

Stock Return Models for Long-Term Embedded Options

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Outline

- Some history
- The models
- Does it matter?
- Traditional model selection
- Bootstrap evidence
- Abusing the bootstrap

History

- Single premium equity linked insurance in North America

Ø Segregated Funds in Canada

Ø Variable Annuities in USA

- Carry guarantees on death and maturity
- Guarantee may be fixed or increasing

History

- 25 years ago, UK faced the same issue
- MGWP published paper in 1980
 - Ø Stochastic simulation of liabilities (and underlying assets)
 - Ø Quantile (VaR) reserve.
 - Ø Early application of early Wilkie Model

Canadian Method

- Stochastic simulation of liabilities
- CTE (Tail-VaR) reserve
- Not much hedging
 - Ø If hedged, simulate and reserve for unhedged risk
- Equity model : ‘freedom with calibration’

Canadian Calibration Method

- Use any model
- Check the left-tail accumulation factor probabilities, using standard data set
- Adjust parameters to meet calibration fatness requirement
- Table calculated using 'Regime-Switching Lognormal -2' model

Accumulation Factors

- Let Y_t represent log return in t^{th} month
- 1-year accumulation factors are $\exp(Y_t + Y_{t+1} + \dots + Y_{t+11})$
- Similarly for 5-year and 10-year
- 40 years data \Rightarrow 4 non-overlapping observations of 10-Year accumulation factor

Canadian Calibration Table

Accumulation Factor	2.5 %ile	5 %ile	10%ile
1-year	0.76	0.82	0.90
5-year	0.75	0.85	1.05
10-year	0.85	1.05	1.35

US approach

- C3P2
- Similar to Canadian approach
- Calibration Table applied to left and right tails
- US table derived from
‘Stochastic Log-Volatility’ model

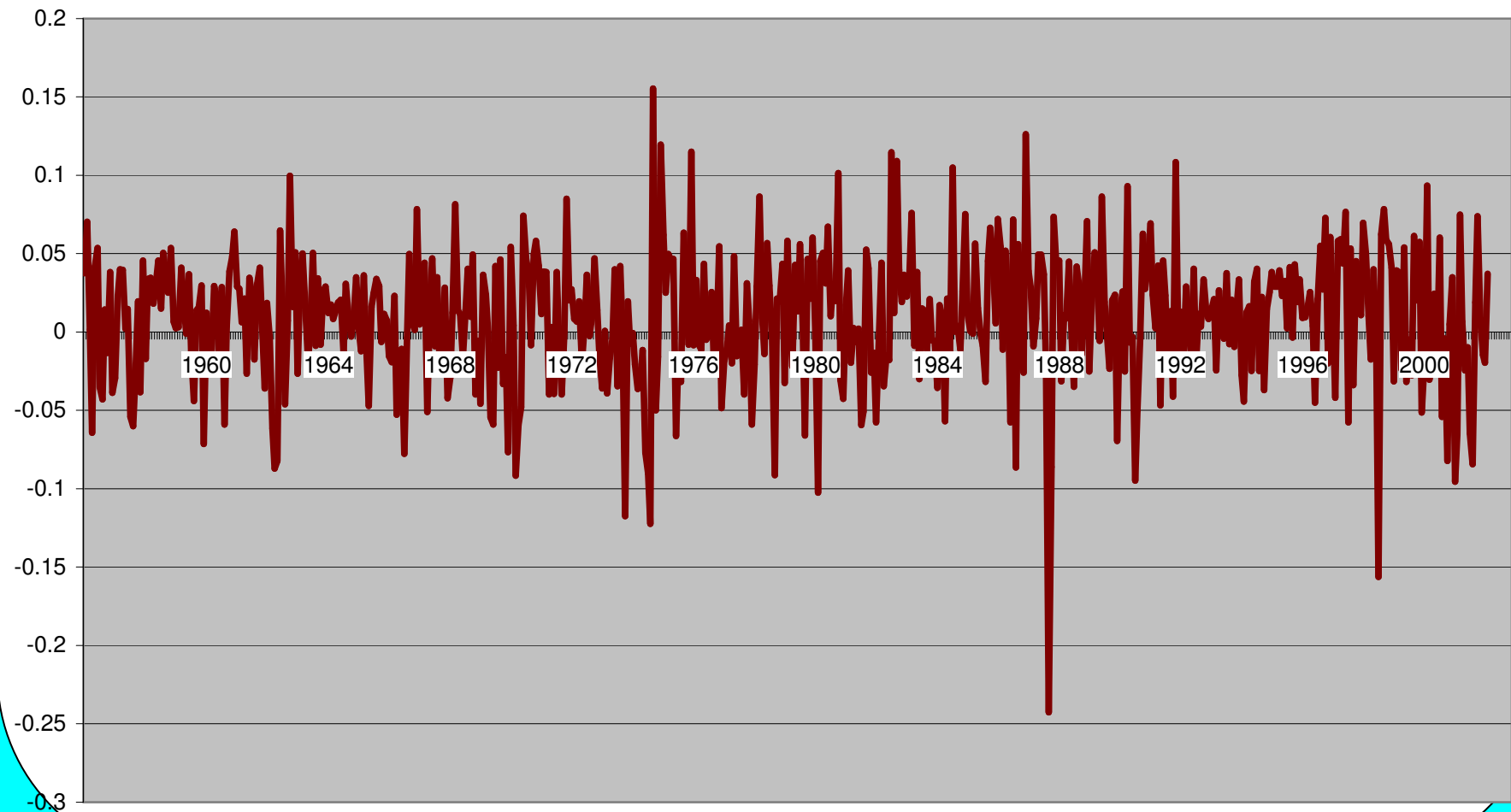
US Calibration

%ile	1-Yr	5-Yr	10-Yr	20Yr
2.5%	0.78	0.72	0.79	N/A
5%	0.84	0.81	0.94	1.51
10%	0.90	0.94	1.16	2.10
90%	1.28	2.17	3.63	9.02
95%	1.35	2.45	4.36	11.70
97.5%	1.42	2.72	5.12	N/A

Some outcomes...

- UK
 - Ø no more maturity guarantees
- Canada
 - Ø cut back on generous guarantees
 - Ø Plethora of equity models proposed
 - Ø Still little hedging
- USA
 - Ø Some hedging...

S&P 500 Total Return Log returns



S&P data

- not much auto-correlation
- but correlation is not always a good measure of independence
- notice bunching of poor returns (eg last 2 years)
- and association of high volatility with crashes
 - Øie large movement down more than up

Some equity models

- Regime Switching Log Normal (Hardy, 2001)
- GARCH(1,1)
- MARCH (Chan and Wong, 2005)
- ‘Stochastic Log Volatility’ (AAA C3-Phase 2)
- Regime Switching Draw Down (Panneton, 2003)
- Regime Switching GARCH (Gray 1996, JFE)

RSLN-2

$$Y_t | \rho_t = \mu_{\rho_t} + \sigma_{\rho_t} \varepsilon_t$$

REGIME 1 ρ_1
Low Volatility σ_1
High Mean μ_1

p_{12}



p_{21}



REGIME 2 ρ_2
High Volatility σ_2
Low Mean μ_2

$$Y_t = \mu_1 + \sigma_1 \varepsilon_t$$

$$Y_t = \mu_2 + \sigma_2 \varepsilon_t$$

The RSLN-2 Model

- The regime process is a hidden Markov process
- 2 Regimes are usually enough for monthly data.
- 2 Regime model has 6 parameters:

$$\Theta = \{\mu_1, \mu_2, \sigma_1, \sigma_2, p_{12}, p_{21}\}$$

- Regime 1: Low Vol, High Mean, High Persistence (small p_{12})
- Regime 2: High Vol, Low Mean, Low Persistence (large p_{21})

GARCH(1,1)

$$Y_t = \mu + \sqrt{h_t} \varepsilon_t$$

$$h_t = \alpha_0 + \alpha_1 (Y_{t-1} - \mu)^2 + \beta h_{t-1}$$

- Where $\varepsilon_t \sim N(0,1)$, iid
- Given F_{t-1} , ε_t is the only stochastic element
- We generally require $\alpha_1 + \beta < 1.0$

MARCH (2;0,0;2,0)

$$Y_t \mid F_{t-1} \sim \begin{cases} Q_1 & \text{w.p. } \alpha_1 \\ Q_2 & \text{w.p. } (1 - \alpha_1) \end{cases}$$

$$Q_1 \sim ARCH(2); \quad Q_2 \sim ARCH(0)$$

$$h_{1,t} = \beta_{10} + \beta_{11}(Y_{t-1} - \phi_1)^2 + \beta_{12}(Y_{t-2} - \phi_1)^2$$

$$h_{2,t} = \beta_{20}$$

MARCH(2;0,0;2,0)

