Seminars on Quantitative Finance October 26, 2005
The Fields Institute
Toronto

A Unifying Framework for Utility Maximization Problems with Unbounded Semimartingales

based on three joint papers with

Sara Biagini

Università degli Studi di Perugia

Marco Frittelli

DiMaD - Università di Firenze

The problem

We are interested in the utility maximization problem:

$$\sup_{H \in \mathcal{H}} E[u(x + (H.X)_T)]$$

u is increasing strictly concave and differentiable

$x \in \mathbb{R}$ is the initial endowment, $T \in (0, +\infty]$,

$X = (X_t)_{t \in [0,T]}$ is an $\mathbb{R}^d-$valued càd-làg semimartingale,

which models the discounted prices of d assets,

$\mathcal{H} \subseteq \left\{ \mathbb{R}^d - \text{valued, predictable, } X - \text{integrable proc. } H \right\}$

is an appropriate class of "**admissible**" integrands

$(H.X)_T = \int_0^T H_s \cdot dX_s$

*

Main issues

- X **is not necessarily locally bounded**

- New concept of $W-$**admissibility**

$$(H \cdot X)_t \geq -cW \quad \forall t \leq T,$$

- **Unified** treatment - via duality - of the cases

$$u : (0, +\infty) \to \mathbb{R} \text{ and } u : (-\infty, +\infty) \to \mathbb{R}.$$

- On **compatibility** conditions on the losses W admitted in trading:

$$\exists \alpha > 0 \text{ such that } E[u(x - \alpha W)] > -\infty.$$

- the dual variables, i.e. "**the sigma martingale measures**" in general belong to

$$ba_+(1)$$

Big losses (given u) $\Longrightarrow$ bubbles

Definition of ”classical” admissible strategies

DEFINITION

A trading strategy H is admissible (we will say $1-$**admissible**) if there exists a constant $c \in \mathbb{R}$ such that, $P-a.s.$,

$$(H \cdot X)_t \geq -c1 \quad \text{for all } t \in [0, T].$$

$\mathcal{H}^1$ is the class of these $1-$**admissible** strategies.

In the **non locally bounded case** it can happen that:

$$\mathcal{H}^1 = \{0\}$$

and this fact forces us to introduce the less restrictive notion of $W-$admissibility, in order to provide a non trivial enlargement of the class $\mathcal{H}^1$.

Motivation

$X = (X_0, X_1)$ one period process with

$$X_0 = 0 \quad \text{and} \quad X_1 \sim N(\mu, \sigma^2)$$

$$\mathcal{H} = \{H \text{ predictable and } X - \text{integrable}\} = \mathbb{R}$$

$$(H \cdot X)_1 = HX_1 \qquad H \in \mathbb{R}$$

HX_1 is not bounded from below, unless $H = 0$.

$$\Longrightarrow \qquad \mathcal{H}^1 = \{0\}\,.$$

Take $u(x) = -e^{-x}$, then

$$\sup_{H \in \mathcal{H}^1} E[u(x + HX_1)] = u(x + 0) = -e^{-x}.$$

$$\sup_{H \in \mathcal{H}} E[u(x + HX_1)] = -e^{-(x + \frac{1}{2}\frac{\mu^2}{\sigma^2})} > -e^{-x}.$$

The maximizer is given by

$$H^* = \frac{\mu}{\sigma} \notin \mathcal{H}^1, \text{ if } \mu \neq 0.$$

Definition of $W - ADMISSIBLE$ strategies

DEFINITION

Let $W \in L^0(P)$ be a fixed random variable s.t.

$W \geq 1$ $P-$ a.s., but possibly unbounded from above.

The $\mathbb{R}^d-$valued predictable $X-$integrable process H is $W-$**admissible**, or it belongs to $\mathcal{H}^W$, if:

- there exists a constant $c \geq 0$ such that, P-a.s.

$$(H \cdot X)_t \geq -cW \quad \forall t \leq T.$$

  (hence $(H \cdot X)_t$ can be unbounded from above and **from below**)

Accepting a greater risk will increase the expected utility.

Note that

$$\mathcal{H}^1 \subseteq \mathcal{H}^W,$$

since $W \geq 1$.

W *is SUITABLE*

(W is sufficiently large)

DEFINITION:

A random variable $W \in L^0(P)$ is **suitable** if $W \geq 1$ $P-$ a.s. and

for all $1 \leq i \leq d$ there exists a process H^i such that:

$$\begin{aligned} -W &\leq (H^i \cdot X^i)_t \leq W, \text{ for all } t \in [0,T] \\ .P(\{\omega \,|\, \exists t &\geq 0\, H_t^i(\omega) = 0\}) = 0. \end{aligned}$$

(“ $H^i \neq 0$ ” and both investments H^i and $-H^i$ in the single asset X^i are “W-admissible”).

