Perturbation Methods in Default Modeling

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References:

Derivatives in Financial Markets with Stochastic Volatility

Cambridge University Press, 2000

Stochastic Volatility Asymptotics

SIAM Journal on Multiscale Modeling and Simulation, 2(1), 2003

Stochastic Volatility Effects on Defaultable Bonds

Applied Mathematical Finance (in press)

Modeling Correlated Defaults: First Passage Model under Stochastic Volatility

Working Paper

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Defaultable Bonds

In the first passage structural approach, the payoff of a defaultable zero-coupon bond written on a risky asset X is

$$h(X) \quad = \quad \mathbf{1}_{\{\inf_{0\leq s\leq T} X_s > B\}}.$$

By no-arbitrage, the value of the bond is

$$\begin{aligned} P^B(t,T) &= \mathbb{E}^\star\left\{e^{-r(T-t)}\mathbf{1}_{\{\inf_{0\leq s\leq T} X_s > B\}} \mid \mathcal{F}_t\right\} \\ &= \mathbf{1}_{\{\inf_{0\leq s\leq t} X_s > B\}} e^{-r(T-t)}\mathbb{E}^\star\left\{\mathbf{1}_{\{\inf_{t\leq s\leq T} X_s > B\}} \mid \mathcal{F}_t\right\}, \end{aligned}$$

Using the **predictable stopping time** $\tau_t = \inf\{s \geq t, X_s \leq B\}$:

$$\mathbb{E}^\star\left\{\mathbf{1}_{\{\inf_{t\leq s\leq T} X_s > B\}} \mid \mathcal{F}_t\right\} \quad = \quad \mathbb{P}^\star\{\tau_t > T \mid \mathcal{F}_t\}.$$

This defaultable zero-coupon bond is in fact a binary down-an-out barrier option where the barrier level and the strike price coincide.

Constant Volatility: Merton's Approach

$$\begin{aligned} dX_t &= rX_t dt + \sigma X_t dW_t^\star \\ X_t &= X_0 \exp\left((r - \frac{1}{2}\sigma^2)t + \sigma W_t^\star\right). \end{aligned}$$

In the Merton's approach, default occurs if $X_T < B$:

Defaultable bond = European digital option

$$\begin{aligned} u^d(t,x) &= \mathbb{E}^\star\left\{e^{-r\tau}\mathbf{1}_{\{X_T > B\}} \mid X_t = x\right\} = e^{-r\tau}\mathbb{P}^\star\left\{X_T > B \mid X_t = x\right\} \\ &= e^{-r\tau}N(d_2(\tau)) \end{aligned}$$

with the usual notation $\tau = T - t$ and the *distance to default*:

$$d_2(\tau) = \frac{\log\left(\frac{x}{B}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

Constant Volatility: Black-Cox Approach

$$\mathbb{E}^\star\left\{\mathbf{1}_{\{\inf_{t\le s\le T} X_s > B\}} \mid \mathcal{F}_t\right\}$$
$$= \mathbb{P}^\star\left\{\inf_{t\le s\le T}\left((r-\frac{\sigma^2}{2})(s-t)+\sigma(W_s^\star - W_t^\star)\right) > \log\left(\frac{B}{x}\right) \mid X_t = x\right\}$$

computed using distribution of minimum, or using PDE's:

$$\mathbb{E}^\star\left\{e^{-r(T-t)}\mathbf{1}_{\{\inf_{t\le s\le T} X_s > B\}} \mid \mathcal{F}_t\right\} = u(t, X_t)$$

where $u(t,x)$ is the solution of the following problem

$$\mathcal{L}_{BS}(\sigma)u = 0 \text{ on } x > B,\, t < T$$
$$u(t,B) = 0 \text{ for any } t \le T$$
$$u(T,x) = 1 \text{ for } x > B,$$

which is to be solved for $x > B$.

Constant Volatility: Barrier Options

Using the European digital pricing function $u^d(t,x)$

$$\mathcal{L}_{BS}(\sigma)u^d = 0 \ \text{ on } \ x > 0,\, t < T$$
$$u^d(T,x) = 1 \ \text{ for } \ x > B, \ \text{ and 0 otherwise}$$

By the method of images one has:

$$\begin{aligned} u(t,x) &= u^d(t,x) - \left(\frac{x}{B}\right)^{1-\frac{2r}{\sigma^2}} u^d\left(t, \frac{B^2}{x}\right) \\ &= e^{-r(T-t)}\left(N(d_2^+(T-t)) - \left(\frac{x}{B}\right)^{1-\frac{2r}{\sigma^2}} N(d_2^-(T-t))\right) \end{aligned}$$

where we denote

$$d_2^\pm(\tau) = \frac{\pm\log\left(\frac{x}{B}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

Yield Spreads Curve

The *yield spread* $Y(0,T)$ at time zero is defined by

$$e^{-Y(0,T)T} = \frac{P^B(0,T)}{P(0,T)},$$

where $P(0,T)$ is the default free zero-coupon bond price given here, in the case of constant interest rate r, by $P(0,T) = e^{-rT}$, and $P^B(0,T) = u(0,x)$, leading to the formula

$$Y(0,T) = -\frac{1}{T}\log\left(N\left(d_2(T)\right) - \left(\frac{x}{B}\right)^{1-\frac{2r}{\sigma^2}} N\left(d_2^-(T)\right)\right)$$