- $\text{MARCH}(K; p_1, \dots, p_K; q_1, \dots, q_K)$ is a mixture of K AR-ARCH models,
- p_j and q_j are the AR-order and ARCH-order of the j^{th} mixture RV
- According to Chan and Wong, provides superior fit to 3rd and 4th moments of monthly log-return disn cf RSLN

SLV

$$v_t = \log \sigma_t = (1 - \varphi)v_{t-1} + \varphi \log \tau + \sigma_v Z_{v,t}$$

$$\mu_t = A + B\sigma_t + C\sigma_t^2$$

$$Y_t = \frac{\mu_t}{12} + \frac{\sigma_t}{\sqrt{12}} Z_{y,t}$$

- $Z_{v,t}$ and $Z_{y,t}$ are standard normal RVs, with correlation ρ
- The v_t process is constrained by upper and lower bounds

SLV

- According to C3P2, SLV

Ø *“Captures the full benefits of stochastic volatility in an intuitive model suitable for real world projections”*

Ø *Stoch vol models are widely used in capital markets to price derivatives...*

Ø *Produces very “realistic” volatility paths*

Regime Switching Draw Down (RSDD)

$$Y_t \mid (\rho_t = s) = \kappa_s + \phi_s D_{t-1} + \sigma_s \varepsilon_t$$

$$D_{t-1} = \min(0, D_{t-2} + Y_{t-1})$$

$$\varepsilon_t \sim N(0,1), \text{ iid}$$

ρ_t is a Markov regime switching process

RSDD

- 2 Regimes proposed by Panneton
- D_t is the draw-down factor
- RSLN-2 is recovered when $\phi_\rho=0$, for $\rho=1,2$
- Captures ‘tendency to recover’

RSGARCH

- Two GARCH regimes
- Markov switching
- After Gray (1995)

Does it matter?

- 6 models, each being championed by someone.
- 2 RS, 2 conditional heteroscedatic, 1 ‘stochastic volatility’.
- Each fitted by MLE (-ish) to S&P500 data
- Does it make any difference to the results for Equity-Linked Capital Requirements?

Two methods for Equity Linked Life Insurance

- Actuarial Approach:
 - Ø Simulate liabilities,
 - Ø apply risk measure,
 - Ø discount at risk-free rate
- Determines the economic capital requirement to write the contract for a given solvency standard.

Two methods for Equity Linked Life Insurance

- Dynamic Hedging Approach
 - Ø Simulate hedge under real world measure
 - Ø Estimate distribution of unhedged liability
 - Ø Apply risk measure and discount at $r-f$ rate
 - Ø Add to cost of initial hedge

Example Contract

- Single Premium GMAB, premium P
- Issue age 50, MER=3% p.y.
- Guarantee risk premium = 0.2% p.y.
- Deterministic mortality and lapses
- Assets in policyholder's fund = F_t at t
- F_t follows model stock returns, less MER

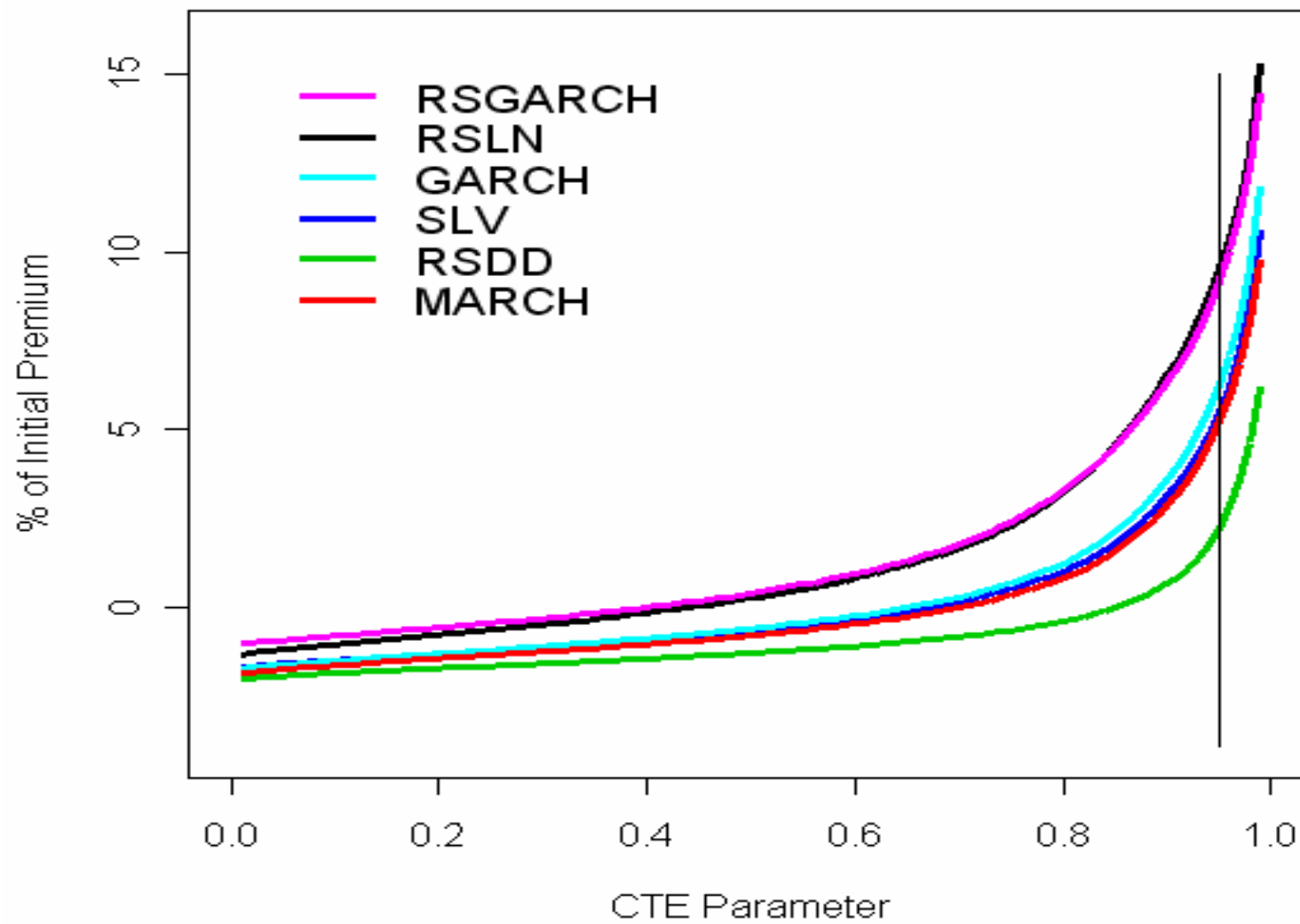
Example Contract

- Benefit on death or maturity is $\max(F_T, G_T)$
- Guarantee G_t at t
 - Ø $G_t = P$ for $0 < t \leq 10$
 - Ø $G_t = \max(F_{10}, P)$ for $10 < t \leq 20$
- Payable on death or maturity

‘Actuarial Approach’

- Use stock return model to generate distribution of PV of guarantee cost,
- $L = e^{-rT} \max(G_T - F_T, 0)$: T is exit date (death or maturity)
- $CTE = E[L | L > Q_\alpha]$,
- Q_α is the α -quantile of L

CTE for 'Actuarial' Risk Management



Risk Measure, % of P; AA

Model	90% CTE	95% CTE
RSDD	0.64 (0.09)	2.25 (0.16)
MARCH	2.85 (0.14)	5.22 (0.19)
SLV	3.12 (0.15)	5.47 (0.20)
GARCH	3.60 (0.19)	6.27 (0.22)
RSLN	6.50 (0.19)	9.53 (0.23)
RSGARCH	6.33 (0.17)	9.18 (0.23)

Does the model matter using
the actuarial approach?

Oh Yes !!!

Using hedging?