PROPOSITION:

If X is locally bounded, then

the constant 1 is suitable.

W is COMPATIBLE with u

(W not too large)

$$\alpha^{W}(x) \triangleq \sup\{\alpha \geq 0 \mid E[u(x-\alpha W)] > -\infty\} \in [0,+\infty].$$

DEFINITION:

A random variable $W \in L^0(P)$ is **compatible** if $W \geq 1$ $P-$ a.s. and

$$\exists \alpha > 0 \text{ such that } E[u(x-\alpha W)] > -\infty. \qquad \text{((*))}$$

or equivalently if

$$\alpha^{W}(x) > 0.$$

REMARK on Cramer's condition. If $u(x) = -e^{-x}$, then (*) holds iff

$$\exists \alpha > 0 \quad \text{such that } E[e^{\alpha W}] < \infty.$$

DEFINITION: $\mathcal{W}$ is the set of all **suitable and compatible** random v.;

$$\mathcal{W}_{\infty} \triangleq \left\{W \in \mathcal{W} : \alpha^{W}(x) = +\infty\right\}.$$

On "compatibility" conditions

$\alpha^W(x) \triangleq \sup\{\alpha \geq 0 \mid E[u(x - \alpha W)] > -\infty\}$.

$$\begin{aligned} W &\in L^\infty, && (1) \\ \forall\alpha &> 0\, E[u(x - \alpha W)] > -\infty, && (2) \\ \exists\alpha &> 0\, E[u(x - \alpha W)] > -\infty. && (3) \end{aligned}$$

Obviously: $(2) \Leftrightarrow \alpha^W(x) = \infty$, $(3) \Leftrightarrow \alpha^W(x) > 0$, $(1) \Rightarrow (2) \Rightarrow (3)$

The strongest condition (1) leads to classical $1-$admissibility: $\mathcal{H}^W = \mathcal{H}^1$

The weaker compatibility condition (2) is studied in Biagini-F. (2004) and leads to:

i) **Uniformity results** w.r.to $W \in \mathcal{W}_\infty$:

the optimal value & the optimal solution do not depend on which W is selected in $\mathcal{W}_\infty$

ii) dual variables that are **probability measures**

The weakest condition (3) will lead to dual variables that are only in $ba_+(1)$.

ASSUMPTION (1):

$u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is increasing strictly concave and differentiable on the interior $\mathcal{I} = (a, +\infty)$ of its effective domain and:

$$u'(a) \triangleq \lim_{x \downarrow a} u'(x) = +\infty,\, u'(+\infty) \triangleq \lim_{x \uparrow +\infty} u'(x) = 0.$$

W.l.o.g., either $\mathcal{I} = (0, +\infty)$ or $\mathcal{I} = (-\infty, +\infty)$.

If $u : \mathbb{R} \to \mathbb{R}$ then: u has Reasonable Asymptotic Elasticity - **RAE(u)** - as introduced by Schachermayer (and Kramkov - Schachermayer)

$$(i)\ AE_{-\infty}(u) \triangleq \liminf_{x \to -\infty} \frac{xu'(x)}{u(x)} > 1$$

$$(ii)\ AE_{+\infty}(u) \triangleq \limsup_{x \to +\infty} \frac{xu'(x)}{u(x)} < 1$$

If $u : (0, +\infty) \to \mathbb{R}$ then: $AE_{+\infty}(u) < 1$.

NOTE: This condition is needed even in the locally bounded case to avoid pathological phenomena.

ASSUMPTION (2) :

$$W \in \mathcal{W} \neq \varnothing$$

(i.e. there exists a suitable and compatible **loss** random variable W)

REMARKS

(i) Suppose X locally bounded. Then:

Assumption (2) is automatically satisfied since $W = 1 \in \mathcal{W}$.

(ii) General X (not nec. positive or loc. bounded),

If $\mathcal{I} = (0, +\infty)$ the assumption (2) implies

$$x > 0 \text{ and } W \in L^\infty,$$

.since $E[u(x - \alpha W)] > -\infty \quad \Rightarrow \quad x - \alpha W \geq 0$

In this case, w.l.o.g. we select

$$W = 1 \quad \text{and} \quad \mathcal{H}^W = \mathcal{H}^1.$$

Definitions of P_Φ *and* M_σ

The convex conjugate $\Phi : \mathbb{R}_+ \to \mathbb{R}$ of u is:

$$\Phi(y) = \sup_{x \in \mathbb{R}} \{u(x) - xy\}.$$

DEFINITION

$$\mathcal{P}_\Phi = \{\xi \in L^1_+(P) \mid E[\Phi(\xi)] < +\infty\}.$$

If $\xi = \frac{dQ}{dP}$ then $Q \ll P$ has *finite entropy*.