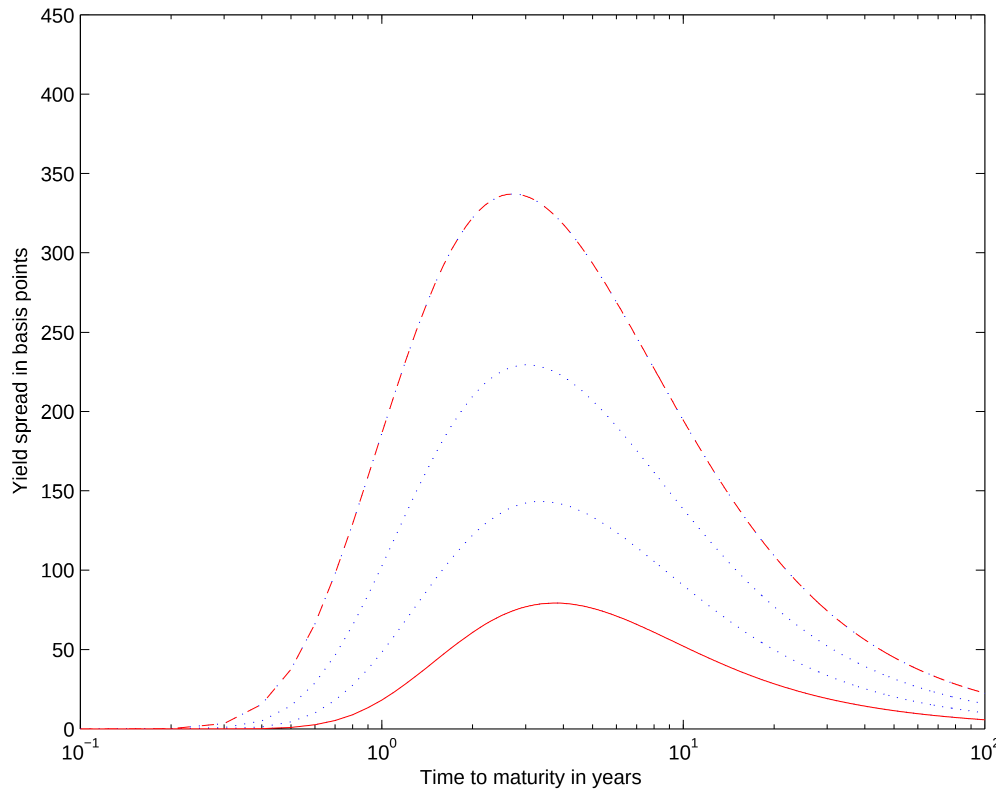


Figure 1: *The figure shows the sensitivity of the yield spread curve to the volatility level. The ratio of the initial value to the default level x/B is set to* 1.3*, the interest rate r is 6% and the curves increase with the values of σ: 10%, 11%, 12% and 13% (time to maturity in unit of years, plotted on the log scale; the yield spread is quoted in basis points)*