- Straight Black-Scholes (LN) delta hedge

Ø $r=0.05$; $\sigma=0.20$

- Simulate additional cost arising from

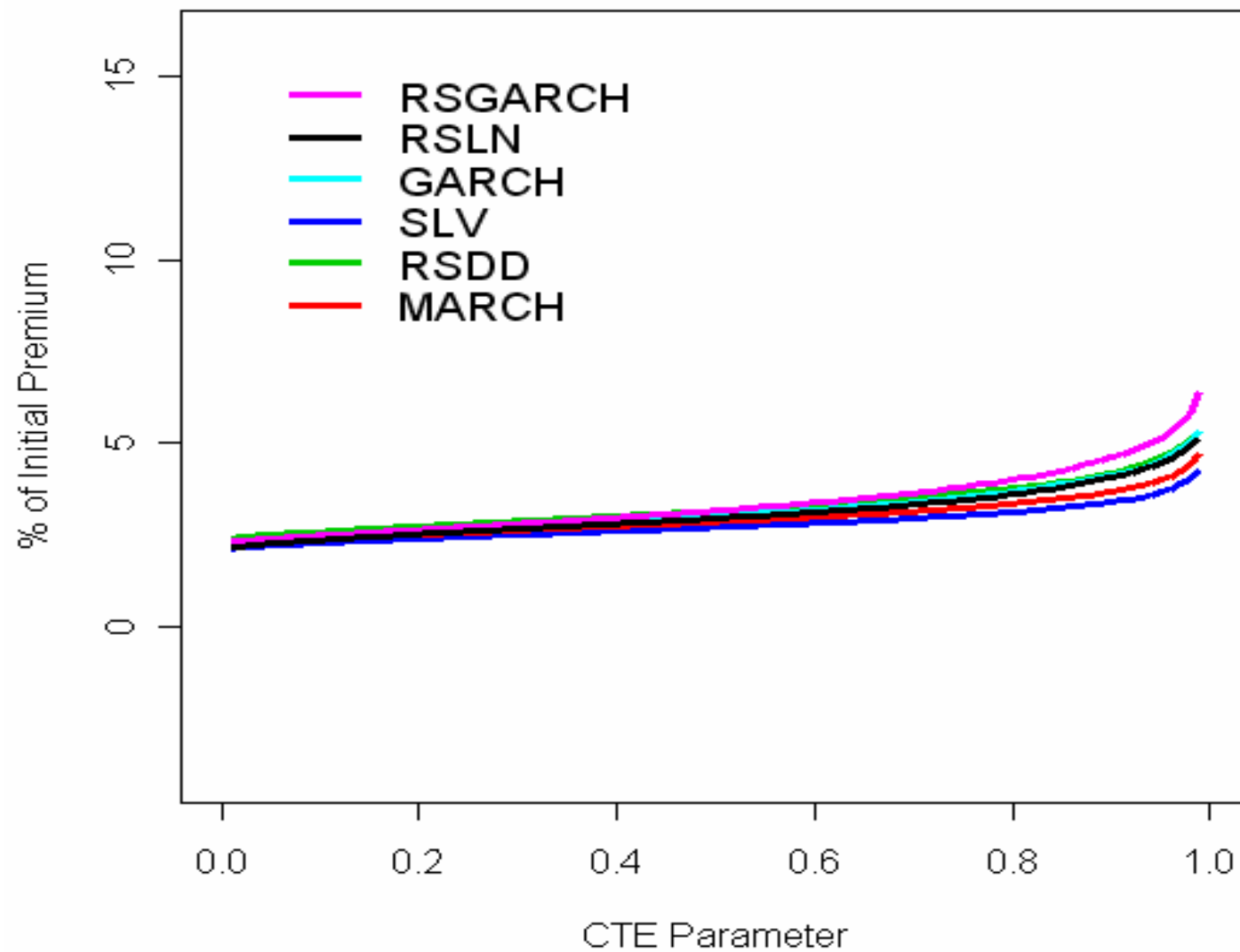
Ø Discrete hedge

Ø Model Error

Ø ie P-measure is GARCH; RSLN etc

Ø Transactions costs

CTE for Dynamic Hedging Risk Management



Risk Measure, % of single premium

Model	90% CTE	95% CTE
RSDD	4.20 (0.08)	4.62 (0.10)
MARCH	3.67 (0.06)	4.00 (0.09)
SLV	3.39 (0.05)	3.67 (0.09)
GARCH	4.12 (0.08)	4.52 (0.11)
RSLN	4.06 (0.08)	4.45 (0.09)
RSGARCH	4.62 (0.12)	5.15 (0.17)

Does the model matter using
the hedging approach?

Not so much....

But

- Many companies are not hedging
- Pressure to adopt models giving lower capital requirements
- Can we use traditional methods to eliminate any of the models?

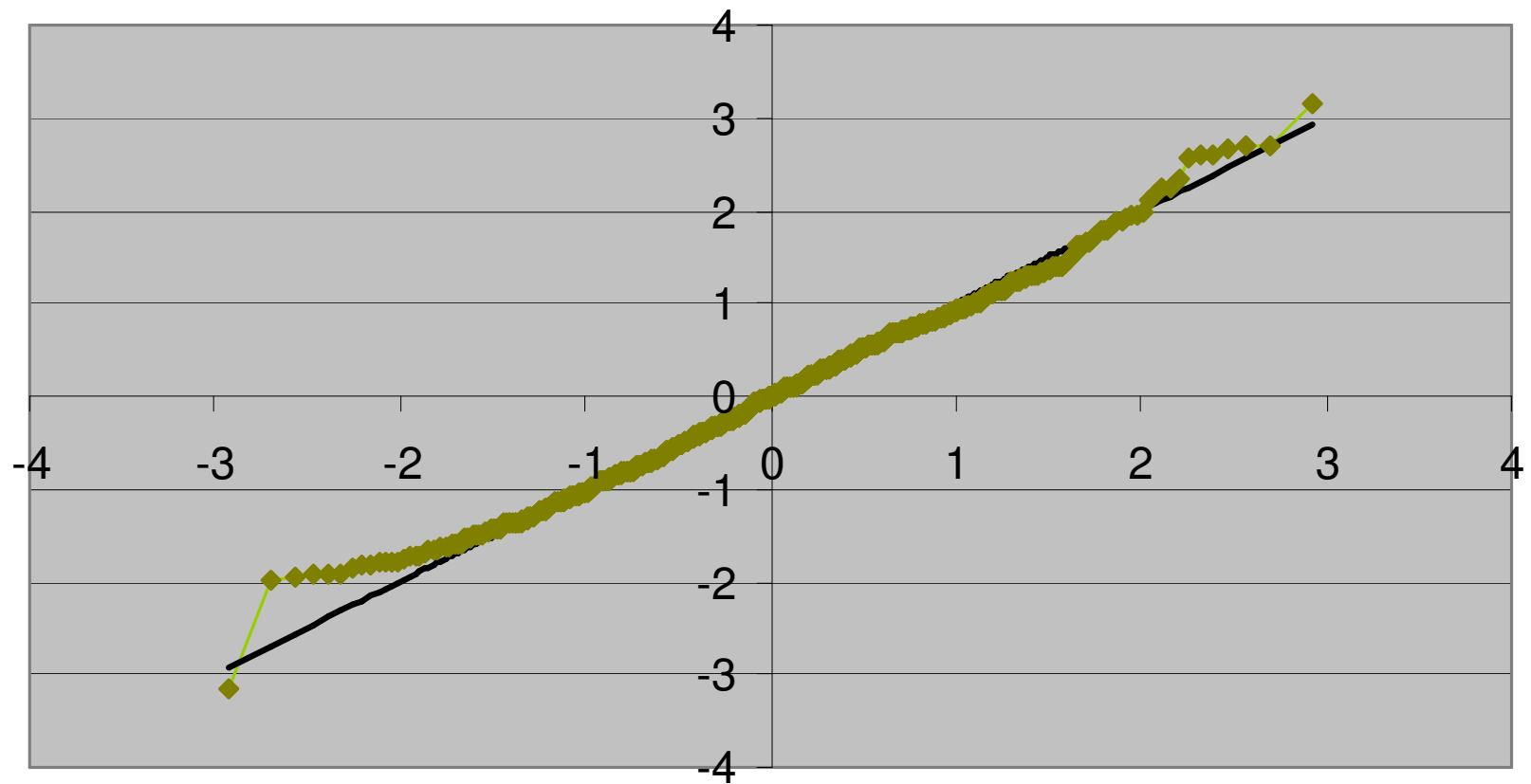
Likelihood Comparison

Model	# parameters	Max LL
RSDD	8	1047.1
MARCH	7	1039.8
SLV	7	1032.9*
GARCH	4	1030.1
RSLN	6	1042.0
RSGARCH	8	1054.9

