terminal wealth with risk**:

$$U(w,S,t;q) = \sup_{\{\pi(s)\}\in\mathcal{S}} \mathbb{E}_t^{\mathbb{P}}\left[u(W^L(T))\right]$$

The Insurer's Indifference Premium

- The **indifference premium** q is defined such that

$$\boxed{U(w, S, t; q) = V(w, t)}$$

- That is, the insurer is **indifferent** between:

  I. Taking on the risk and receiving premiums

  II. Not taking on the risk, and receiving no premiums

The Insurer's Indifference Premium

- First, we solve for the value function V

- Applying the dynamic programming principal to **V** leads to the **HJB equation**

$$\begin{cases} V_t + r\,w\,V_w + \max_\pi \left[(\mu - r)\,\pi\,V_w + \frac{1}{2}\sigma^2\pi^2\,V_{ww}\right] = 0, \\ V(w,T) = u(w). \end{cases}$$

The Insurer's Indifference Premium

- For **exponential utility** the value function has an affine structure

$$V(w,t) = -\frac{1}{\hat{\alpha}} e^{-\alpha(t)\,w+\beta(t)}$$

- The **optimal investment** is independent of wealth

$$\pi^*(t) = -\frac{(\mu - r)V_w}{\sigma^2 V_{ww}}$$

- This is the famous **Merton result**

The Insurer's Indifference Premium

- Next, we solve for the value function U

- Applying the dynamic programming principal to **U** leads to a similar **HJB equation**

$$
\begin{aligned}
0 \quad &= U_t + (rW + \mathbf{q})U_w + \mu\,\mathbf{S}\,\mathbf{U_s} + \tfrac{\mathbf{1}}{\mathbf{2}}\sigma^{\mathbf{2}}\,\mathbf{S}^{\mathbf{2}}\,\mathbf{U_{ss}} \\
&\quad + \lambda(\mathbf{t})(\mathbf{U}(\mathbf{w} - \mathbf{g}(\mathbf{S},\mathbf{t}),\mathbf{S},\mathbf{t}) - \mathbf{U}(\mathbf{w},\mathbf{S},\mathbf{t})) \\
&\quad + \max_\pi \left\{ \tfrac{1}{2}\sigma^2 U_{ww}\,\pi^2 + \pi\left[(\mu - r)U_w + \sigma^{\mathbf{2}}\mathbf{S}\,\mathbf{U_{ws}}\right]\right\}
\end{aligned}
$$

$$
\text{subject to } U(w, S, T; q) = u(w)
$$

The Insurer's Indifference Premium

- The **optimal investment** is still independent of wealth, but now may depend on the asset's price

$$\boxed{\pi^*(t) = -\frac{(\mu - r)U_w + \sigma^2\,\mathbf{S(t)}\,\mathbf{U_{ws}}}{\sigma^2 U_{ww}}}$$

- We **can solve** the HJB equation for **exponential utility** by writing

$$U(w, S, t; q) = V(w, t)\ \exp\{-\eta(t)\, q + \gamma(S, t)\ \}$$

The Insurer's Indifference Premium

- We find that γ satisfies the **linear PDE**

$$\begin{cases} 0 &= \lambda(t)\left(e^{\alpha(t) g(S,t)} - 1\right) + rS\gamma_s + \frac{1}{2}\sigma^2 S^2 \gamma_{ss} + \gamma_t \ , \\ \gamma(S,T) &= 0 \ . \end{cases}$$

- **Feyman-Kac** with a source provides the solution

$$\boxed{\gamma(S(t),t) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \lambda(u)\left(e^{\alpha(u)\, g(S(u),u)} - 1\right) du\right]}$$

The Insurer's Indifference Premium

- Another interpretation of the pricing result

$$\gamma(S(t),t) = \mathbb{E}_t^{\widetilde{\mathbb{Q}}}\left[N(T)\right] - \mathbb{E}_t^{\mathbb{Q}}\left[N(T)\right]$$

$$\widetilde{\lambda}(S(t),t) = \lambda(t)\; e^{\alpha(t)\, g(S(t),t)}$$

- With constant losses and zero interest rates the measure $\widetilde{\mathbb{Q}}$ is the minimizer of

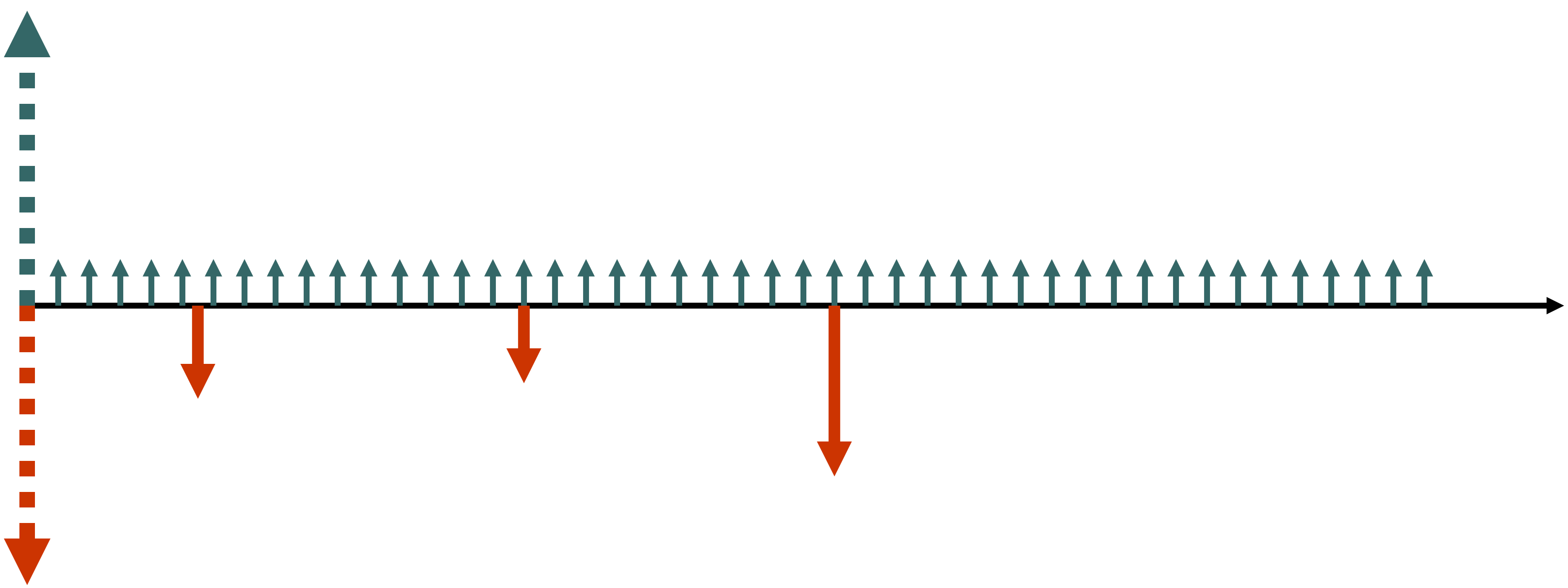
$$\inf_{\widetilde{\mathbb{Q}} \ll \mathbb{P}} \mathbb{E}^{\widetilde{\mathbb{Q}}}\left[\ln \frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} - \alpha\, g\, N(T)\right]$$