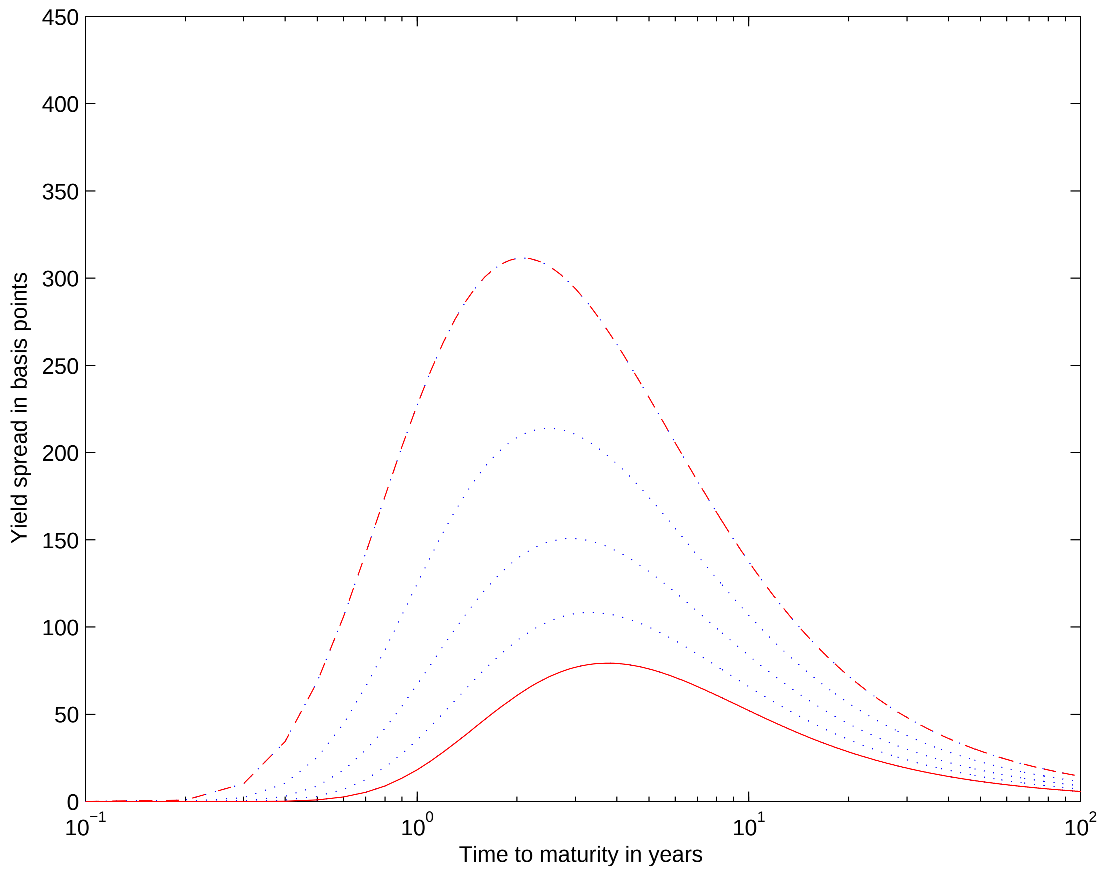


Figure 2: *This figure shows the sensitivity of the yield spread to the leverage level. The volatility level is set to 10%, the interest rate is 6%. The curves increases with the decreasing ratios* x/B*:* $(1.3, 1.275, 1.25, 1.225, 1.2)$.

Challenge: Yields at Short Maturities

As stated by Eom et.al. (empirical analysis 2001), **the challenge for theoretical pricing models is to raise the average predicted spread relative to crude models such as the constant volatility model, without overstating the risks associated with volatility or leverage.**

Several approaches (**within structural models**) have been proposed that aims at the modeling in this regard. These include

- Introduction of jumps (Zhou,...)
- Stochastic interest rate (Longstaff-Schwartz,...)
- Imperfect information (on X_t) (Duffie-Lando,...)
- Imperfect information (on B) (Giesecke)

Stochastic Volatility Models

$$
\begin{aligned}
dX_t &= \mu X_t dt + f(Y_t) X_t \, dW_t^{(0)} \\
dY_t &= \alpha(m - Y_t)dt + \nu\sqrt{2\alpha}\, dW_t^{(1)}
\end{aligned}
$$

where we assume that

- f non-decreasing, $0 < c_1 \leq f \leq c_2$
- Invariant distribution of Y: $\mathcal{N}(m, \nu^2)$ independent of α
- $\alpha > 0$ is the *rate of mean reversion* of Y
- The standard Brownian motions $W^{(0)}$ and $W^{(1)}$ are correlated

$$
d\left\langle W^{(0)}, W^{(1)} \right\rangle_t = \rho_1 \, dt
$$

Stochastic Volatility Models under $I\!\!P^\star$

In order to price defaultable bonds under this model for the underlying we rewrite it under a risk neutral measure $I\!\!P^\star$, chosen by the market through the market price of volatility risk Λ_1, as follows

$$\begin{aligned} dX_t &= rX_t dt + f(Y_t)X_t\, dW_t^{(0)\star}, \\ dY_t &= \left(\alpha(m - Y_t) - \nu\sqrt{2\alpha}\Lambda_1(Y_t)\right) dt + \nu\sqrt{2\alpha}\, dW_t^{(1)\star}. \end{aligned}$$

Here $W^{(0)\star}$ and $W^{(1)\star}$ are standard Brownian motions under $I\!\!P^\star$ correlated as $W^{(0)}$ and $W^{(1)}$. We assume that the market price of volatility risk Λ_1 is bounded and a function of y only.