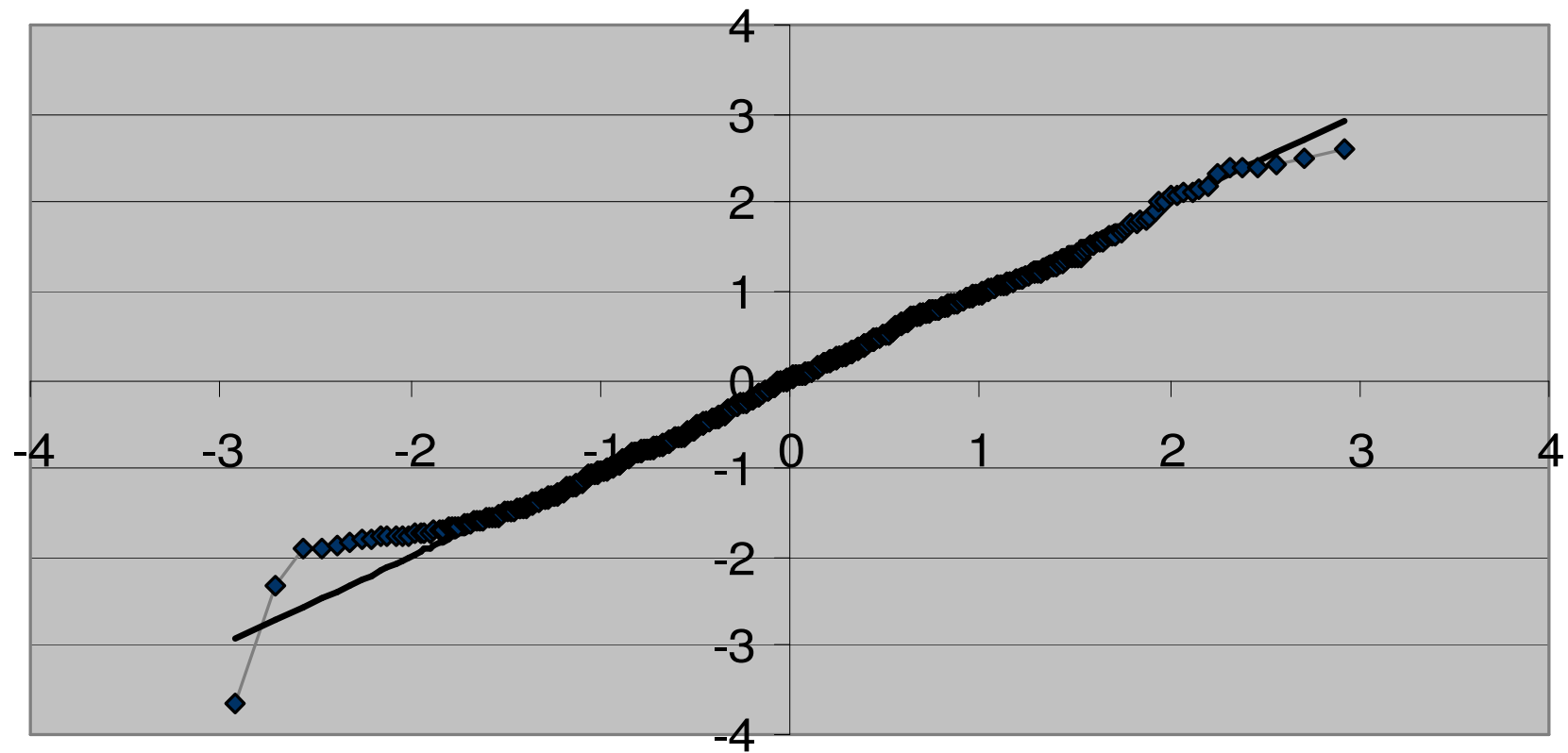
Residual analysis

- Residuals for GARCH are easy
- For MARCH, use same weights as original mixture
- Residuals for RS models – weighted from individual regimes $\Pr(\rho_t | y_1, \dots, y_t)$
- Residuals for SLV – using simulated volatility paths