Insurer's Problem :
Insurance Risk - Examples

Near risk-neutral Insurers

- A **risk-neutral insurer** charges a premium of

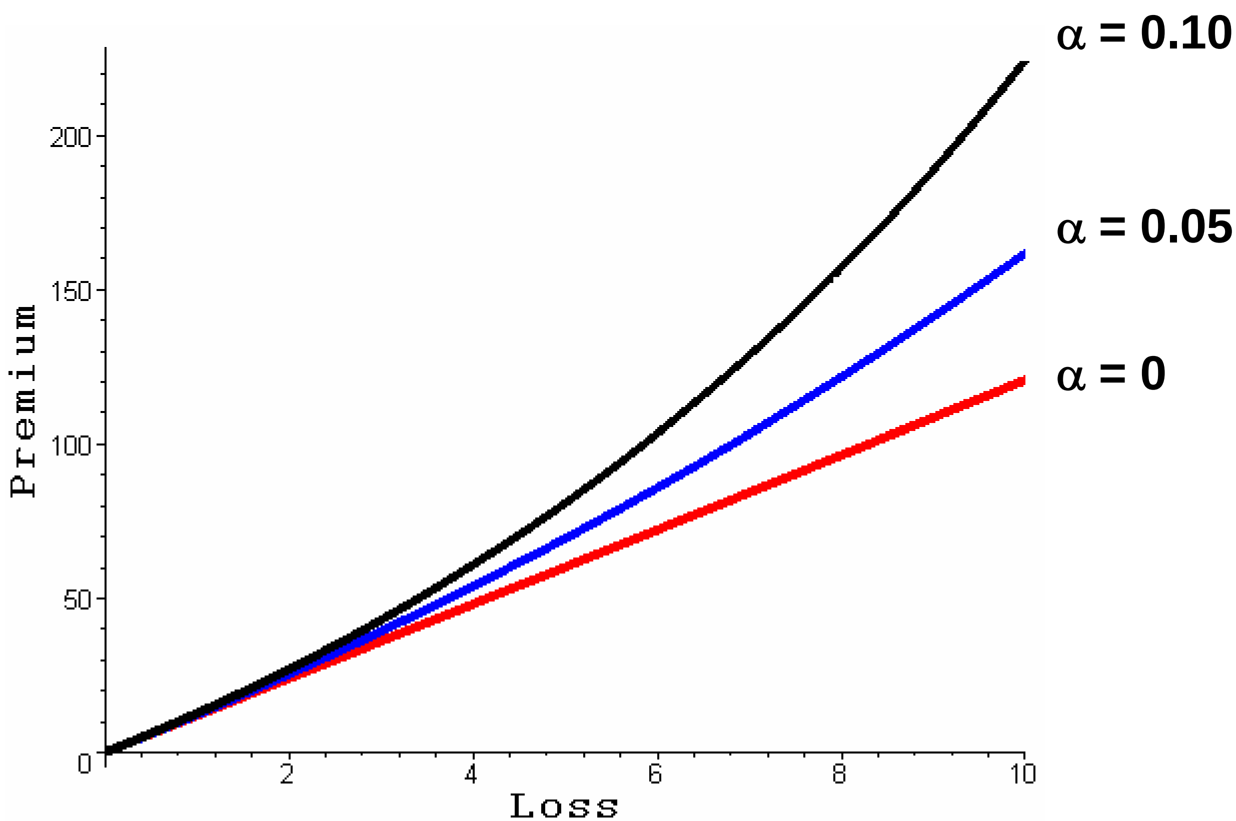
$$q_t = \frac{1}{\int_t^T e^{-r\,u} du} \, \mathbb{E}_t^{\mathbb{Q}} \left[\sum_{n=N(t)+1}^{N(T)} e^{-r\,t_n} g(S(t_n), t_n) \right]$$



The Insurer's Indifference Premium

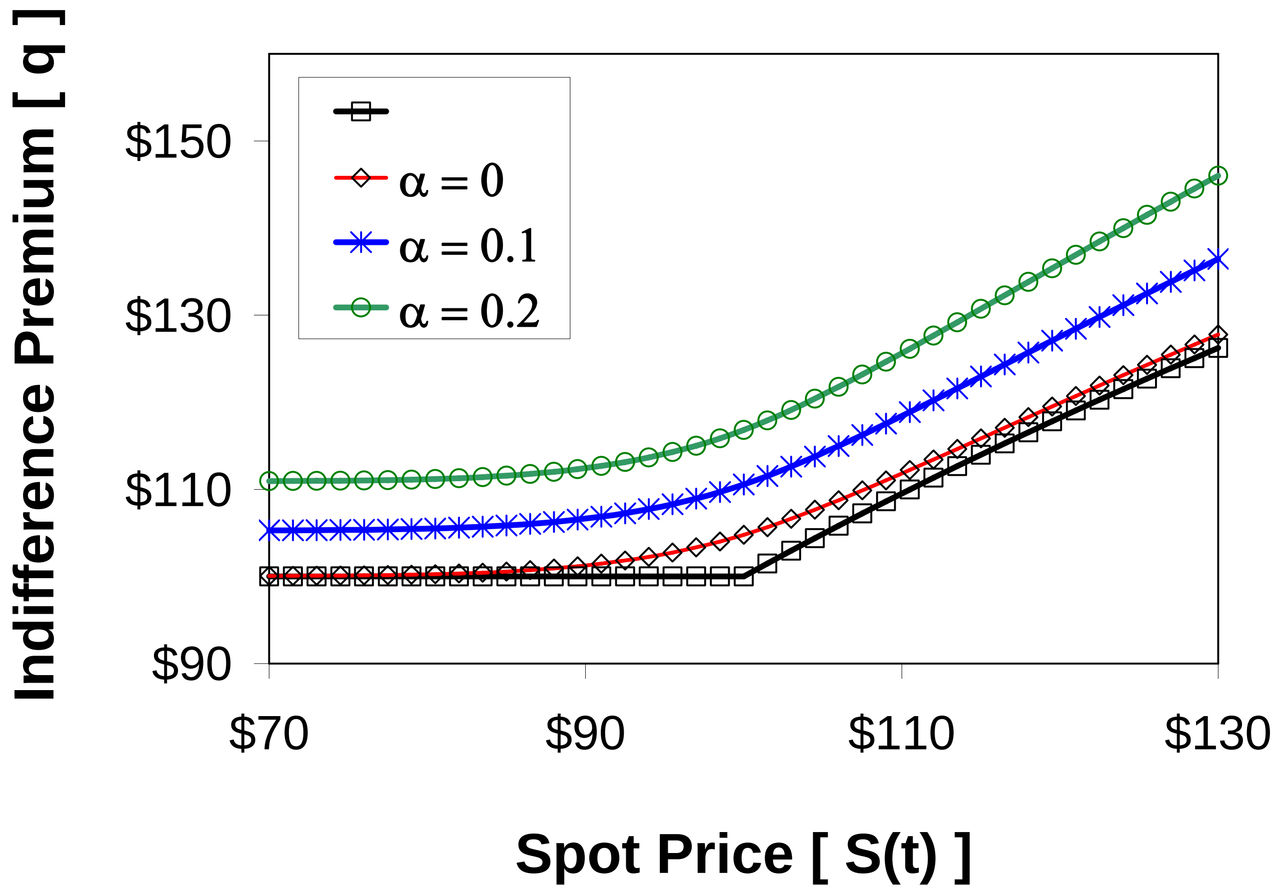
- **Constant losses** and activity rate we find the exact result

$$q = \frac{\lambda}{\hat{\alpha}\left(e^{r(T-t)} - 1\right)} \left(Ei(\hat{\alpha}\, l\, e^{r(T-t)}) - Ei(\hat{\alpha}\, l) - (T-t)r \right) \ ,$$