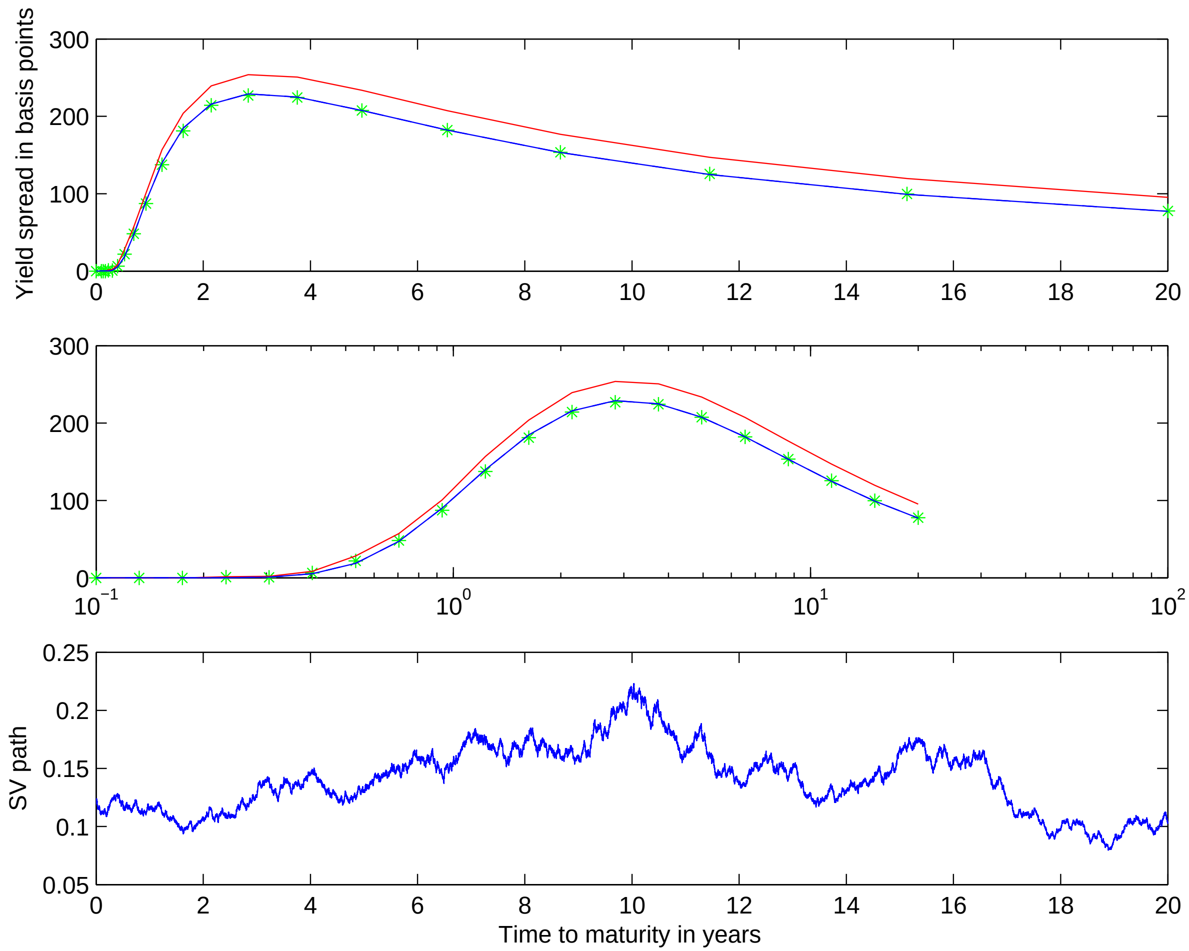


Figure 3: **Uncorrelated slowly mean-reverting stochastic volatility**: $\alpha = 0.05$ *and* $\rho_1 = 0$.