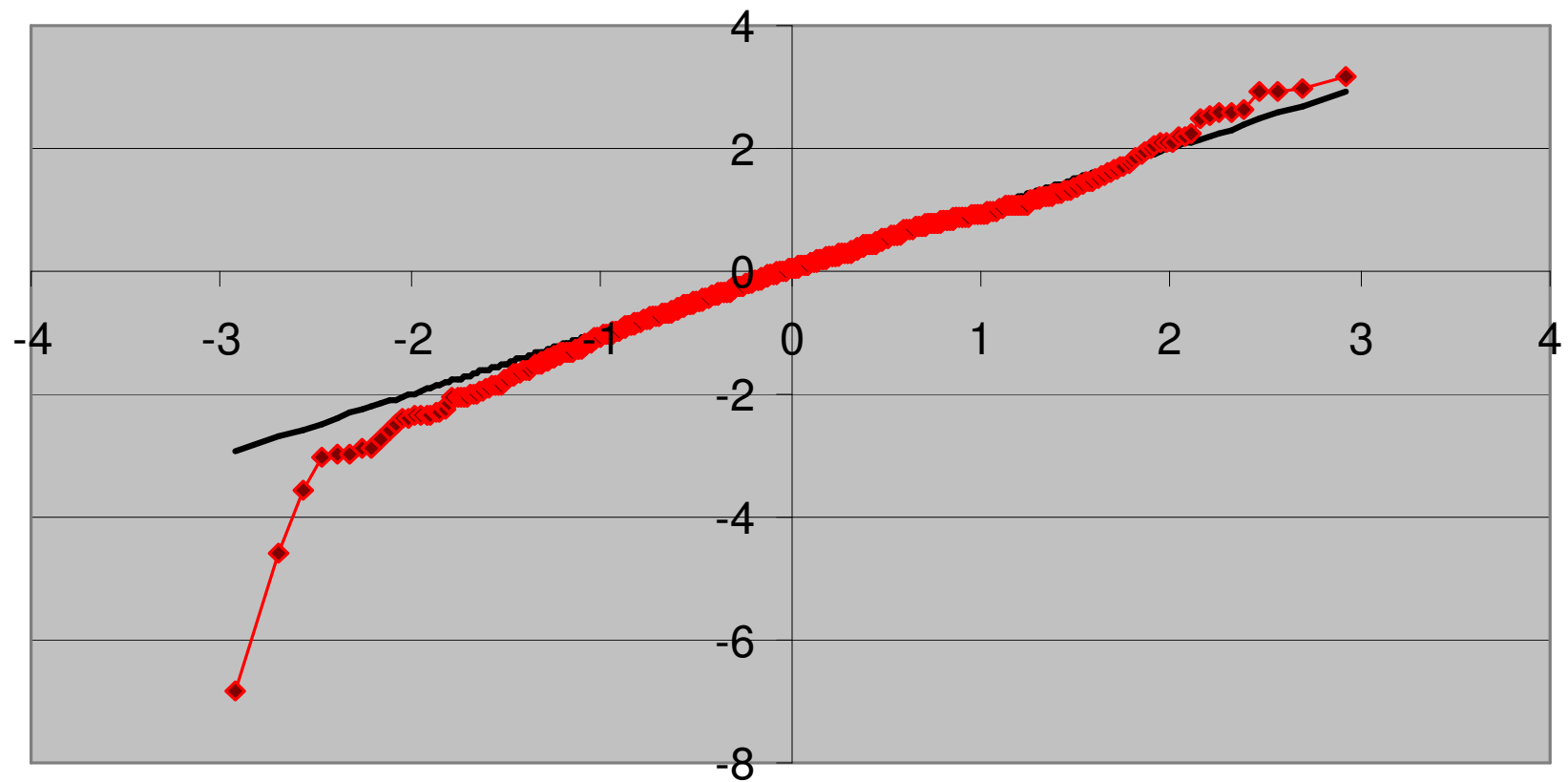
RSDD Residuals q-q Plot



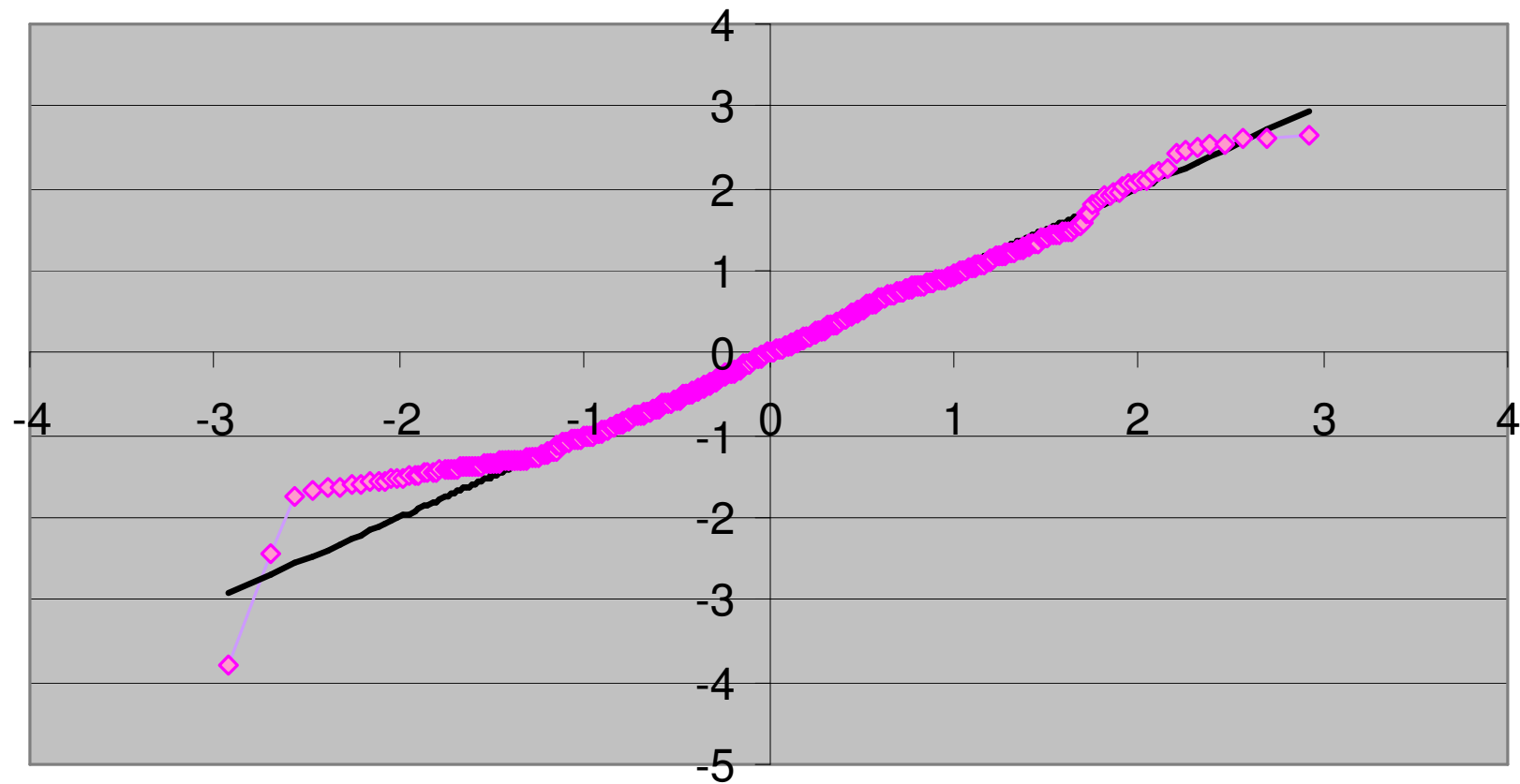
RSLN Residuals q-q Plot



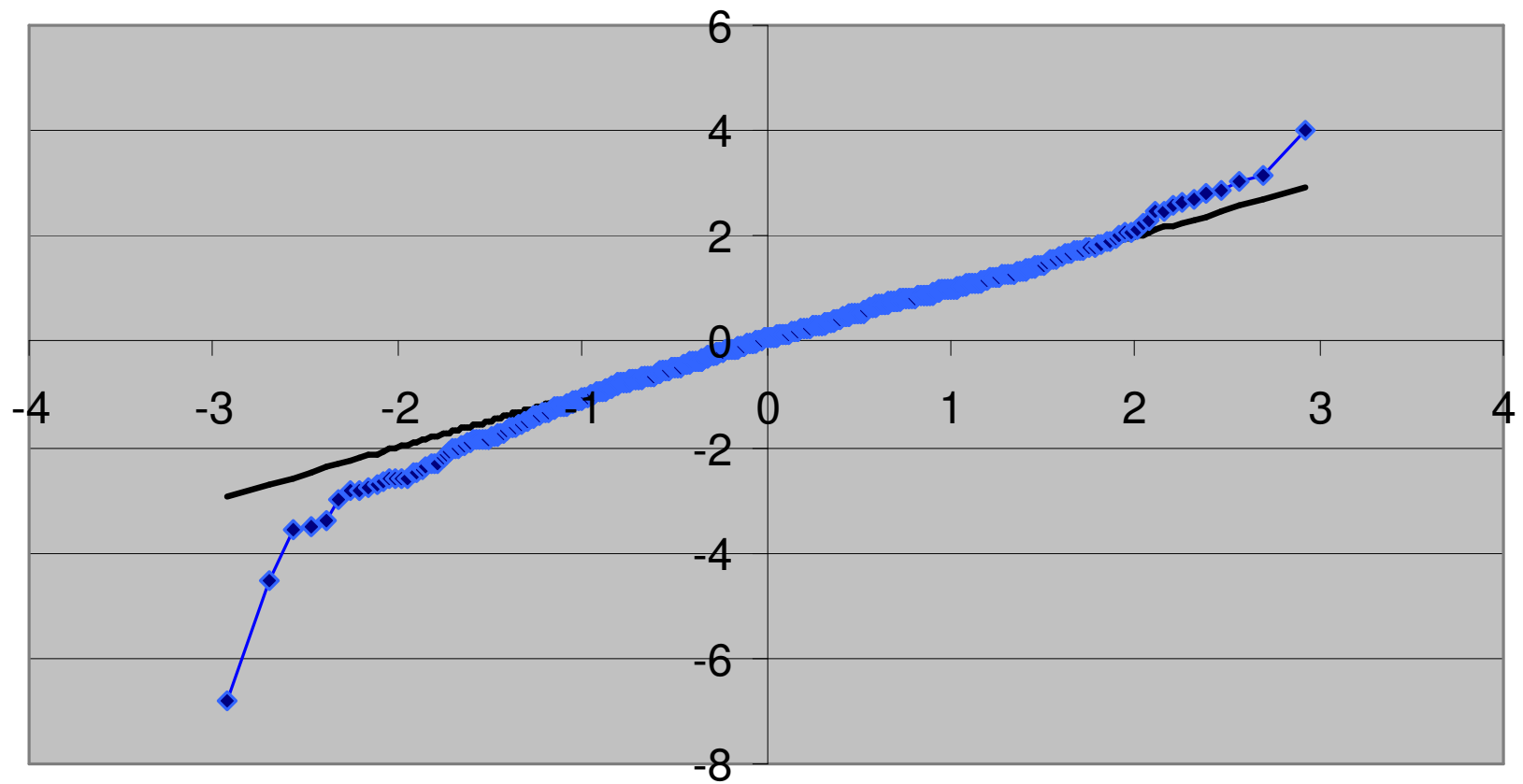
MARCH Residuals q-q Plot



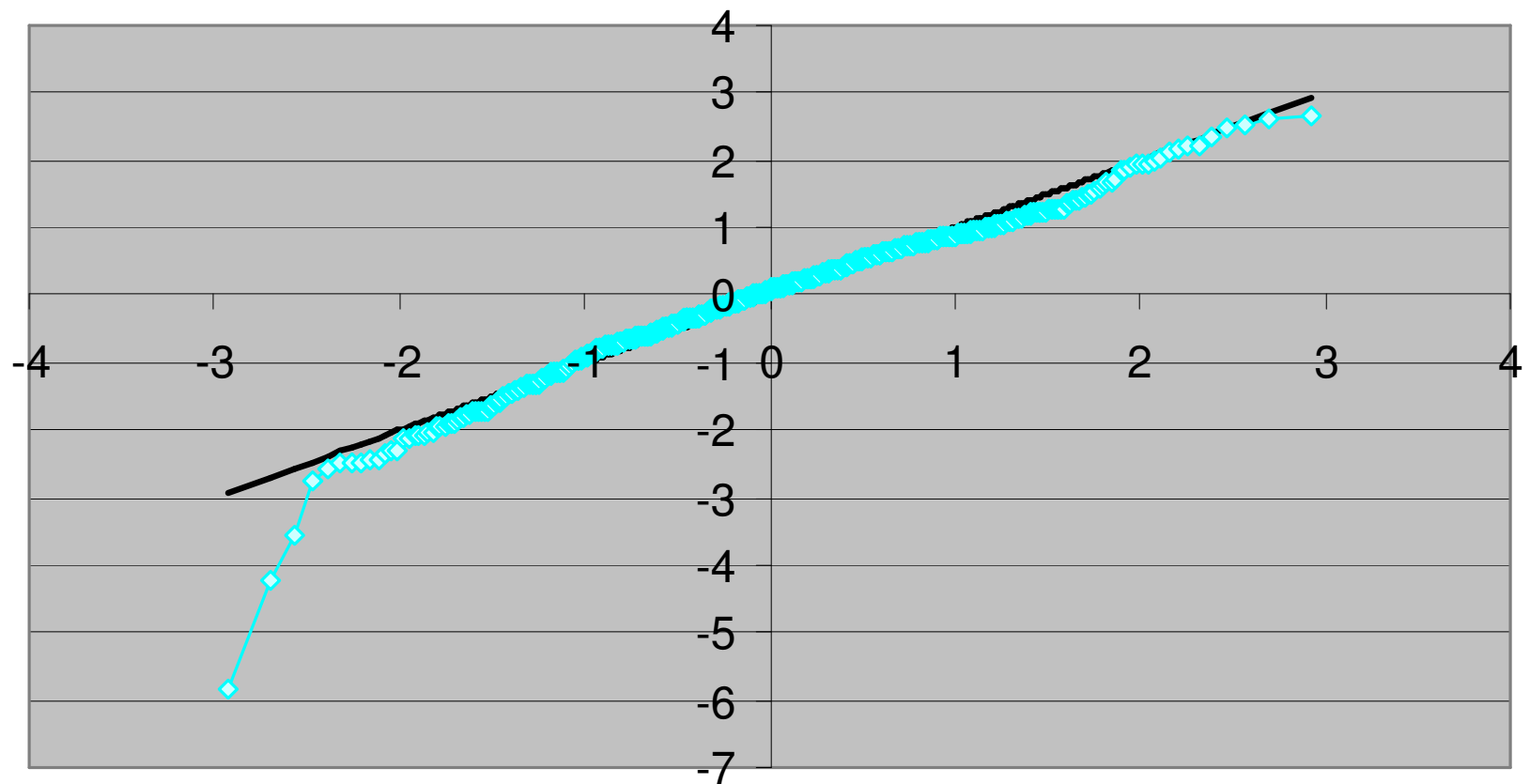
RSGARCH Residuals q-q Plot



SLV Residuals q-q Plot



GARCH Residuals q-q Plot



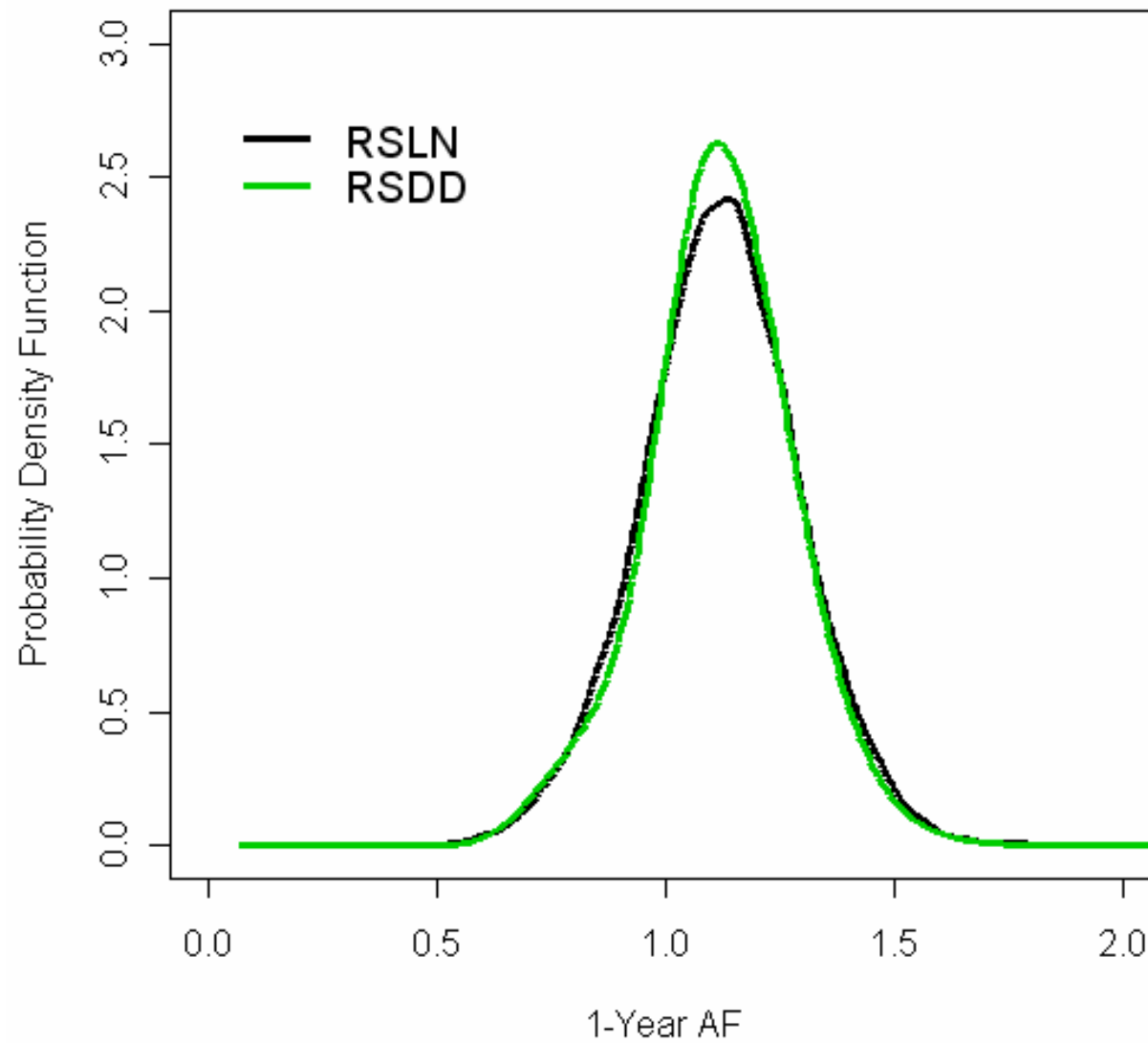
So far ...

- Likelihood based selection doesn't help much
- AIC is too simple, BIC depends on sample size, LRT has technical limitations
- Residuals can be useful, but are tricky in multifactor cases