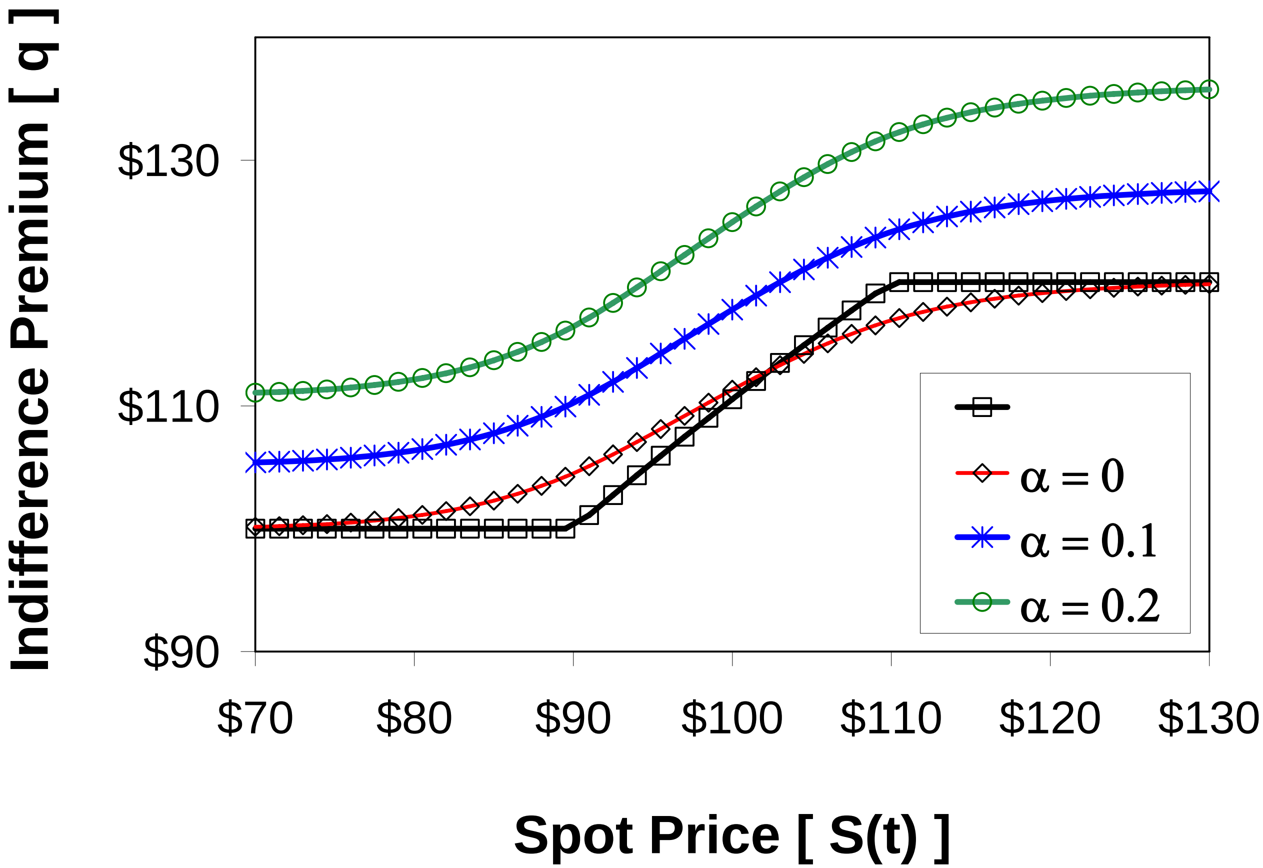
The Insurer's Indifference Premium

- **Minimum guaranteed benefit plus index participation**



The Insurer's Indifference Premium

- Minimum guaranteed benefit plus **capped** index participation



Insurer’s Problem :
Insurance Risk - Hedging

The Insurer's Hedge

- The risky asset investment **without** insurance risk is :

$$\pi^*(t) = \frac{\mu - r}{\hat{\alpha}\,\sigma^2} e^{-r(T-t)}$$

- The risky asset investment **with** insurance risk is :

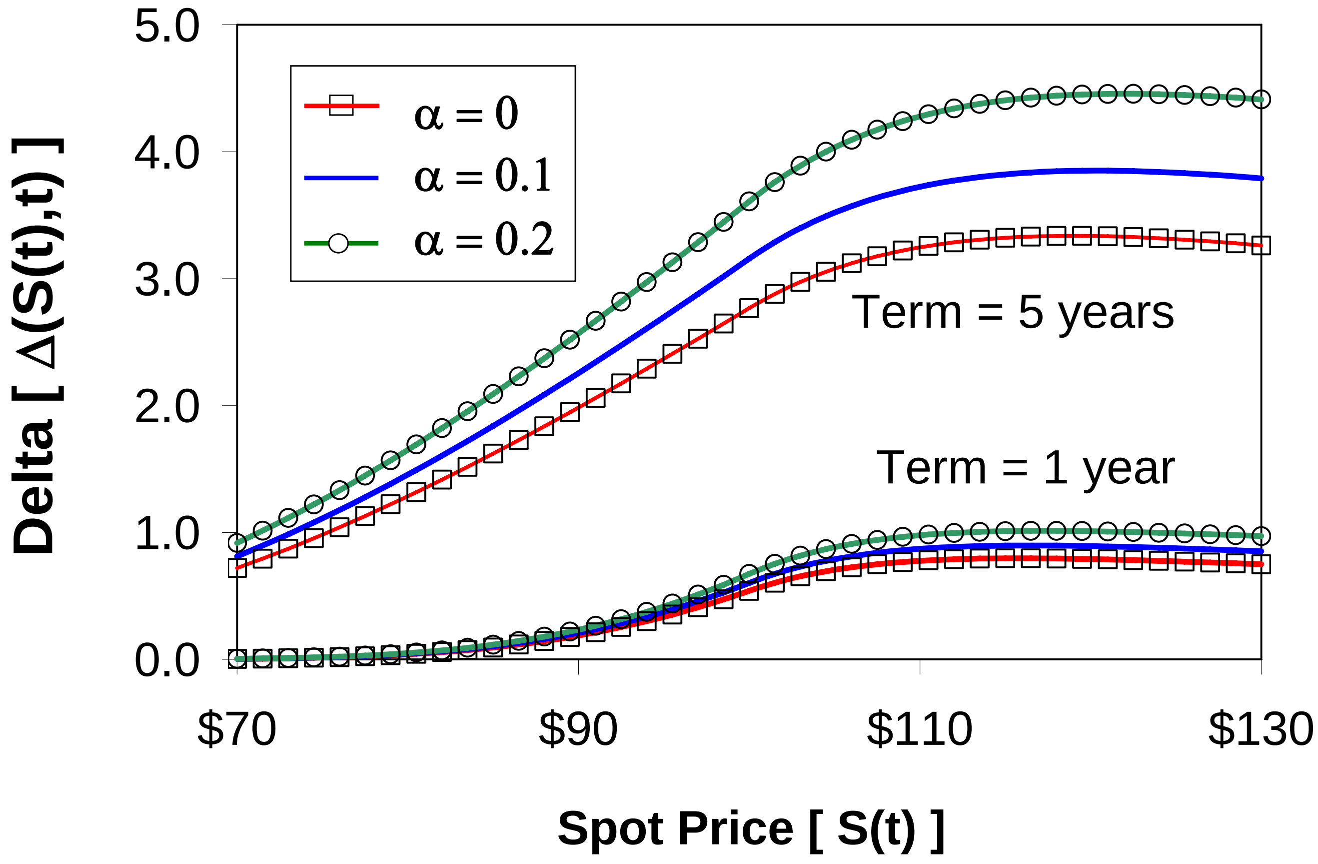
$$\pi^*(t) = -\frac{(\mu - r)U_w + \sigma^{\mathbf{2}}\,\mathbf{S(t)}\,\mathbf{U_{ws}}}{\sigma^2 U_{ww}}$$

- The hedge is the additional units of asset :