DEFINITION

$$M_\sigma = \{Q \ll P : X \text{ is a } \sigma - \text{martingale w.r.to } Q\}$$

is the set of **Sigma martingale probability m**.

Essentially, X is a $\sigma-$martingale if each component X^i can be written as a stochastic integral of a local martingale N^i.

M_σ replaces the set **of loc. martingale measures**, that was adequate when X was assumed locally bounded.

Nice mathematical properties when $W \in W$

Let $W \in L^0(P)$ and define:

$$M_{\sigma,W} \triangleq \{Q \in M_\sigma \mid E_Q[W] < +\infty\};$$

$$M_{sup,W} \triangleq \{Q \ll \quad \begin{array}{l} P \,|\, E_Q[W] < +\infty \text{ and} \\ H \cdot X \text{ is a } Q\text{-supermart. } \forall H \in \mathcal{H}^W\}; \end{array}$$

$$M_{T,W} \triangleq \{Q \ll \quad \begin{array}{l} P \,|\, E_Q[W] < +\infty \;(H \cdot X)_T \in L^1(Q), \\ E_Q[(H \cdot X)_T] \leq 0 \,\forall H \in \mathcal{H}^W\}. \end{array}$$

LEMMA:

Let $W \in \mathcal{W}$ and suppose that $M_{\sigma,W} \neq \varnothing$ Then:

$$M_{\sigma,W} = M_{sup,W} = M_{T,W}.$$

and

$$M_{\sigma,W} \cap \mathcal{P}_\Phi = M_\sigma \cap \mathcal{P}_\Phi.$$

$M_\sigma \cap \mathcal{P}_\Phi$ will be the relevant set of ”pricing” probability measures, when $\alpha^W(x) = +\infty$

Dual Varables

$$(L^\infty(P), ba(P))$$

$$\begin{aligned} H &\in \mathcal{H}^W \text{ if } (H\cdot X)_t \geq -cW \quad \forall t \leq T. \\ K^W &= \left\{ (H\cdot X)_T \mid H \in \mathcal{H}^W \right\} \end{aligned}$$

The following set replaces the set of bounded super-replicable claims:

$$\begin{aligned} \mathcal{C}_W &\triangleq \left(\frac{K^W}{W} - L^0_+(P)\right) \cap L^\infty(P) \\ &= \left\{ f \in L^\infty(P) \mid f \geq k \text{ and } k \in \frac{K^W}{W} \right\}. \end{aligned}$$

By Fatou:

$$\begin{aligned} \sup_{k\in K^W} E[u(x+k)] &= \sup_{k\in\frac{K^W}{W}} E[u(x+kW)] \\ &= \sup_{f\in\mathcal{C}_W} E[u(x+fW)] \\ &= \inf_{z\in(\mathcal{C}_W)^0} \ldots \end{aligned}$$

The set of dual variables is:

$$\mathcal{Z}_W \triangleq \mathcal{C}_W^0$$

$$\mathcal{Z}_W \triangleq \{z \in ba \mid z(f) \leq 0 \text{ for all } f \in \mathcal{C}_W\} \subseteq ba_+$$

REMARK:

$$z \in \mathcal{Z}_W \quad \Longleftrightarrow \quad z_r(\frac{k}{W}) + z_s(\frac{k}{W}) \leq 0 \text{ for all } k \in K^W,$$

where, by Yosida-Hewitt,

$$z = z_r + z_s \in ca \oplus pa$$

How can we interpret the dual variables $z \in \mathcal{Z}_W$ as pricing operators ?

After normalization, each dual variable $z \in \mathcal{Z}_W$ having zero singular component is a sigma martingale measure.

$$z \in \mathcal{Z}_W \; \Leftrightarrow z_r(\frac{k}{W}) + z_s(\frac{k}{W}) \leq 0 \text{ for all } k \in K^W.$$

Let $\mathcal{Z}_W^r$ be the set of **true** measures in $\mathcal{Z}_W$, that is

$$\mathcal{Z}_W^r = \mathcal{Z}_W \cap L^1(P)$$

(possibly empty)

PROPOSITION:

If $Q \in M_{\sigma,W}$ then $z \triangleq Q(W\bullet) \in \mathcal{Z}_W^r$.