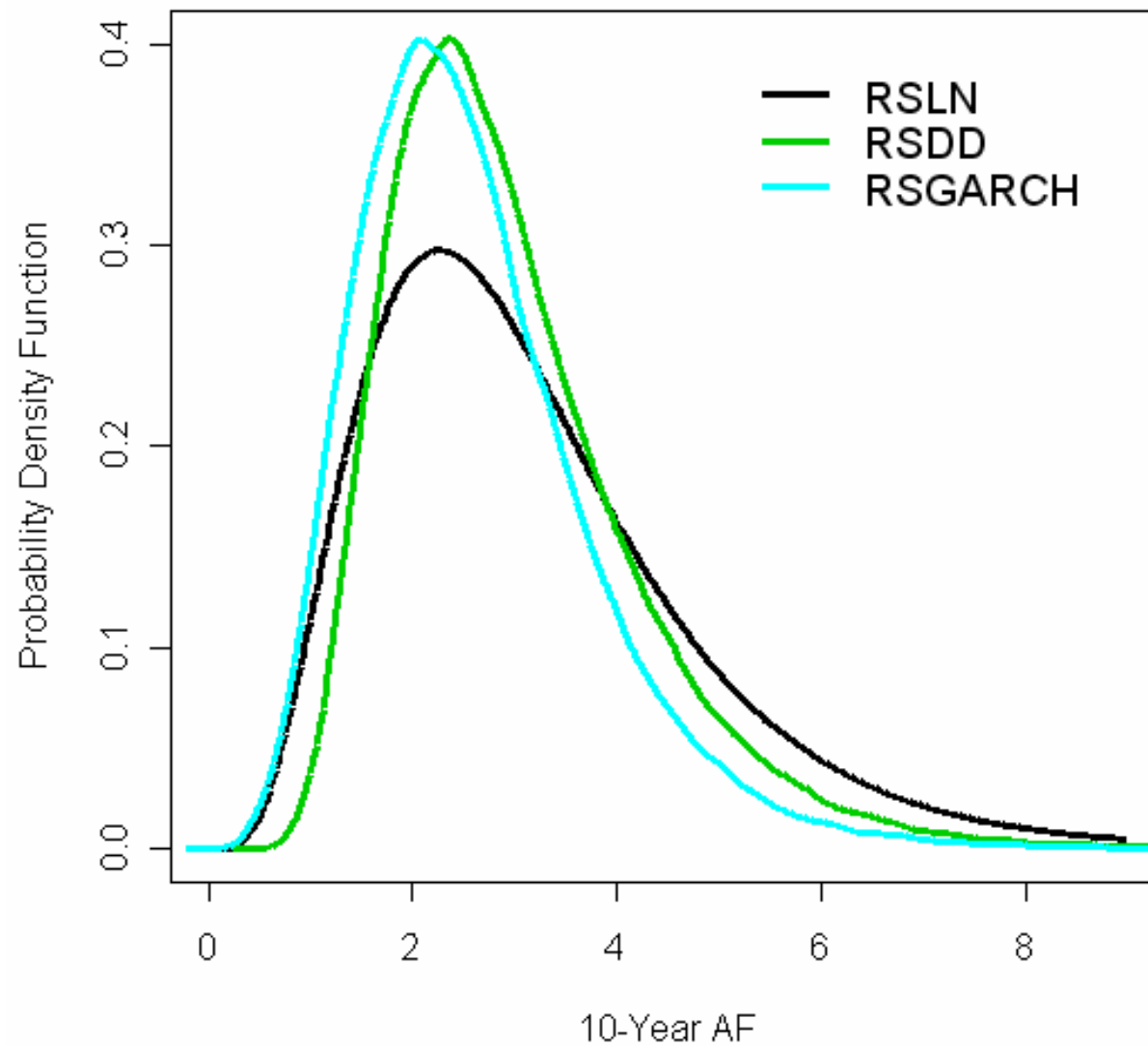
so good...

- The regime switching models look good on likelihood and on residuals (all pass J-B test)
- But -- big difference in application between rsdd and rsln or rsgarch
- What causes the big difference?
- Which rs model should we believe?

1-Year Accumulation Factors



10-Year Accumulation Factors



Bootstrapping time series

- The traditional bootstrap is applied to independent observations.
- Dependent time series require different treatment.
- Order matters.