$$\boxed{\Delta = \frac{U_{ws}}{U_{ww}}\,S = \frac{\gamma_S}{\alpha(t)}\,S}$$

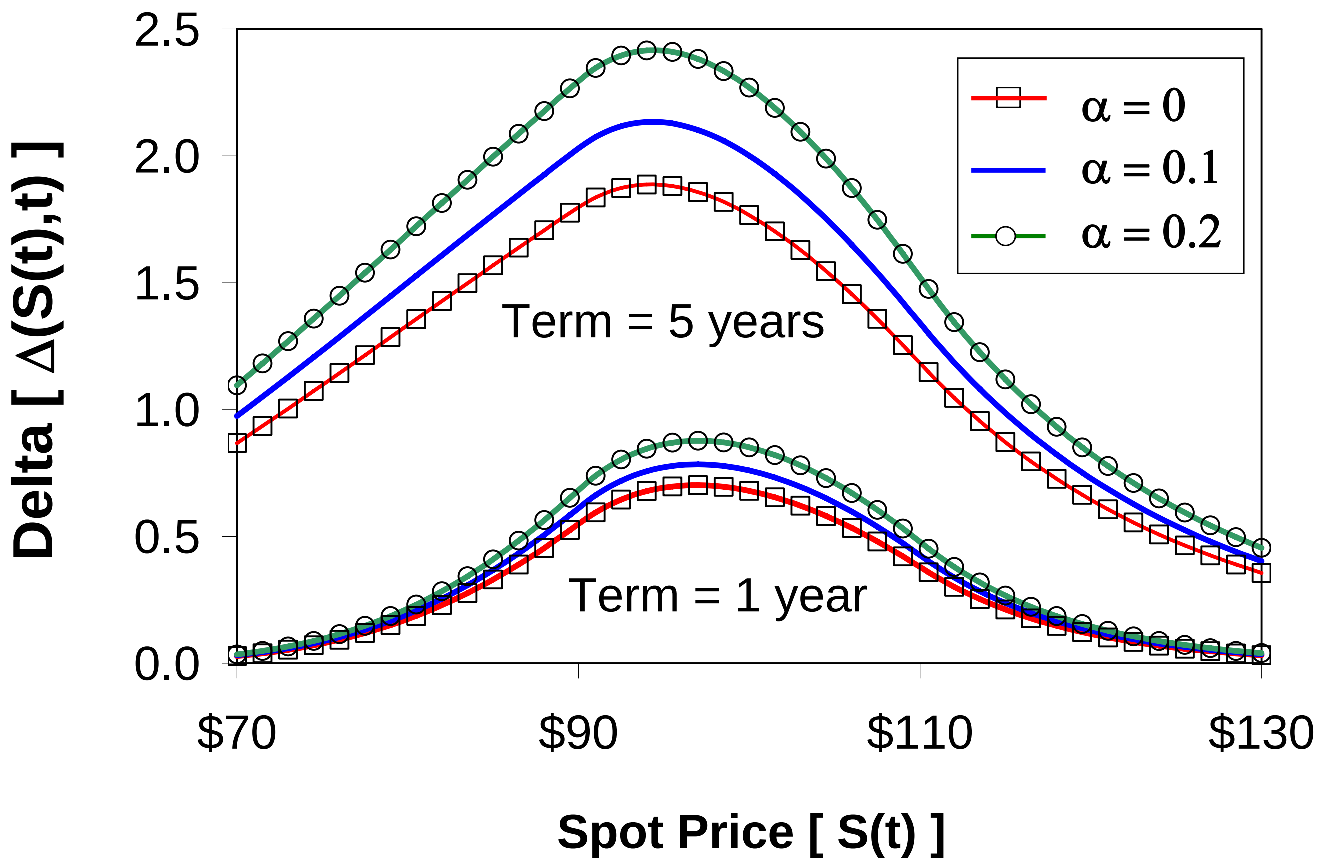
The Insurer's Hedge

- **Minimum guaranteed benefit plus index participation**



The Insurer’s Hedge

- Minimum guaranteed benefit plus **capped** index participation



Insurer's Problem :
Valuing the Reinsurance Contract

The Insurer's Reinsurance Indifference Price

- The insurer maximizes expected utility of terminal wealth **with insurance risk** and receives a **reinsurance payment**

$$U^R(w, L, S, t) = \sup_{\{\pi(s)\} \in \mathcal{S}} \mathbb{E}_t^{\mathbb{P}} \left[u(W^L(T) + \mathbf{h}(\mathbf{S}(\mathbf{T}), \mathbf{L}(\mathbf{T}))) \right]$$

- The insurer's **indifference price** P for the reinsurance is

$$\boxed{U^R(w - \mathbf{P}, S, t; q) = U(w, S, t; q)}$$

The Insurer's Reinsurance Indifference Price

- **U^R** satisfies a **similar HJB equation** as **U** with new boundary conditions

$$
\begin{aligned}
0 \;=\;& U_t^R + (rW+q)U_w^R + \mu\, S\, U_S^R + \frac{1}{2}\sigma^2\, S^2\, U_{SS}^R \\
& + \lambda(t)\left(U^R(w-g(S,t),\mathbf{L}+\mathbf{g}(\mathbf{S},\mathbf{t}),S,t) - U^R(w,\mathbf{L},S,t)\right) \\
& + \max_{\pi}\left\{\frac{1}{2}\sigma^2 U_{ww}^R\,\pi^2 + \pi\left[(\mu-r)U_w^R + \sigma^2 S(t) U_{ws}^R\right]\right\},
\end{aligned}
$$

subject to $\mathbf{U^R(w,L,S,t;q) = u(w+h(L,S))}$

The Insurer's Reinsurance Indifference Price

- **Counterparty risk** can be treated easily
- The HJB equation takes on an additional term

$$
\begin{aligned}
0 \;=\; & U_t^R + (rW + q)U_w^R + \mu\, S\, U_S^R + \frac{1}{2}\sigma^2\, S^2\, U_{SS}^R \\
& + \kappa(\mathbf{t})\left(\mathbf{U}(\mathbf{w},\mathbf{L},\mathbf{S},\mathbf{t}) - \mathbf{U}^{\mathbf{R}}(\mathbf{w},\mathbf{L},\mathbf{S},\mathbf{t})\right) \\
& + \lambda(t)\left(U^R(w - g(S,t), \mathbf{L} + \mathbf{g}(\mathbf{S},\mathbf{t}), S, t) - U^R(w, \mathbf{L}, S, t)\right) \\
& + \max_{\pi}\left\{\frac{1}{2}\sigma^2 U_{ww}^R\, \pi^2 + \pi\left[(\mu - r)U_w^R + \sigma^2 S(t) U_{ws}^R\right]\right\},
\end{aligned}
$$

subject to $U^R(w, L, S, t; q) = u(w + h(L, S))$

The Insurer's Reinsurance Indifference Price

- We demonstrate that the indifference **price** satisfies the **non-linear PDE**

$$
\begin{cases}
r\,P = & P_t + r\,S\,P_S + \frac{1}{2}\sigma^2\,S^2\,P_{SS} \\
& -\frac{\kappa(\mathbf{t})}{\alpha(\mathbf{t})}\left(\mathbf{1} - \mathbf{e}^{-\alpha(\mathbf{t})\,\mathbf{P}(\mathbf{L},\mathbf{S})}\right) \\
& +\frac{\lambda(\mathbf{t})}{\alpha(\mathbf{t})}\mathbf{e}^{\alpha(\mathbf{t})\mathbf{g}(\mathbf{S},\mathbf{t})}\left(\mathbf{1} - \mathbf{e}^{-\alpha(\mathbf{t})[\mathbf{P}(\mathbf{L}+\mathbf{g}(\mathbf{S},\mathbf{t}),\mathbf{S})-\mathbf{P}(\mathbf{L},\mathbf{S})]}\right) \\
P(L,S,T) = & h(L,S)\,.
\end{cases}
$$

Probabilistic Interpretation

- For loss-independent pay-off functions (and no-counterparty risk) i.e. **h(S,L) = *h(S)*** the indifference price is the **Black-Scholes price**

$$\boxed{P(S,L,t) = e^{-r(T-t)}\,\mathbb{E}_t^{\mathbb{Q}}\left[h(S(T))\right]}$$

- The measure **Q** is the **minimal entropy measure**

Probabilistic Interpretation

- For more general pay-off structures, can perform a **perturbative expansion** in the risk-aversion parameter

$$P(L,S,t) = P^0(L,S,t) + \hat{\alpha} P^1(L,S,t) + o(\hat{\alpha})\,,$$

- The **zeroth order** price satisfies

$$\begin{cases} (r+\kappa(t))\,P^0 = & P^0_t + r\,S\,P^0_S + \frac{1}{2}\sigma^2\,S^2\,P^0_{SS} + \lambda(t)\Delta P^0 \\ P^0(L,S,T) = & h(L,S)\,, \end{cases}$$

- Feynman-Kac solves the PDE

$$\boxed{P^0(L,S,t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r+\kappa(u))\,du} h(L(T),S(T)) \middle| \mathcal{F}_t \right]}$$