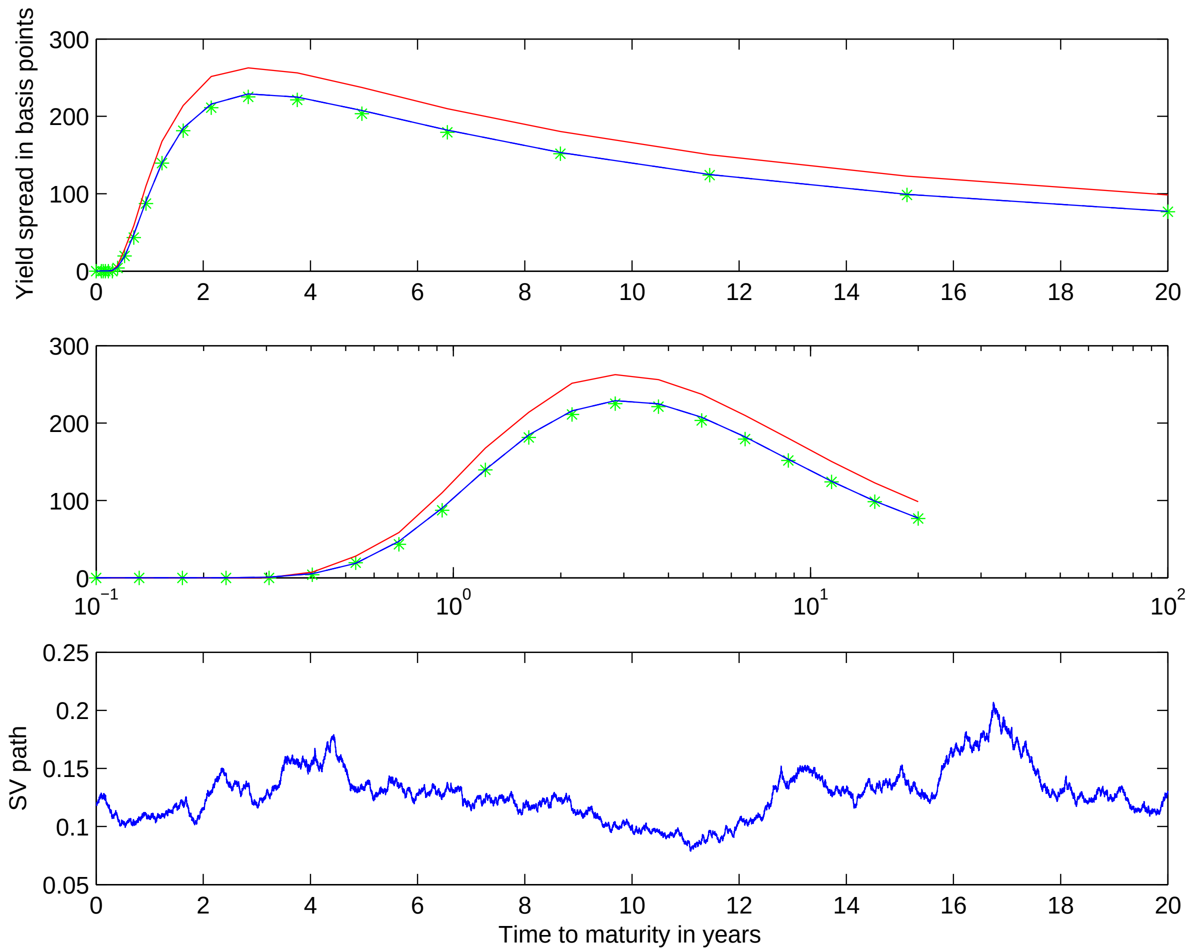


Figure 4: **Correlated slowly mean-reverting stochastic volatility***: $\alpha = 0.05$ and $\rho_1 = -0.05$.*

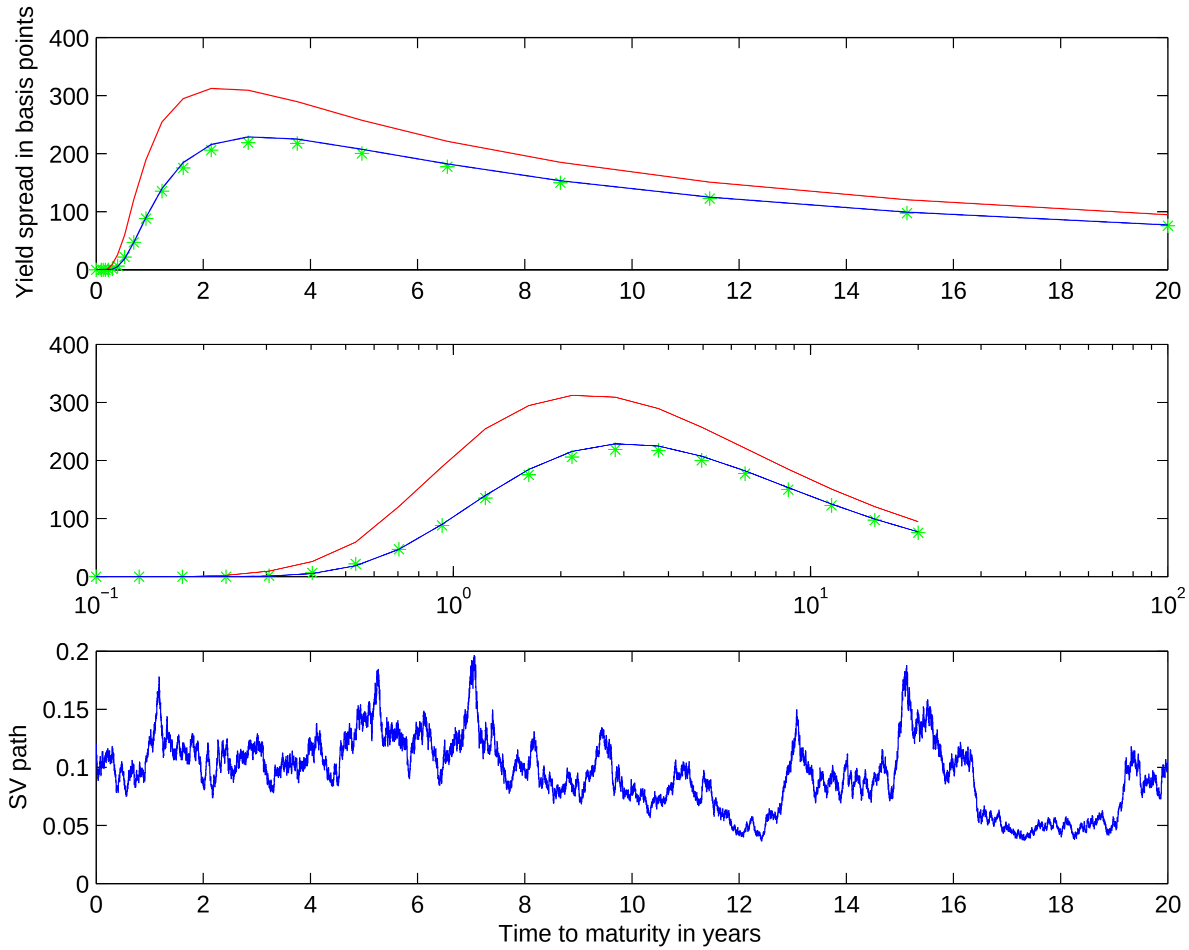


Figure 5: **Uncorrelated stochastic volatility:** $\alpha = 0.5$ **and** $\rho_1 = 0$**.**