Viceversa, if $z \in \mathcal{Z}_W^r$, then $Q \triangleq \frac{z(\bullet \frac{1}{W})}{z(1/W)} \in M_{\sigma,W}$.

RESULTS

1. Existence (and uniqueness) and properties of the optimal solution to primal and dual problems when X is a general semimartingale

2. Unifying framework for:

$u : (0, +\infty) \to \mathbb{R}$ and $u : (-\infty, +\infty) \to \mathbb{R}$

(a) same class of admissible integrands

(b) same class of dual variables

3. $\mathcal{I} = (0, +\infty)$ we recover known results (CSW)

4. $\mathcal{I} = (-\infty, +\infty)$ $\alpha^W(x) = +\infty$

   (a) $Q_s^* = 0$

   (b) uniformity with respect to $W \in \mathcal{W}_\infty$.

   (c) supermart. property of the optimal wealth proc.

5. $\mathcal{I} = (-\infty, +\infty)$ $\alpha^W(x) < +\infty$

   (a) sufficient conditions for $Q_s^* = 0$

   (b) examples show that $Q_s^* \neq 0$

6. When big losses are admitted (w.r.to u) then the pricing functionals have a singular component (even with no random endowment).

First simple result

$$U^{\mathbf{W}}(x) \triangleq \sup_{H \in \mathcal{H}^W} E[u(x + (H \cdot X)_T)]$$

By Fatou, Fenchel, Rockafellar:

THEOREM

If $W \in \mathcal{W}$ and $U^W(x) < u(+\infty)$ then:

$$\begin{aligned} & \sup_{H \in \mathcal{H}^W} E[u(x + (H \cdot X)_T)] = \sup_{k \in K^W} E[u(x + k)] \\ = & \sup_{k \in \frac{K^W}{W}} E[u(x + kW)] = \sup_{f \in \mathcal{C}_W} E[u(x + fW)] \\ = & \min_{z \in \mathcal{Z}_W} \left\{ E[\frac{x}{W}\frac{dz_r}{dP}] + E\left[\Phi(\frac{1}{W}\frac{dz_r}{dP})\right] + G(z_s) \right\} < \infty, \end{aligned}$$

and the min is reached by an element $z^* \in \mathcal{Z}_W \subseteq ba_+$ such that $z_r^*(\Omega) > 0$.

$$\begin{aligned} \mathcal{D}_W & \triangleq \{ f \in L^\infty \mid E[u(x + fW)] > -\infty \} \\ G(z_s) & \triangleq \sup_{f \in \mathcal{D}_W} \{-z_s(f)\}, \; z_s \in ba_+. \end{aligned}$$

$$z \in \mathcal{Z}_W \longleftrightarrow \mathcal{M}_W(\Phi, G) \ni Q$$

$$Q(\bullet) = \frac{z_r(\frac{1}{W}\bullet) + z_s(\bullet)}{z_r(\frac{1}{W})}, \; z \in \mathcal{Z}_W$$

$$\begin{aligned} \mathcal{M}_W \;\; &\triangleq \;\; \{Q \in ba_+ \mid Q_r(W) < +\infty, Q_r(\Omega) = 1, \\ & \quad Q_r(k) + Q_s(\frac{k}{W}) \leq 0 \text{ for all } k \in K^W\}. \end{aligned}$$

Notice:

$$\mid Q_r \mid = 1$$

$$\begin{aligned} & \sup_{H \in \mathcal{H}^W} E[u(x + (H \cdot X)_T)] \\ = \; & \min_{\lambda > 0, Q \in \mathcal{M}_W} x\lambda + E\left[\Phi(\lambda \frac{dQ_r}{dP})\right] + \lambda G(Q_s) < \infty \end{aligned}$$

DEFINE:

$$\mathcal{M}_W(\Phi, G) \triangleq \{z \in \mathcal{M}_W : G(Q_s) < +\infty, \; Q_r \in P_\Phi\}$$

Main result (all utility functions)

$$K_\Phi^G \triangleq \left\{ f \in L^1(Q_r) \text{ and } E_{Q_r}[f] \leq G(Q_s)\ \forall Q \in \mathcal{M}_W(\Phi, G) \right\}$$

THEOREM

If $\sup_{H\in\mathcal{H}^W} E[u(x + (H \cdot X)_T)] < u(+\infty)$ then:

$\mathcal{M}_W(\Phi, G) \neq \varnothing$,

the optimal solution to:

$$U_\Phi(x) \triangleq \sup \left\{ E[u(x + f)] \mid f \in K_\Phi^G \right\}$$

exists, it is given by

$$f_x \triangleq -x - \Phi'(\lambda^* \frac{dQ_r^*}{dP}) \in K_\Phi^G$$

where Q^* and λ^* are optimal for the dual problem and

$$
\begin{aligned}
& \sup_{H \in \mathcal{H}^W} E[u(x + (H \cdot X)_T)] \\
= & \min_{\lambda > 0, Q \in \mathcal{M}_W(\Phi, G)} x\lambda + E\left[\Phi(\lambda \frac{dQ_r}{dP})\right] + \lambda G(Q_s) \\
= & U_\Phi(x) = E[u(x + f_x)],
\end{aligned}
$$

$$
E_{Q_r^*}[f_x] = G(Q_s^*).
$$

THREE CASES

$$\alpha^W(x) \triangleq \sup\{\alpha \geq 0 \mid E[u(x - \alpha W)] > -\infty\} \in [0, +\infty].$$

1. $\mathcal{I} = (0, +\infty)$.

$\alpha^W(x) > 0$ implies $x > 0$ and $W \in L^\infty$.

1. Then:

$$W = 1, \;\; \mathcal{H}^W = \mathcal{H}^1$$

2. $\mathcal{I} = (-\infty, +\infty)$ and $\alpha^W(x) = +\infty$.

3. $\mathcal{I} = (-\infty, +\infty)$ and $\alpha^W(x) < +\infty$.

FIRST CASE: $I = (0, +\infty)$ *- As in CSW*

Then $\alpha^W(x) > 0$ implies $x > 0$ and $W \in L^\infty$. Then:

$$W = 1 \text{ and } \alpha^1(x) = x$$

Normalize $z \in \mathcal{Z}_1 \subseteq ba_+$: $R(\cdot) \triangleq \frac{z(\cdot)}{z(\Omega)}$, so that$| R |= 1$

Dual variables as in Cvitanic Schachermayer Wang.

$$f_x \triangleq -x - \Phi'(\lambda^* \frac{dR_r^*}{dP}) \in K^1$$

is the optimal solution to

$$\sup \left\{ E[u(x+f)] \mid f \in K^1 \right\}$$

$$= \sup \left\{ E[u(x+f)] \mid f \in K_\Phi^G \right\}$$

and satisfies

$$R_r^*(x + f_x) = x.$$

(we recover known results).

SECOND CASE: $I = (-\infty, +\infty)$ *and* $\alpha^W(x) = +\infty$

Back to $Q \in \mathcal{M}_W(\Phi, G)$, where$\mid Q_r \mid = 1$.

Since $E[u(x - \alpha W)] > -\infty$ **for all** $\alpha > 0$

$$\begin{aligned} \mathcal{D}_W &\triangleq \{f \in L^\infty(P) \mid E[u(x + fW)] > -\infty\} = L^\infty(P) \\ G(Q_s) &\triangleq \sup_{f \in \mathcal{D}_W} \{-Q_s(f)\} = +\infty, \text{unless } Q_s = 0. \end{aligned}$$

Hence:

$$G(Q_s^*) = 0, \quad Q_s^*(\Omega) = 0, \quad Q_r^*(\Omega) = 1.$$

The optimal Q^* is a **true probability**!!!

$$\mathcal{M}_W(\Phi, G) = M_\sigma \cap \mathcal{P}_\Phi$$

$$K_\Phi^G = K_\Phi^0 = \left\{ f \in L^1(Q) : E_Q[f] \leq 0 \ \forall Q \in M_\sigma \cap \mathcal{P}_\Phi \right\}$$

The set $M_\sigma \cap \mathcal{P}_\Phi$ **does not depend on** $W \in \mathcal{W}_\infty$

$$E_{Q_r^*}[x + f_x] = x,$$

a true expectation.

THEOREM (CASE $I = (-\infty, +\infty)$ and $\alpha^W(x) = +\infty$)

Suppose that there exists $W_0 \in \mathcal{W}_\infty$ and $x_0 \in \mathbb{R}$ such that $U^{W_0}(x_0) < u(+\infty)$. Then:

(a) $M_\sigma \cap \mathcal{P}_\Phi \neq \varnothing$;

(b) For all $W \in \mathcal{W}_\infty$ and all $x \in \mathbb{R}$, $U^W(x) < u(+\infty)$;

(c) $U^W(x)$ does **not** depend on $W \in \mathcal{W}_\infty$, and

$$U^W(x) = \min_{\lambda > 0, Q \in M_\sigma \cap \mathcal{P}_\Phi} \lambda x + E\left[\Phi\left(\lambda \frac{dQ}{dP}\right)\right];$$