Probabilistic Interpretation

- The linear correction to the price satisfies

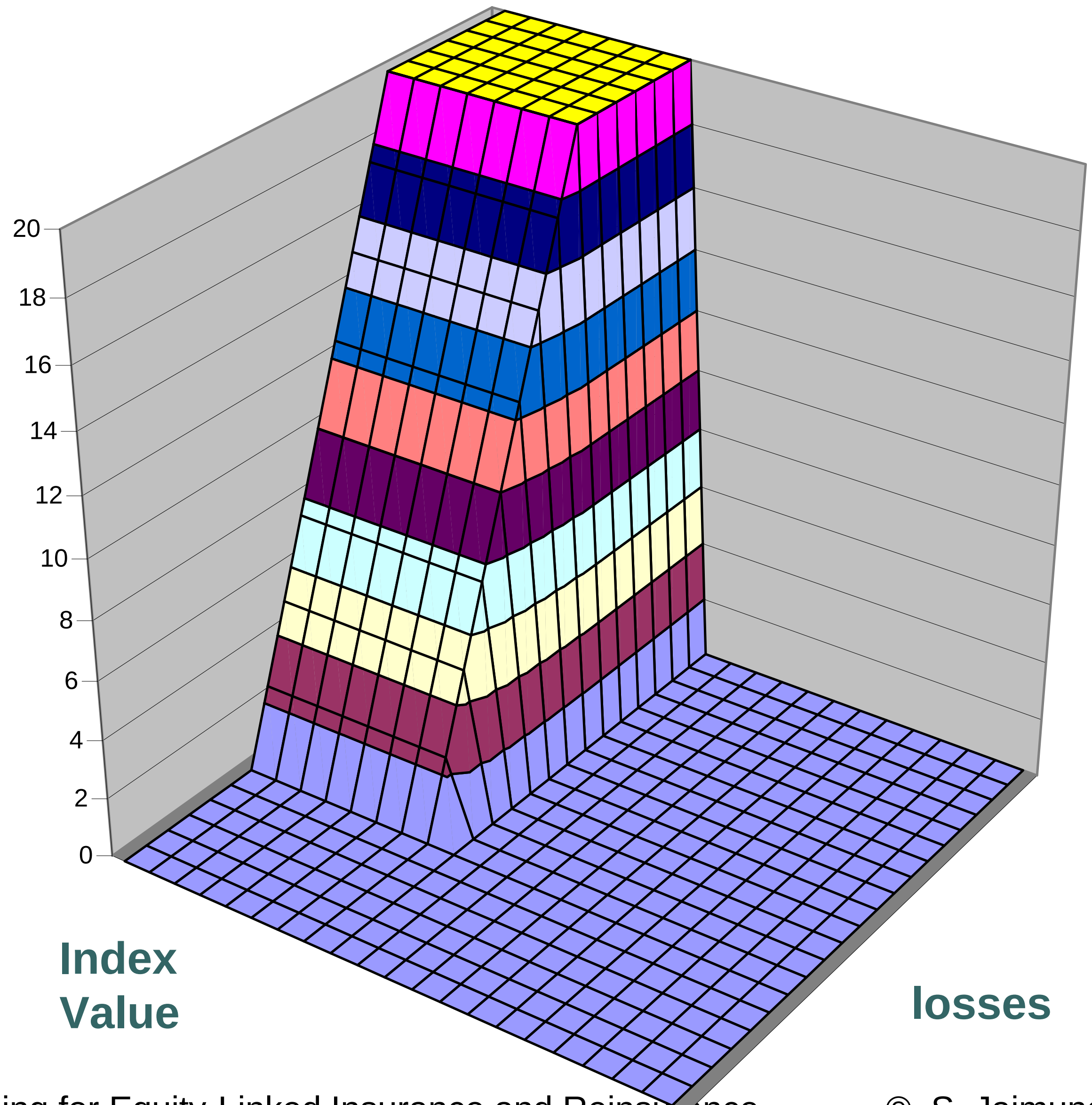
$$\begin{cases} (r+\kappa(t))\,P^1 = \; P^1_t + r\,S\,P^1_S + \frac{1}{2}\sigma^2\,S^2\,P^1_{SS} + \lambda(t)\Delta P^1 \\ \qquad\qquad\qquad\qquad +\frac{1}{2}\kappa(t)(P^0(L,S,t))^2 \\ \qquad\qquad\qquad\qquad +\lambda(t)\,e^{r(T-t)}\left\{g^2(S,t) - \left[\Delta P^0(L,S,t) - g(S,t)\right]^2\right\}, \\ \quad P^1(L,S,T) = \; 0\,. \end{cases}$$

Probabilistic Interpretation

- Feynman-Kac (with source) solves the PDE

$$P^1(L,S,t) \;=\; \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \frac{1}{2}\kappa(u)(P^0(L(u),S(u),u))^2 + \lambda(u)\;\left\{g^2(S(u),u) - \left[\Delta P^0(L(u),S(u),u) - g(S(u),u)\right]^2\right\}\,du\,\middle|\,\mathcal{F}_t\right]$$

Numerical Examples : Fixed Loss Sizes



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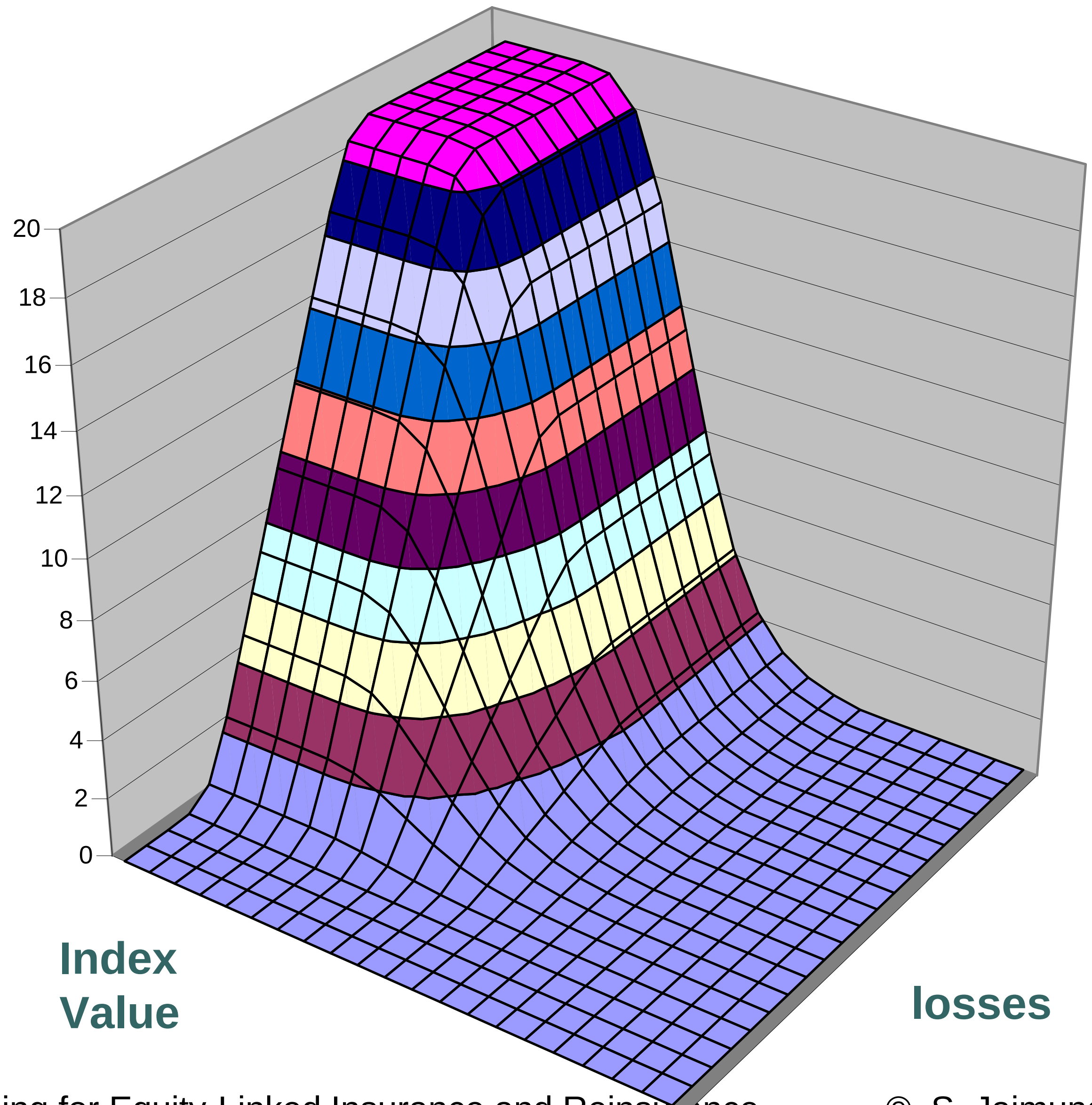