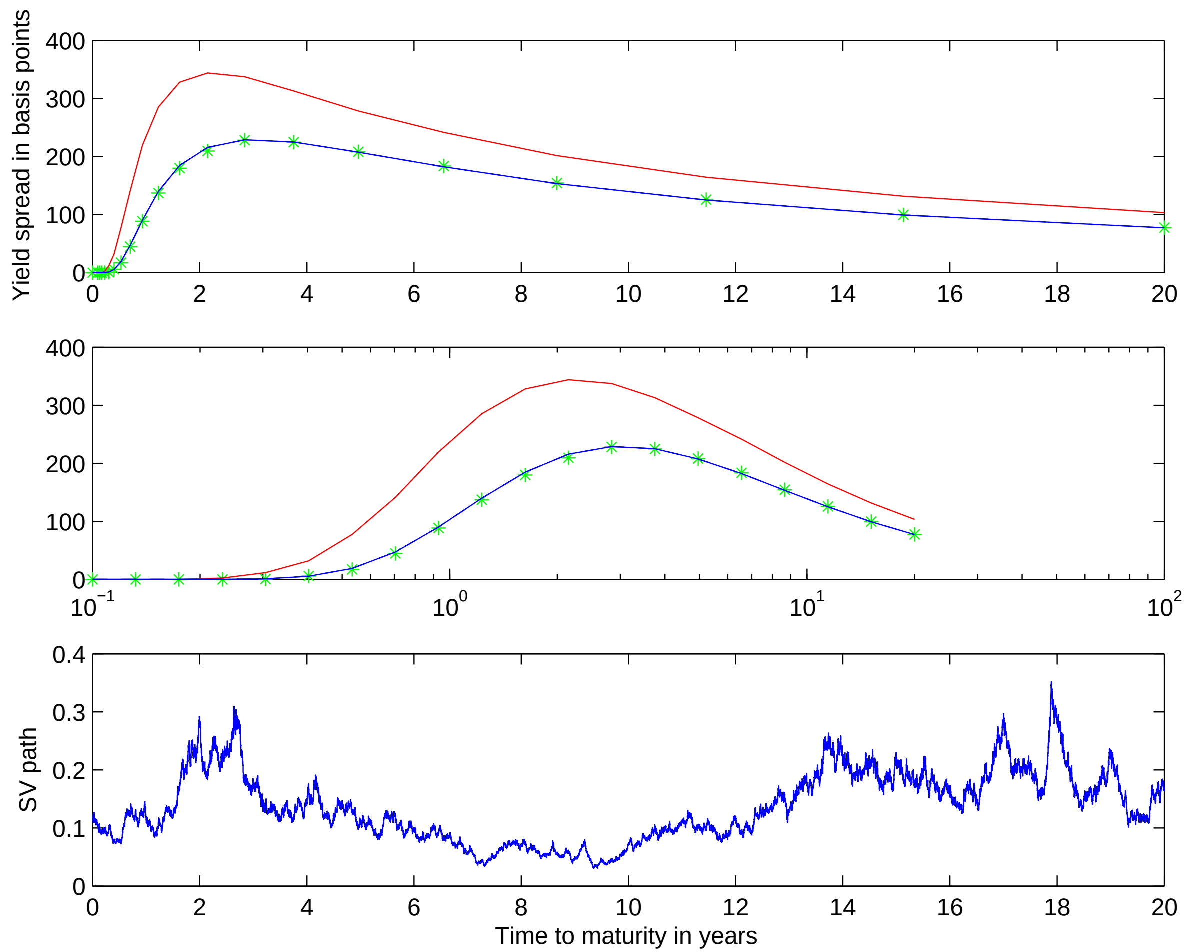


Figure 6: **Correlated stochastic volatility:** $\alpha = 0.5$ **and** $\rho_1 = -0.05$**.**