S&P 1-year Acc Factors

- If we take 1-year factors starting in January, empirical percentiles are (from 48 observations):

Ø2.5%ile – 0.84

Ø5%ile – 0.85

Ø10%ile – 0.94

S&P 1-year Acc Factors

- If we take 1-year factors starting in September, empirical percentiles are (47 observations):

Ø2.5%ile – 0.75

Ø5%ile – 0.87

Ø10%ile – 0.89

- Ranges are:
 - 2.5%ile (0.74, 0.89)
 - 5%ile (0.83, 0.91)
 - 10%ile (0.89, 0.95)

1-year Acc factors

- Can't use all 1-year factors because of dependence
- If we only use (eg) January series, we are ignoring information
- Bootstrap the percentiles using time series bootstrap.

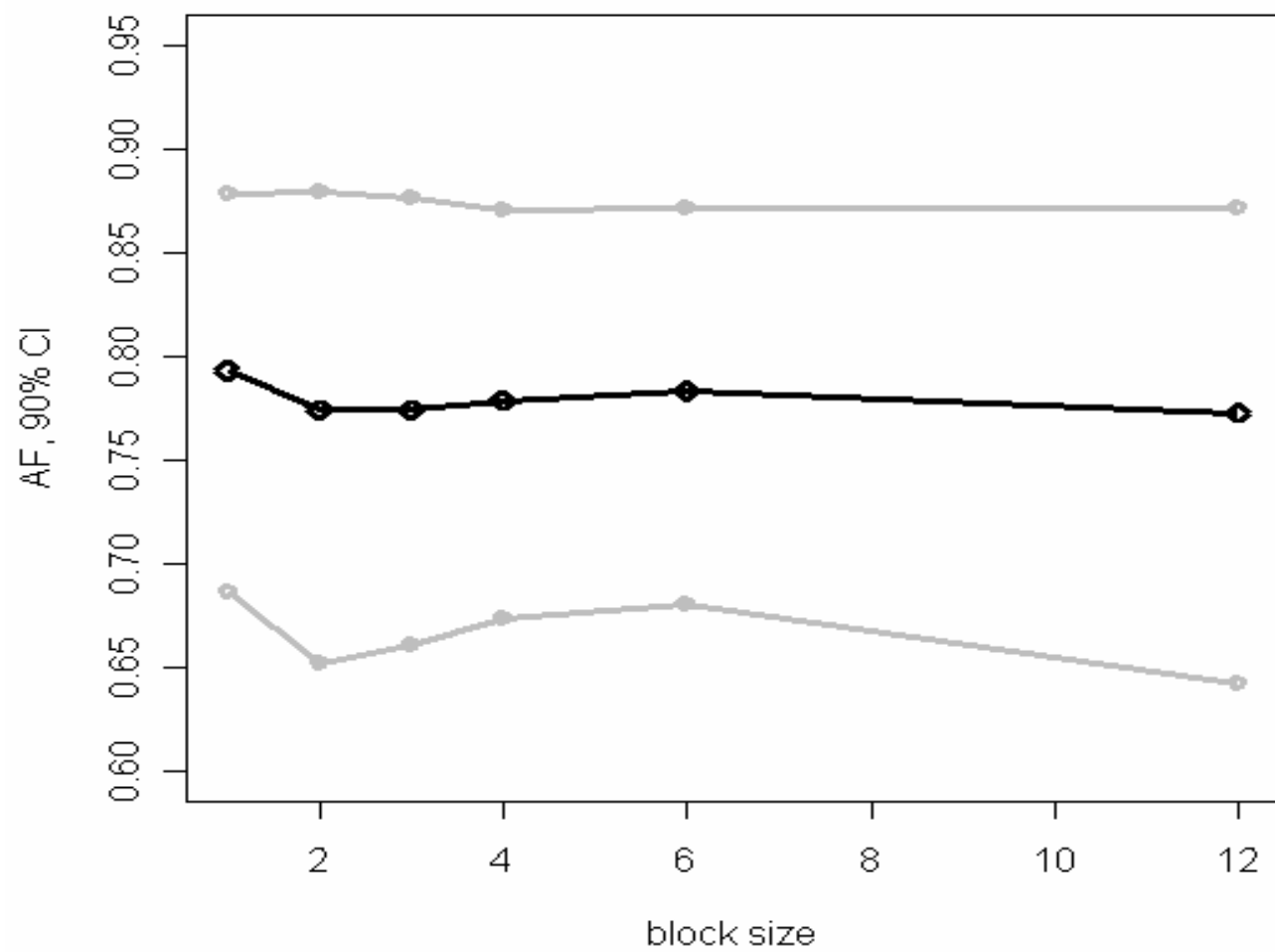
Time series bootstrap

- Bootstrap from original observations in blocks of b consecutive values.
- If the blocks are too small, lose dependence factor \Rightarrow results too thin tailed (if +vely autocorrelated)
- If blocks are too large lose data, \Rightarrow results too thin tailed (extreme results averaged out)

Block size

- So choose block size to maximize tail thickness.
- Other ways of selecting block size.
- No general agreement – see references.
- Randomized block length suggested.
- block resampling reduces exposure of end points → cycle from end to start.

1-Year AF Bootstrap; 2.5%ile



Bootstrap Quantile Estimates 1-Year Accumulation

Model	2.5%ile	5%ile	10%ile
Bootstrap 90% CI	0.67→0.87	0.76 →0.91	0.84 →0.97
RSDD	0.768	0.831	0.901
RSLN	0.764	0.829	0.908
RSGARCH	0.792	0.847	0.910

This doesn't help us much.

10-year accumulation factor

- We can do the same thing
- But the original data only has 4 non-overlapping observations
- minimum 10-year observed AF is estimate of $1/5=20\%$ ile
- So we bootstrap B samples of 4 observations

Bootstrap Quantile Estimates 10-Year Accumulation

Model	20 %ile
Bootstrap 90% CI	0.95→2.83 1.706
RSDD	1.953 (<i>1.92,1.97</i>)
RSLN	1.773
RSGARCH	1.660 (<i>1.63,1.68</i>)

And this doesn't help us much either.

Oversampling

- Bootstrapping re-samples from original data
- \Rightarrow Four 10-year accumulation factors from 584 observations
- What happens if we break the rules and keep sampling?
- The ‘empirical distribution’

Oversampling

- If data are independent or +vely autocorrelated then oversampling \rightarrow thin tails
 - Ø positive bias for low quantiles; negative for high quantiles
 - Ø Bias should be small for large original sample

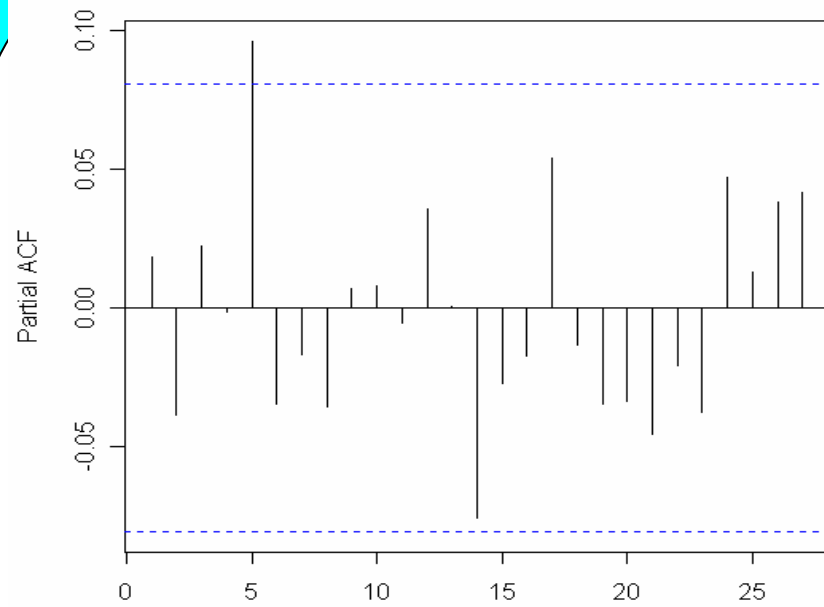
Oversampling

- If data are +vely auto-correlated and block size is not large enough to capture long down or up periods → even thinner tails

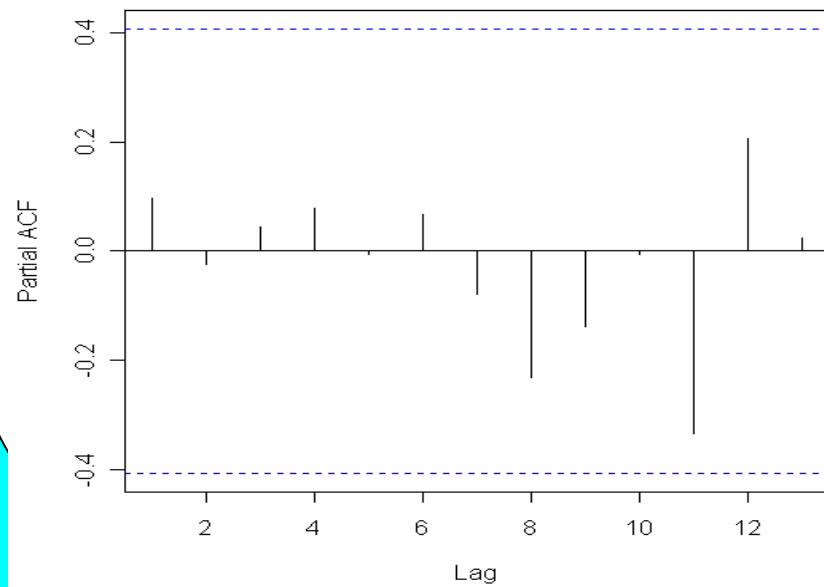
Oversampling

- If data are –vely auto-correlated
 - Ø Oversampling with small block size will fatten tails
 - Ø Overall effect depends on correlation
- But we are estimating AFs so we also look at these correlations.