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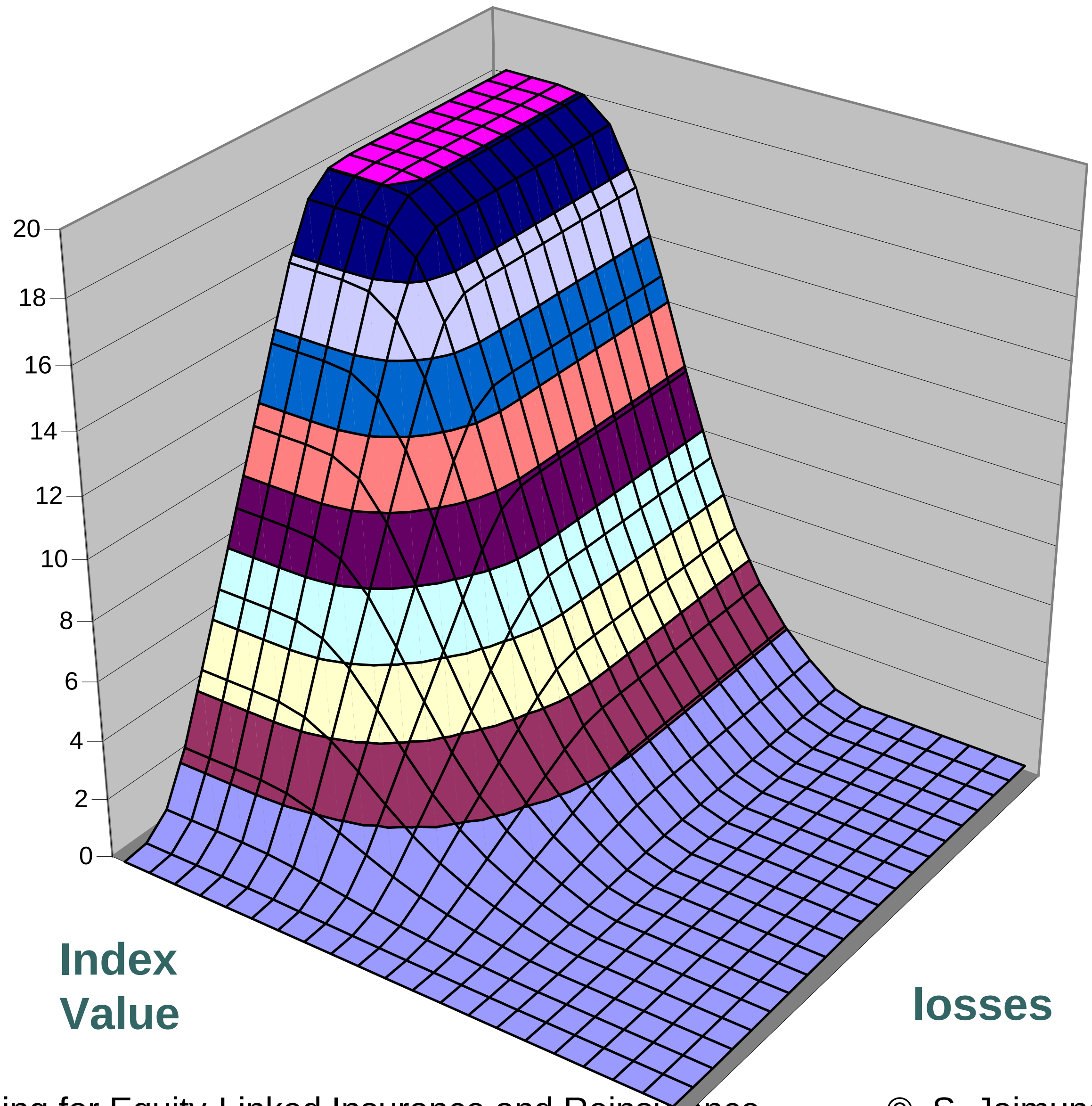
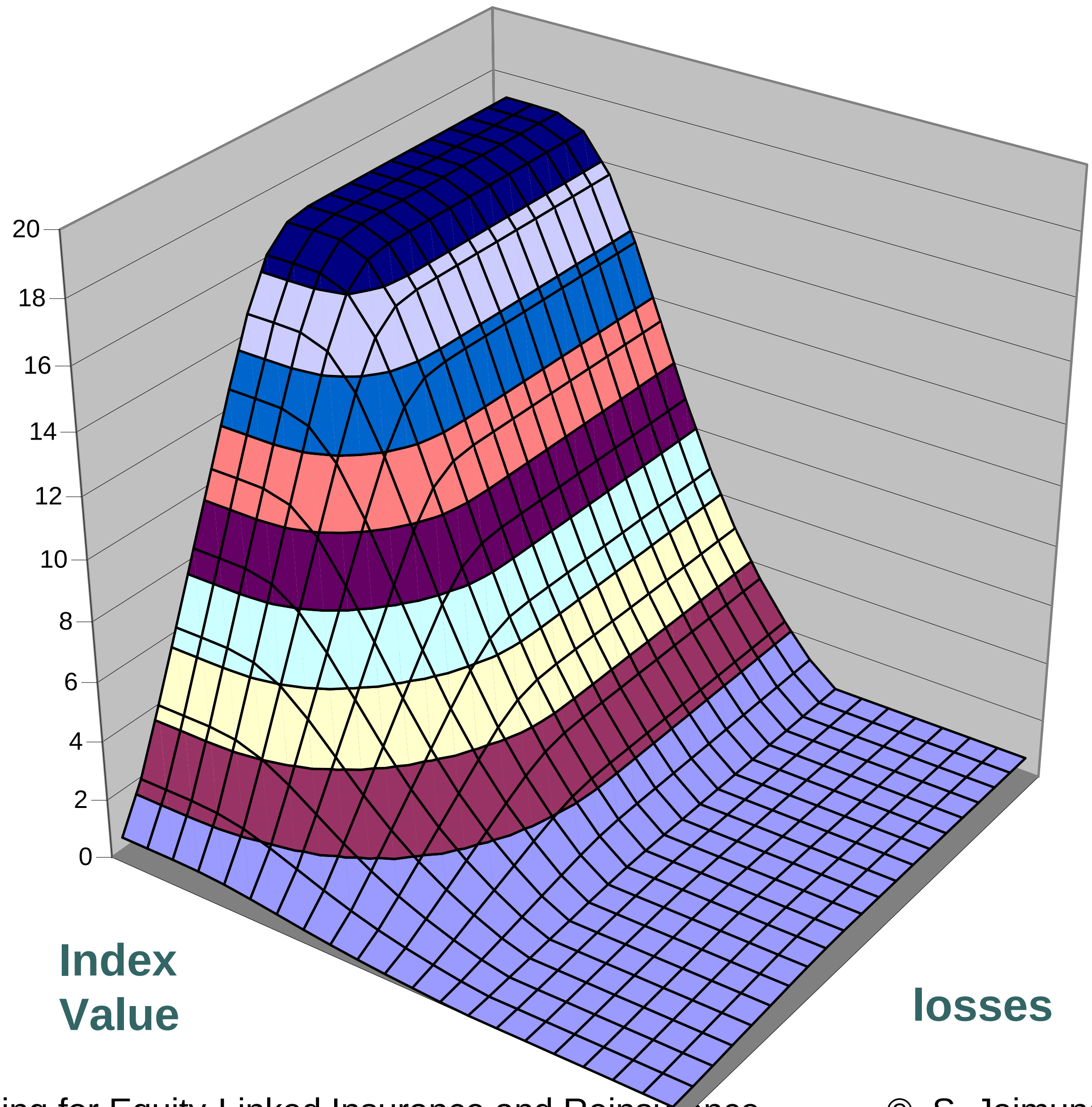
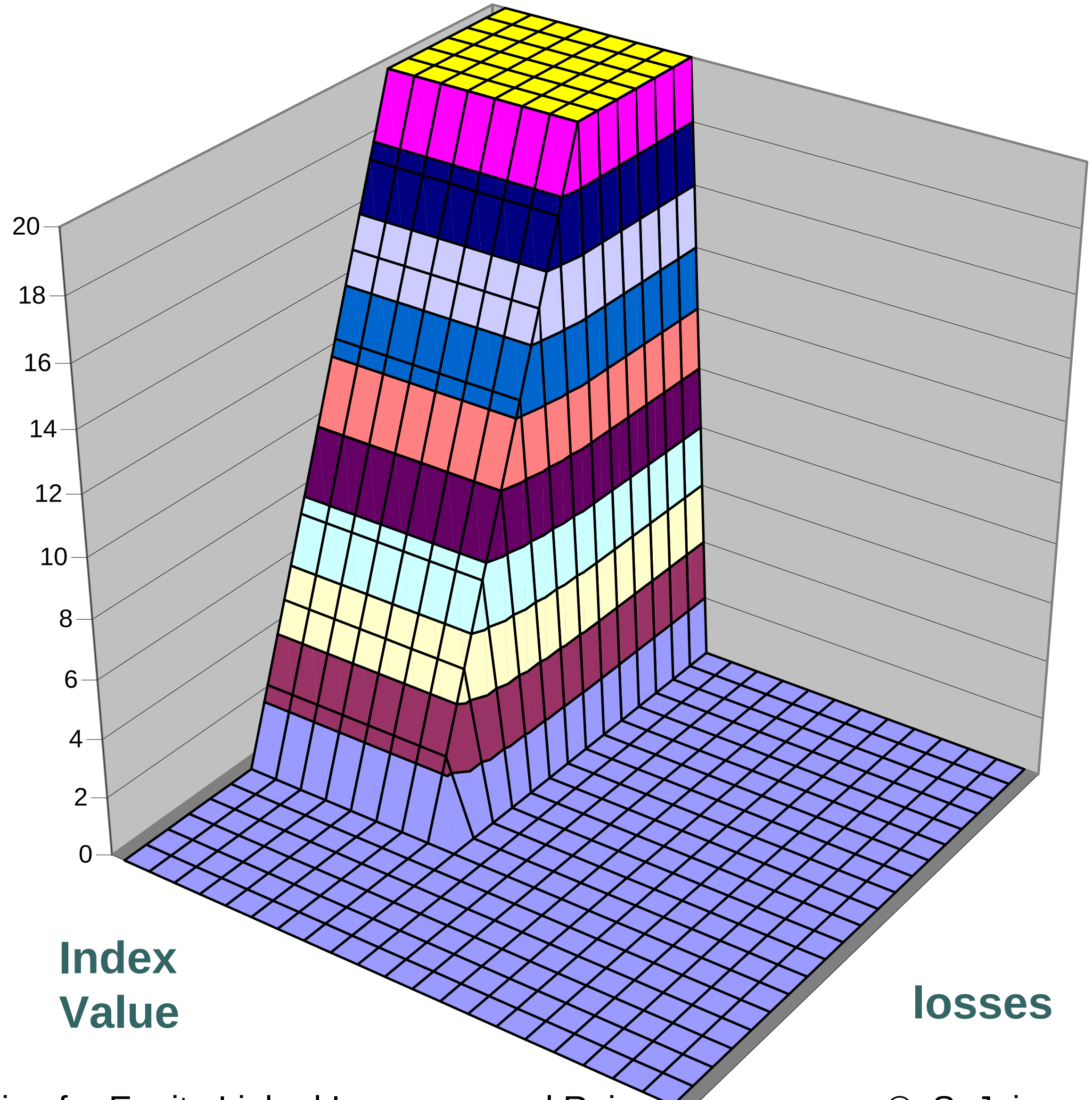