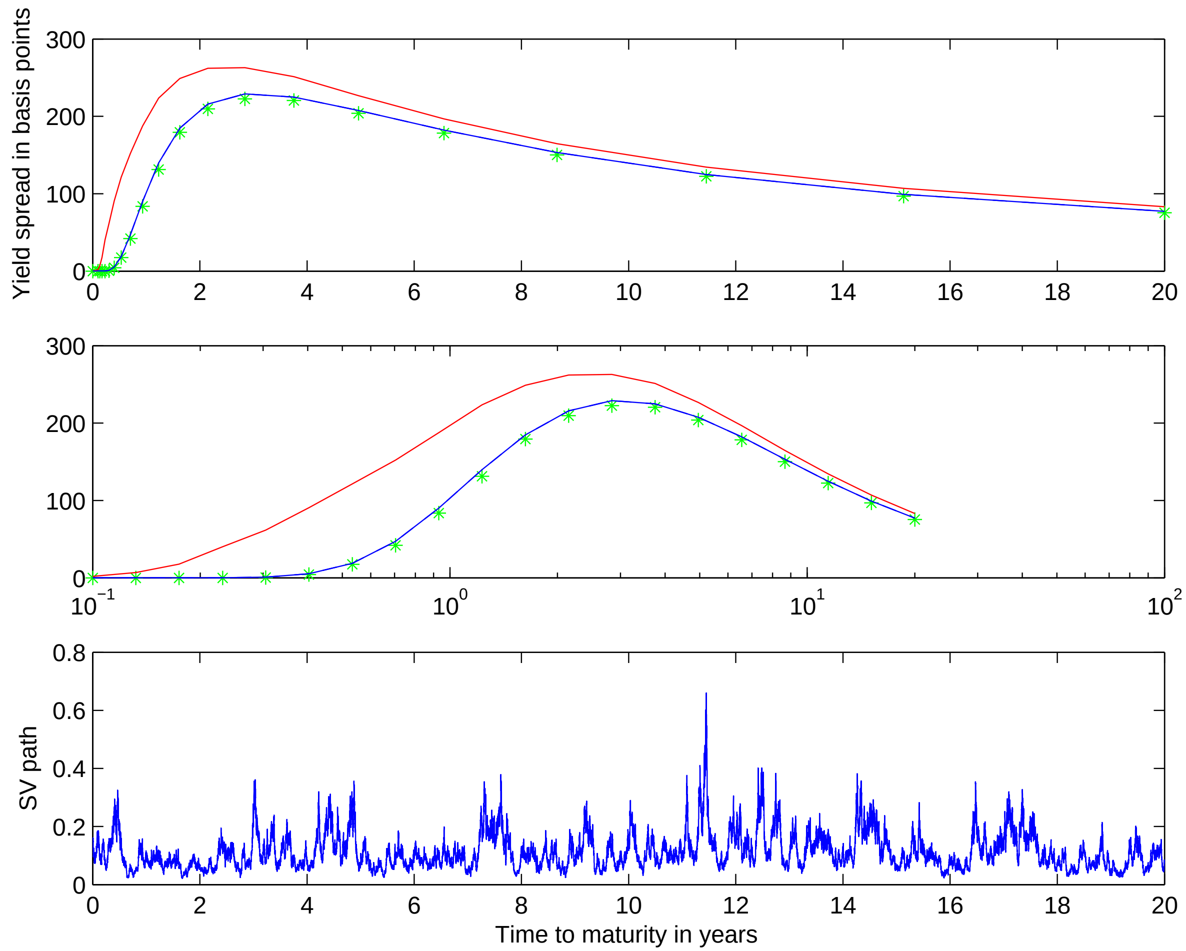


Figure 7: **Uncorrelated fast mean-reverting stochastic volatility:** $\alpha = 10$ **and** $\rho_1 = 0$**.**