(d) $\forall x \in \mathbb{R}$ there exists the optimal solution $f_x \in K^0_\Phi$:

$$\max\left\{E[u(x+f)] \mid f \in K^0_\Phi\right\} = E[u(x+f_x)] = U_\Phi(x) < u(\infty)$$

and

$$U_\Phi(x) = U^W(x) \text{ for all } W \in \mathcal{W}_\infty;$$

(e) If λ_x, Q_x is the optimal solution in (c), then:

$$u'(x+f_x) = \lambda_x \frac{dQ_x}{dP};$$

(f) There exists a $\mathbb{R}^d-$valued predictable $X-$integrable process H^x such that

$$f_x = (H^x \cdot X)_T \quad Q_x - a.s.$$

and $H^x \cdot X$ is a Q_x-uniformly integrable martingale.

(g) **Supermartingale property of** $H^x \cdot X$:

If $Q_x \sim P$ then the optimal process

$$H^x \cdot X \text{ is a supermatingale wrt each } M_\sigma \cap P_\Phi.$$

(h) Let $V_\Phi(\lambda) = \min_{Q \in M_\sigma \cap \mathcal{P}_\Phi} E[\Phi(\lambda \frac{dQ}{dP})]$ and let Q_λ attain the minimum.

$$U_\Phi(x) = \inf_\lambda \{\lambda x + V_\Phi(\lambda)\}$$

V_Φ and U_Φ are cont. differentiable and:

$$V'_\Phi(\lambda) = E[\Phi'(\lambda \frac{dQ_\lambda}{dP}) \frac{dQ_\lambda}{dP}]$$

$$x U'_\Phi(x) = E[u'(x + f_x)(x + f_x)]$$

WHY K^0_Φ *?*

$$K^0_\Phi = \left\{ f \in L^1(Q) : E_Q[f] \le 0 \ \forall Q \in M_\sigma \cap \mathcal{P}_\Phi \right\}$$

EXAMPLE: We show what may go wrong:

(1) Arbitrage free market (NFLVR holds true); (2) $\mathcal{W}_\infty \neq \varnothing$

(3) $U^W(x) < u(+\infty)$ for all $W \in \mathcal{W}_\infty$; (4) For each $W \in \mathcal{W}_\infty$ the problem

$$\sup_{H \in \mathcal{H}^W} E[u(x + (H \cdot X)_T)]$$

does **not** admit an optimal solution $H^* \in \mathcal{H}^W$.

Therefore, the domain K^0_Φ larger than K^W is really needed. In general

$$f_x \in K^0_\Phi \quad \text{but} \quad f_x \notin K^W.$$

However,

$$\sup_{k \in K^W} E[u(x+k)] = \sup_{k \in K^0_\Phi} E[u(x+k)].$$

The supermartingale property of the optimal process

(Biagini-F. 2004)

$H^x \cdot X$ is a supermatingale w.r.to each $M_\sigma \cap P_\Phi$.

A bit of history

The supermartingale property of the optimal portfolio process for general semimartingales can be seen as the fourth point in the following list, concerning the case X **locally bounded**:

1. Six Authors' paper.

   When $u(x) = -e^{-x}$ and the *reverse Holder inequality* holds, it was proved that the optimal wealth process is a **true martingale** wrt every loc. mart. meas. Q with finite entropy.

2. Kabanov and Stricker removed the RHI;

3. Schachermayer proved that if $Q_x \sim P$, then

   $H_x \cdot X$ is a **supermartingale** under every loc. mart. meas. with finite entropy

   (the true martingale property is lost for general u).

- We proved that this supermartingale property holds even for **unbounded** semi-martingales.

Example 1 (Merton)

We consider a Black Scholes market with an **exponential** utility maximizer agent.

$$dX_t = \mu X_t dt + \sigma X_t dB_t \quad 0 \leq t \leq T < +\infty,$$

where B is the standard Brownian motion.

Here the process is continuous (hence locally bounded) and the hypotheses of the Theorem are satisfied with $W_0 = 1$, x arbitrary, so that:

$$U^W(x) = U^1(x) \quad \text{for any} \quad W \in \mathcal{W}_\infty.$$

Let $Z_t = B_t + \frac{\mu}{\sigma} t$ be the Brownian motion under the unique martingale measure Q.

It is widely known that

$$U^1(x) = \sup_{k \in K^1} E[u(x+k)] = E[-e^{-(x+\frac{\mu}{\sigma} Z_T)}],$$

However, the function

$$f_x = \frac{\mu}{\sigma} Z_T$$

does not belong to K^1, because it is unbounded, and no optimal solution exists in K^1.