Numerical Examples : Fixed Loss Sizes



Numerical Examples : Digital Loss Sizes



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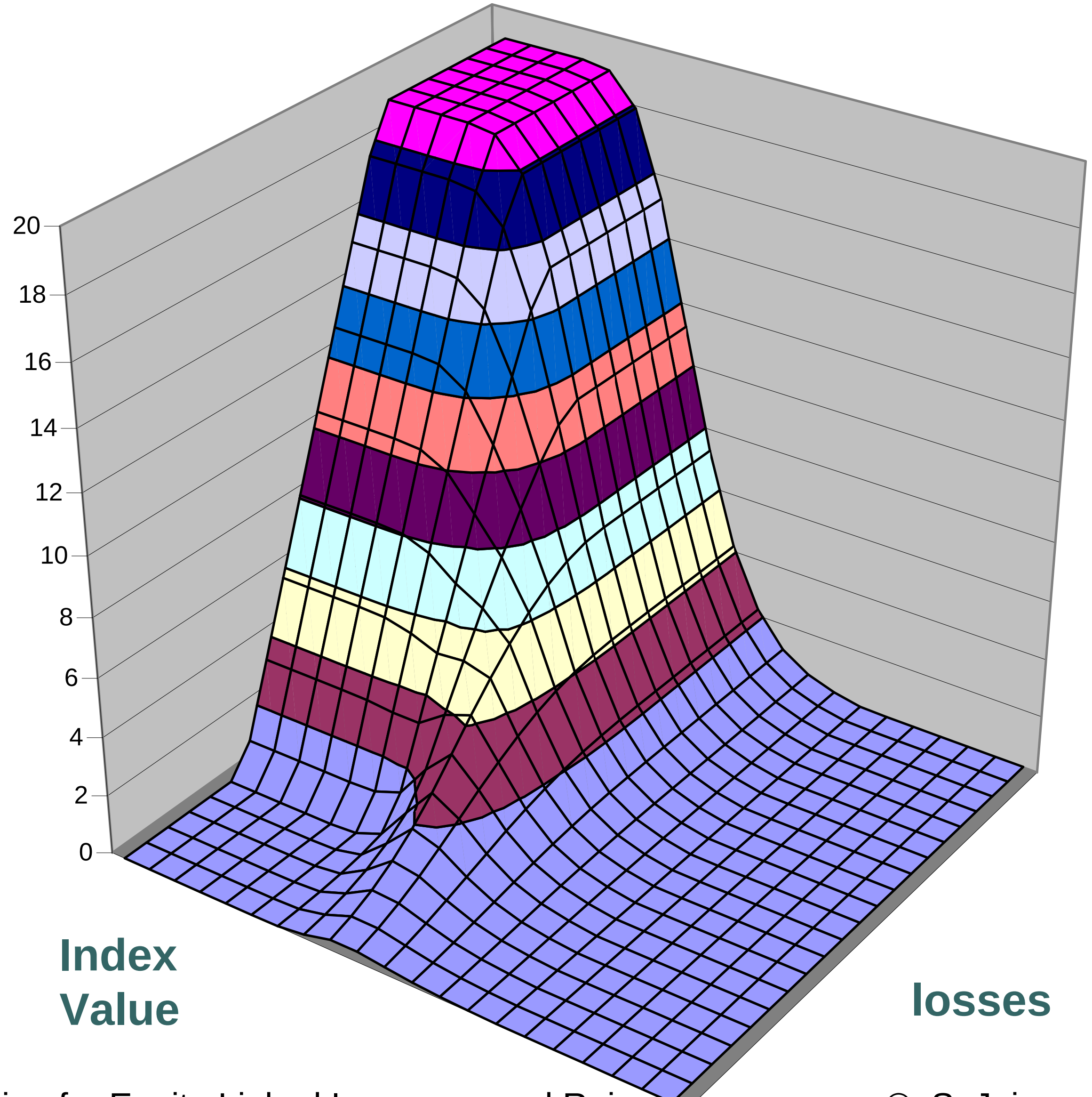