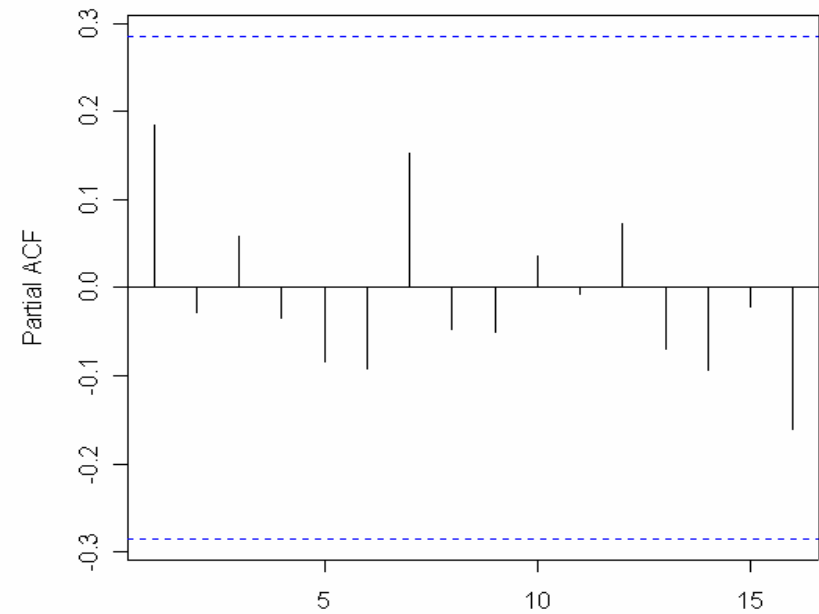
Series data



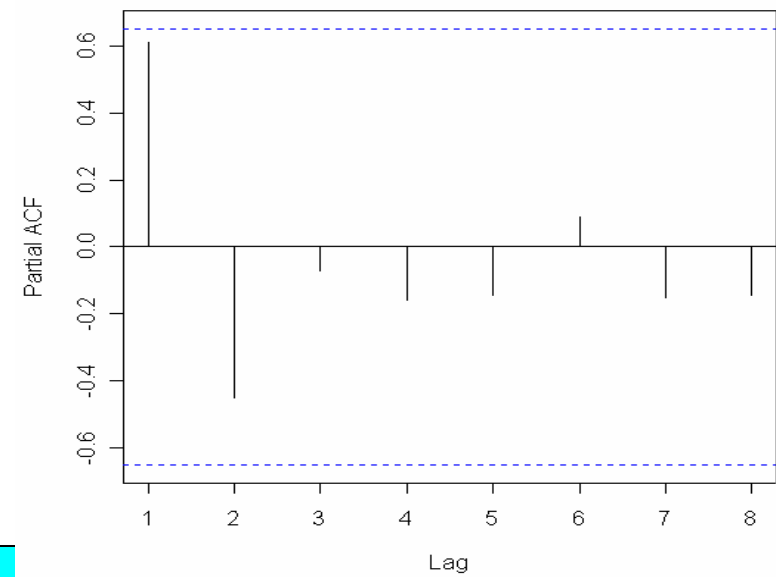
Series AF24m



Series AF12m



Series AF60m

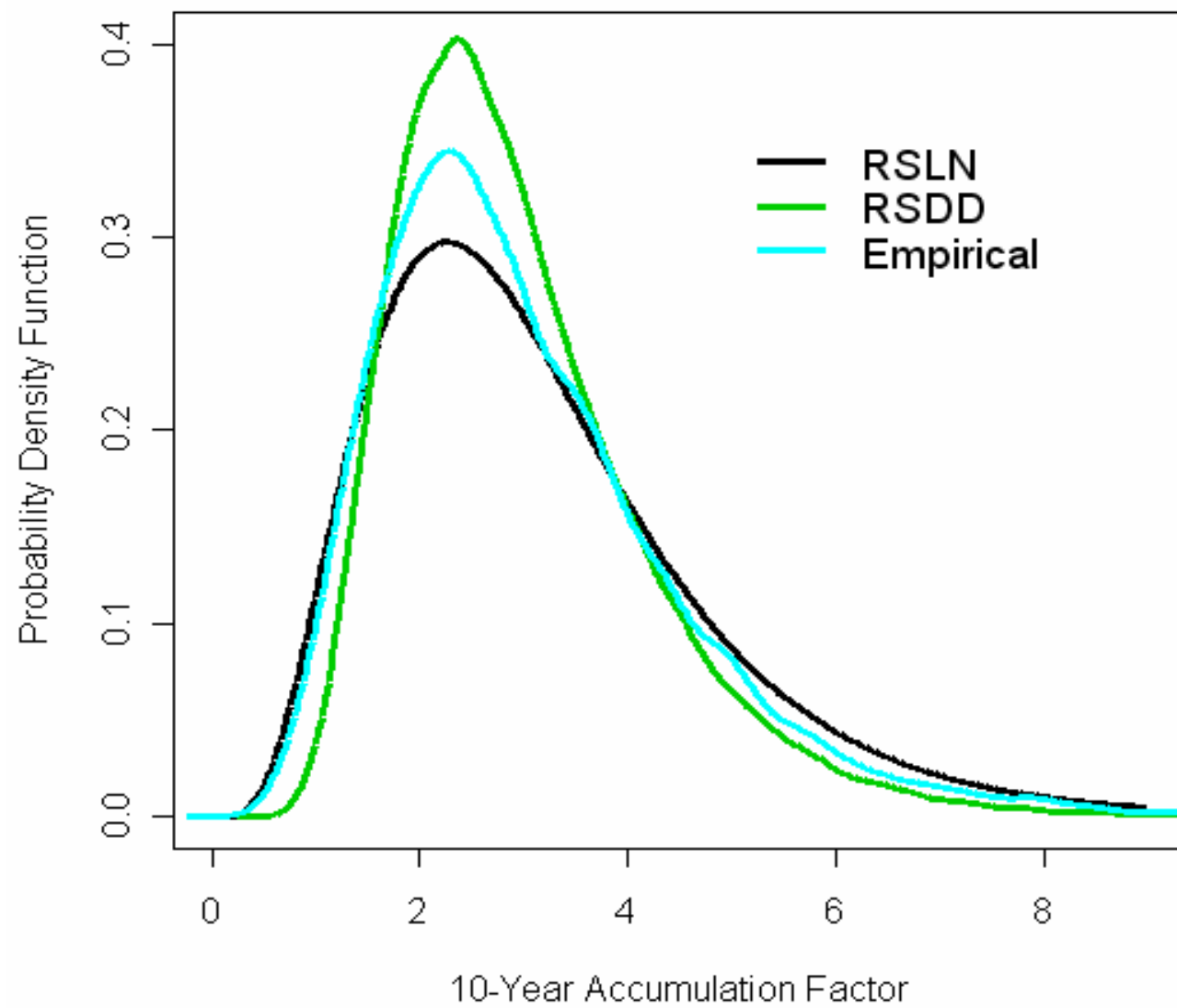


Back to the data

- No significant negative autocorrelations...
- So oversampling should over-estimate left tail quantiles (on average)
- And underestimate right tail quantiles

Left tail, 10-Year AFs

Model	2.5%	5%	10%
Bootstrap (sort of...)	1.041 <i>(1.03, 1.06)</i>	1.228 <i>(1.20,1.25)</i>	1.478 <i>(1.47,1.49)</i>
RSDD	1.277	1.439	1.653
SLV	1.082	1.254	1.468
RSGARCH	0.905	1.086	1.315
RSLN	0.914	1.105	1.378



Summing up

- We need to pay attention to model econometrics
- Huge financial implications – especially with traditional actuarial methods
- Abusing the bootstrap offers some info
- Multiple state models for equity returns.

References

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