But if we take $W' = 1 - \mathrm{inf}_{t \le T} Z_t$, then:

$$W' \in \mathcal{W}_\infty \quad \text{and} \quad f_x \in K^{W'}.$$

Indeed:

$$f_x = \frac{\mu}{\sigma} \int_0^T \frac{1}{\sigma X_t} dX_t \quad \text{with} \quad H' = \frac{\mu}{\sigma^2 X} \in \mathcal{H}^{W'}$$

This classic setup provides an example in which:

(1) $\mathcal{H}^1$ is strictly contained in $\mathcal{H}^{W'}$,

(2) $U^1(x) = U^{W'}(x)$.

(3) There exists an optimal solution in $\mathcal{H}^{W'}$, but not in $\mathcal{H}^1$.

This enlargement of the strategies does not increase the maximum, but it is necessary to catch the optimal solution

Example 2 (not locally bounded price process)

Let $u(x) = -e^{-x}$, $\Phi(z) = z \ln z - z$, and consider the price process:

$$X_t = V\, I_{\{\tau \le t\}}$$

which consists of one jump of size V at the stopping time τ.

Suppose $V \sim N(\mu, \sigma^2)$, $\mu \neq 0$, and V and τ are $P-$independent.

$$\mathcal{H}^1 = \{0\}, \qquad K^1 = \{0\}$$

$$U^1(x) = \sup_{k \in K^1} E[-e^{-(x+k)}] = -e^{-x}.$$

Note that the constant 1 is NOT $X-$suitable, hence:

$$1 \notin \mathcal{W}_\infty.$$

PROPOSITION:

(1) $W' \triangleq (1 + |V|) \in \mathcal{W}_\infty \neq \emptyset$

(2) $M_\sigma \cap \mathcal{P}_\Phi \neq \emptyset$.

(3) For all $x \in \mathbb{R}$, $U^{W'}(x) < 0 = u(+\infty)$ and

$$\sup_{k \in K^{W'}} E[-e^{-(x+k)}] = \min_{y>0, Q \in M_\sigma \cap \mathcal{P}_\Phi} \left\{ xy + E[\Phi(y \frac{dQ}{dP})] \right\} \tag{4}$$

(4) the supremum in the primal problem is a maximum, the optimal solution is

$$f^* = \frac{\mu}{\sigma^2} V \in K^{W'}$$

and the optimal value

$$U^{W'}(x) = -e^{-(x+\frac{\mu^2}{2\sigma^2})} > -e^{-x}$$

is strictly bigger than $-e^{-x}$, which is the optimal value of the maximization on the trivial domain $K^1 = \{0\}$.

Similar results can be obtained in a model with infinitely many jumps: take a Compound Poisson process on $[0, T]$:

$$X_t = \sum_{j \leq N_t} V_j,$$

where the jumps V_j are unbounded (i.e.:$V_j \sim N(m, \sigma^2)$, with $m \neq 0$) and N_t is a Poisson process independent from $(V_j)_j$.

THIRD CASE: $I = (-\infty, +\infty)$ *and* $\alpha^W(x) < +\infty$

$$\mathcal{D}_W \triangleq \{f \in L^\infty(P) \mid E[u(x + fW)] > -\infty\}$$

Note that

$$f_n \triangleq -n1_{\{W \leq n\}} \in \mathcal{D} \text{ for all } n \geq 1.$$

Hence: if $z \in ba_+$ satisfy $G(z_s) < +\infty$, then

$$z_s(\{W \leq n\}) = 0 \text{ for all } n \geq 1.$$

$$z_s(f) = z_s(f1_{W>n}),\ f \in L^\infty.$$

Define for $f \in K^W$

$$c_f \triangleq \lim_n c_n,$$

where:

$$c_n(f) = \min\{c \mid f1_{\{W>n\}} \geq -cW1_{\{W>n\}}\}$$

is the minimal c such that $f \geq -cW$ for $W > n$.

$$
\begin{aligned}
c_n(f) &\triangleq \min\{c \mid f I_{\{W>n\}} \geq -cW I_{\{W>n\}}\}, \;\; c_n(f) \downarrow c_f \\
\alpha^W(x) &\triangleq \sup\{\alpha \geq 0 \mid E[u(x-\alpha W)] > -\infty\} > 0.
\end{aligned}
$$

PROPOSITION

Suppose that the optimal $f_x \in K^W$. Then

$$
\alpha^W z_s^*(\Omega) \leq G(z_s^*) = z_s^*(-\frac{f_x}{W}) \leq c_{f_x} z_s^*(\Omega)
$$

As a consequence:

$$
c_{f_x} < \alpha^W \Rightarrow z_s^* = 0.
$$

Interpretation:

When the maximum utility is reached without becoming too close to the maximum tolerated risk, the optimal charge is a true measure.