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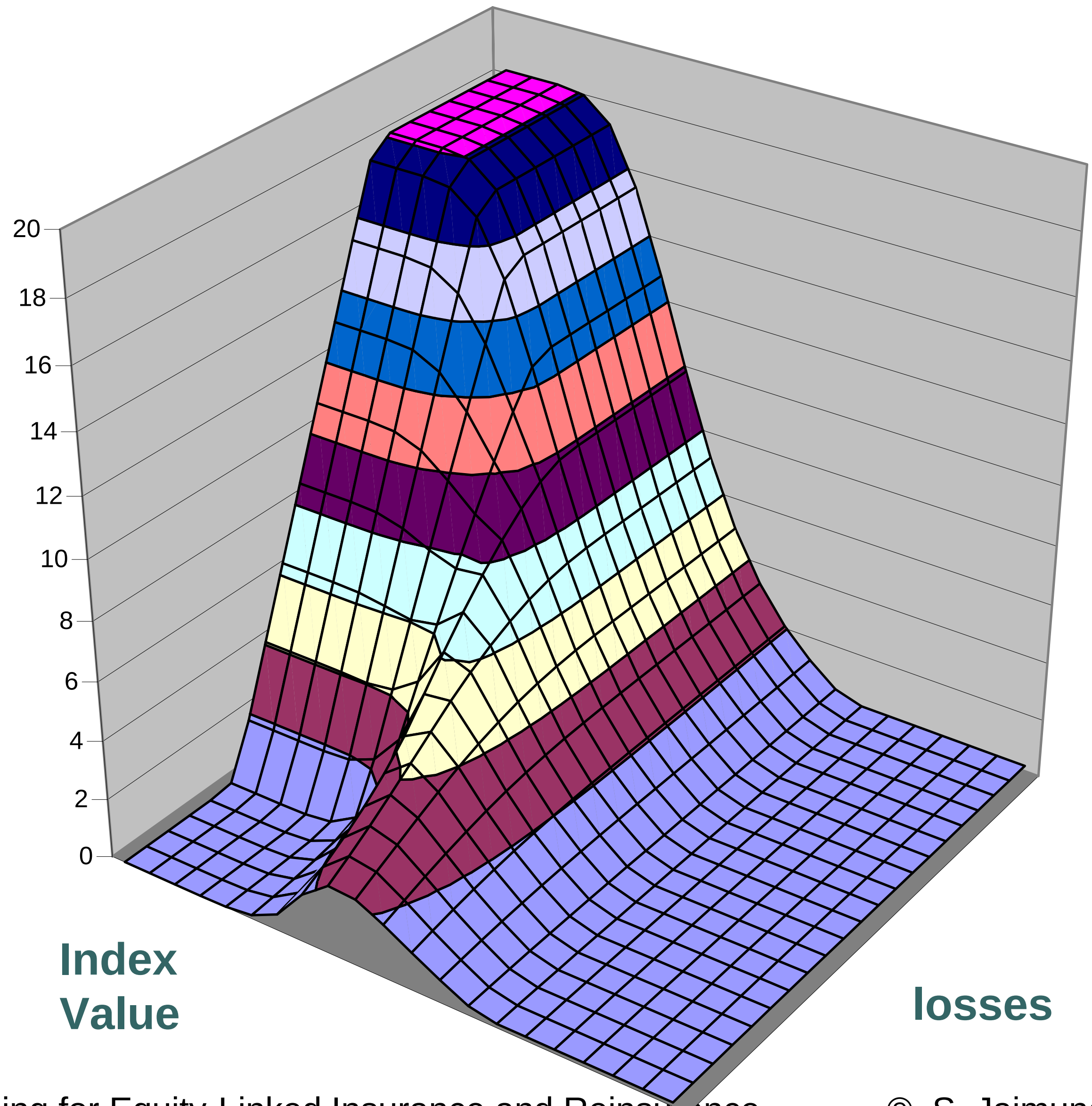
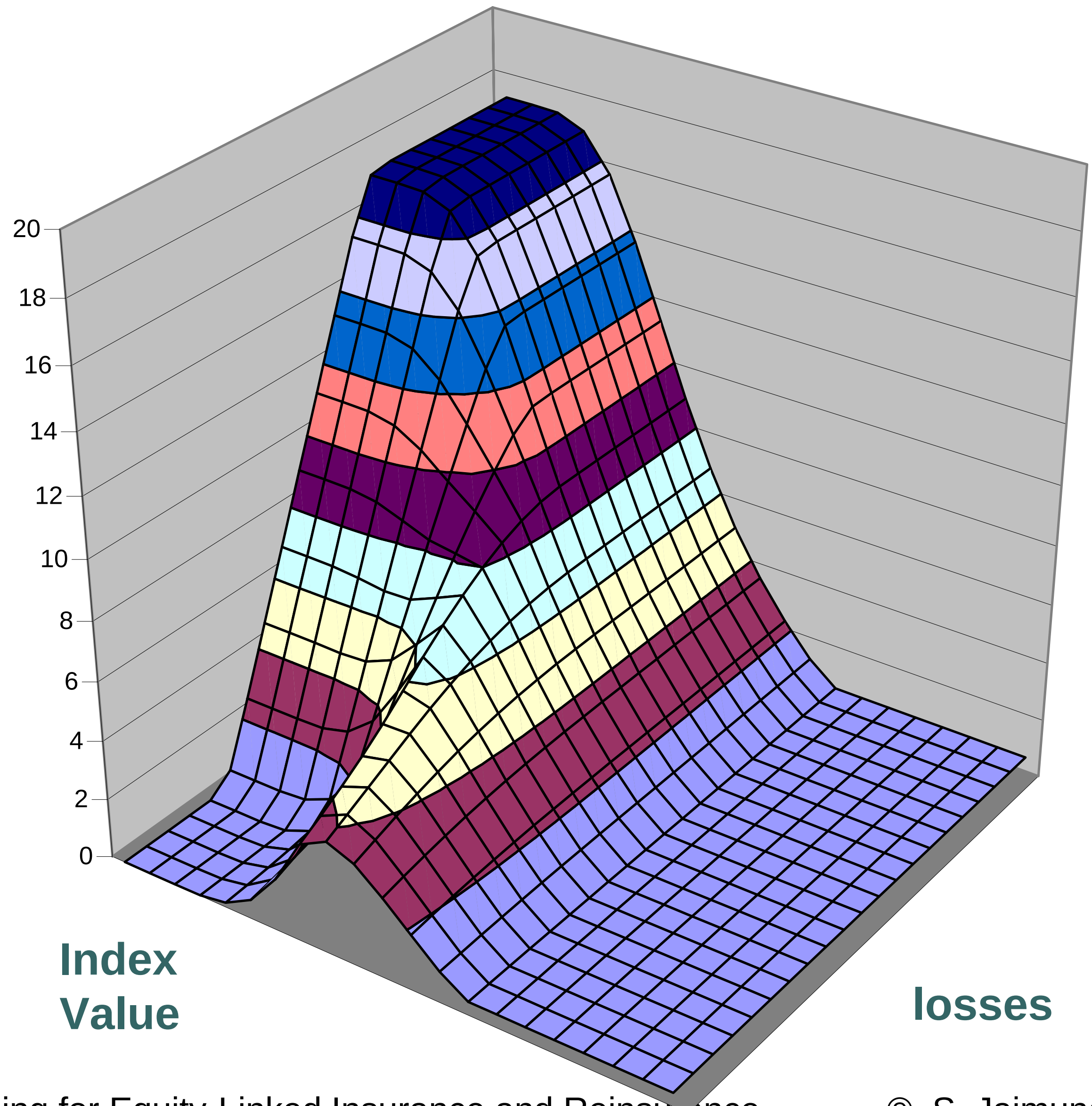


Numerical Examples : Digital Loss Sizes



Conclusions and Future Work

- Obtained **Indifference premium** for insurers exposed to equity-linked losses as an **expectation**
  - Performed **perturbative expansions**
  - Solved **numerically**

- Obtained the **PDE** for the **indifference price** of a reinsurance contract issued to such an insurer
  - Performed **perturbative expansions**
  - Recast as a **dual optimization problem**
  - Solved **numerically**

- Ongoing work includes
  - Incorporating **stochastic interest rates**
  - Applying to **Equity-Linked Notes**, **CDSs** and **CDOs**

Thank you for your attention

My Relevant Papers

- S.J. "***Utility Indifference for Catastrophe Options***", working paper

- S.J. and Suhas Nayak, "***Valuing Equity-Linked Insurance and Reinsurance Products***", working paper

- S.J. and Tao Wang "***Catastrophe Options with Stochastic Interest Rates and Compound Poisson Losses***", to appear in Insurance: Mathematics and Economics

- S.J. and V.R. Young "***Pricing Equity Indexed Pure Endowments with Risky Assets that follow Levy Processes*** ", Insurance: Mathematics and Economics, vol *36, issue 2, pg. 329-346 (2005)*