When the optimal claim 'tends' to the maximum risk, a singular part may or may not appear in the optimal Q^*: this depends also on the market model, as shown in the examples.

Example: exponential utility without bubble: $c_{f_x} < \alpha^W$

One period market model, $X_0 = 0$, X_1 doubly exponential with density: $\frac{\sqrt{3}}{2} e^{-\sqrt{3}|x-1|}$.

X is unbounded from both sides

$W = 1 + |X_1|$ is suitable , $\mathcal{H}^W = \mathbb{R}$ and $\alpha^W(x) = \alpha^W = \sqrt{3}$.

$$\sup_{a \in \mathbb{R}} E[-e^{-aX_1}] = \sup_{|a|<\sqrt{3}} E[-e^{-aX_1})] = E[-e^{-X_1}] = -\frac{3}{2e} < 0 = u(+\infty).$$

$f_x = X_1 \in K^W$ and $c_{f_x} = 1 < \alpha^W$, then

$$Q_s^* = 0$$

$$\begin{aligned} \sup_{a \in \mathbb{R}} E[-e^{-aX_1}] &= \min_{Q \in ba_+ : E_{Q_r}[X_1] + Q_s(\frac{X_1}{W}) = 0} -e^{-H(Q_r,P) - \sqrt{3} Q_s(\Omega)} \\ &= -e^{-H(Q_r^*,P)} = -\frac{3}{2e} \end{aligned}$$

where $dQ_r^* = e^{-\ln(\frac{3}{2e})} e^{-X_1} dP$ is the optim. marting. m.:

$$E_{Q_r^*}[X_1] = 0.$$

Example (continuation)

The relevant $Q \in \mathcal{M}_W(\Phi, G)$ satisfy $(K \geq 0)$:

$$Q_r(X_1) + Q_s([K, +\infty)) - Q_s((-\infty, -K]) = 0$$

and Q_s is null on each bounded set.

Examples of $Q \in \mathcal{M}_W(\Phi, G)$:

$Q_r = P$

Q_s pure charge such that:

- on the positive halfline and on every compact it is null
- it gives mass 1 to the whole negative halfline.

Then

$$\begin{aligned} Q_r(X_1) + Q_s\left(\frac{X_1}{1 + |X_1|}\right) &= Q_r(X_1) - Q_s((-\infty, 0]) \\ &= 1 - 1 = 0 \end{aligned}$$

Example: exponential utility with bubble, $c_{f_x} = \alpha^W$

$\Omega_1 = \{\omega_0^1, \omega_1^1, \omega_2^1, \cdots, \omega_n^1, \cdots\}$ and $\Omega_2 = \mathbb{R}$.

Fix a doubly exponential variable Y on Ω_2 with density:

$$ce^{-|x|}$$

and take $W = 1 + |Y|$.

$\alpha^W(x) = \alpha^W = 1.$

Let $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F}_0$ trivial and $\mathcal{F}_1 = \mathcal{P}(\Omega_1) \otimes \sigma(W)$.

Define: $X_0 = 0$ and $X_1 = ZW$, where $Z \in L^\infty(\Omega_1)$

$$Z = \begin{cases} 1 & \text{on } \omega_0^1 \\ \frac{1}{n} - 1 & \text{on } \omega_n^1,\ n \geq 1 \end{cases}$$

Let $P = P_1 \otimes P_2$, where P_2 gives Y the doubly-exp distribution and P_1 is identified with the numbers

$$p_n = P_1(\omega_n^1) > 0, n \geq 0.$$

The investor has exponential utility, hence we face ($x = 0$):

$$\sup_{h \in R} E[-e^{-hX_1}]$$

Selecting p_n appropriately, we show:

$g(h) \triangleq E[-e^{-hX_1}]$ is finite iff $-1 < h \leq 1$;

$g'(h) > 0$ for all $-1 < h \leq 1$

Then the maximum of g is reached when $h = 1$.

$$f_x = X_1, \quad \frac{dQ_r^*}{dP} = \frac{e^{-X_1}}{E[e^{-X_1}]}$$

Since $g'(1) > 0$,

$$G(Q_s^*) = E_{Q_r^*}[f_x] = \frac{g'(1)}{E[e^{-X_1}]} > 0$$

and $Q_s^*(\Omega) = E_{Q_r^*}[f_x] > 0$

Note that $c_{f_x} = 1 = \alpha^W$.