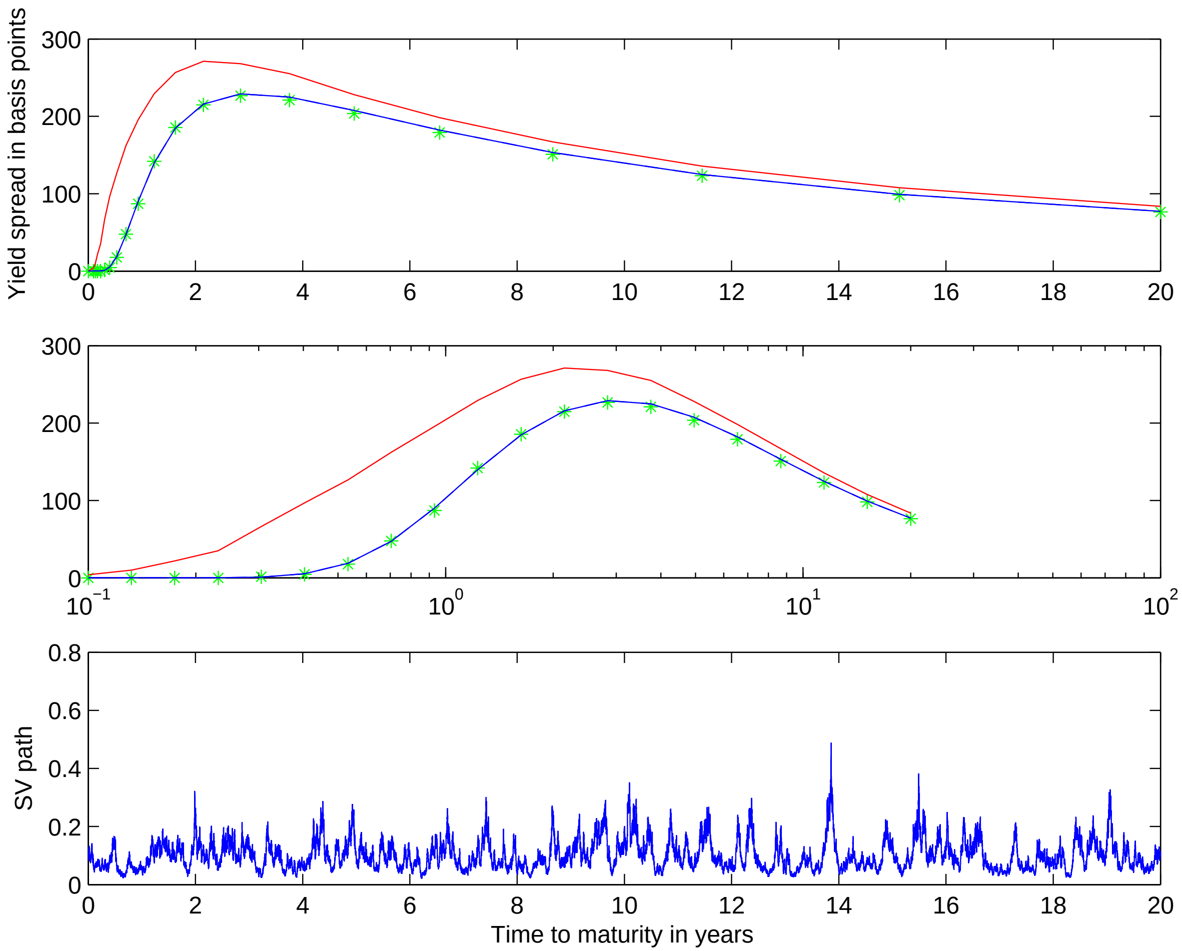


Figure 8: **Correlated fast mean-reverting stochastic volatility:** $\alpha = 10$ **and** $\rho_1 = -0.05$**.**

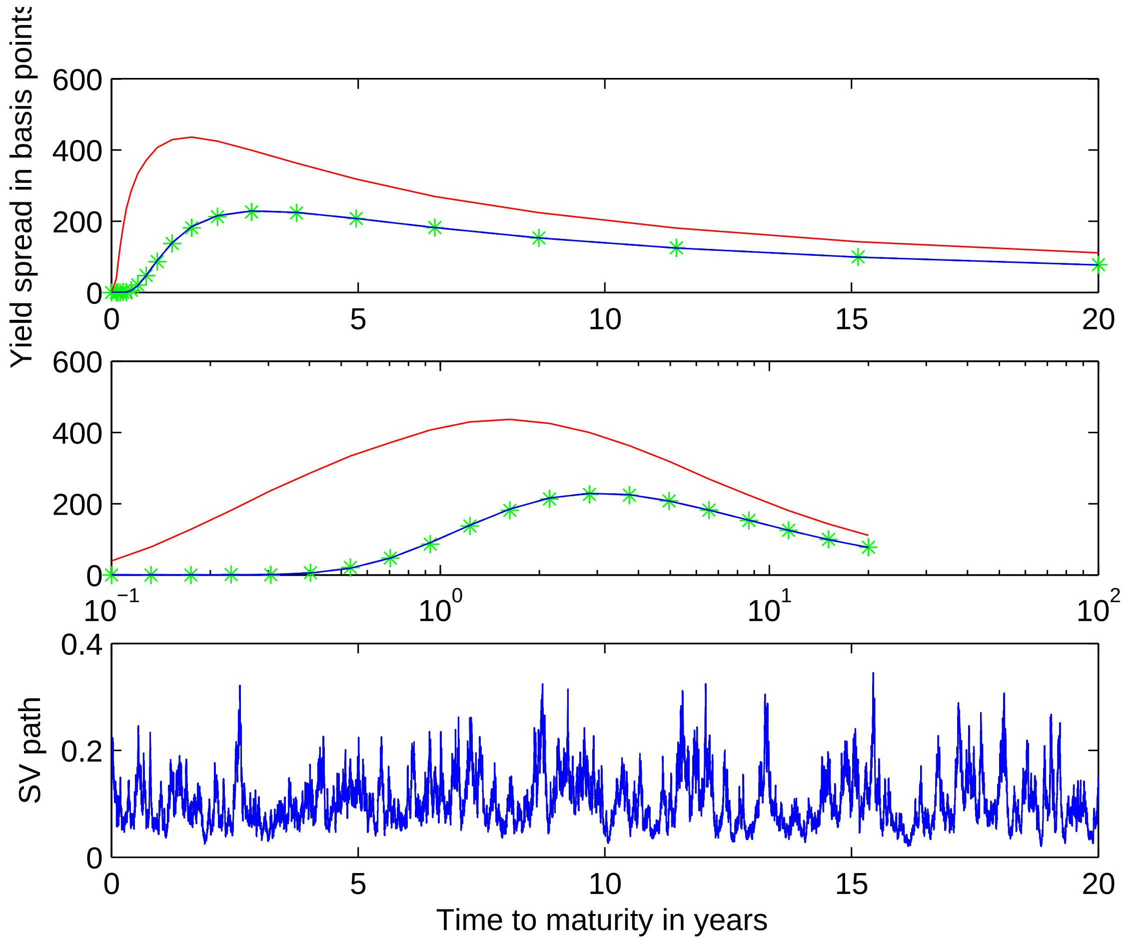


Figure 9: **Highly correlated fast mean-reverting stochastic volatility:** $\alpha = 10$ **and** $\rho_1 = -0.5$**.**

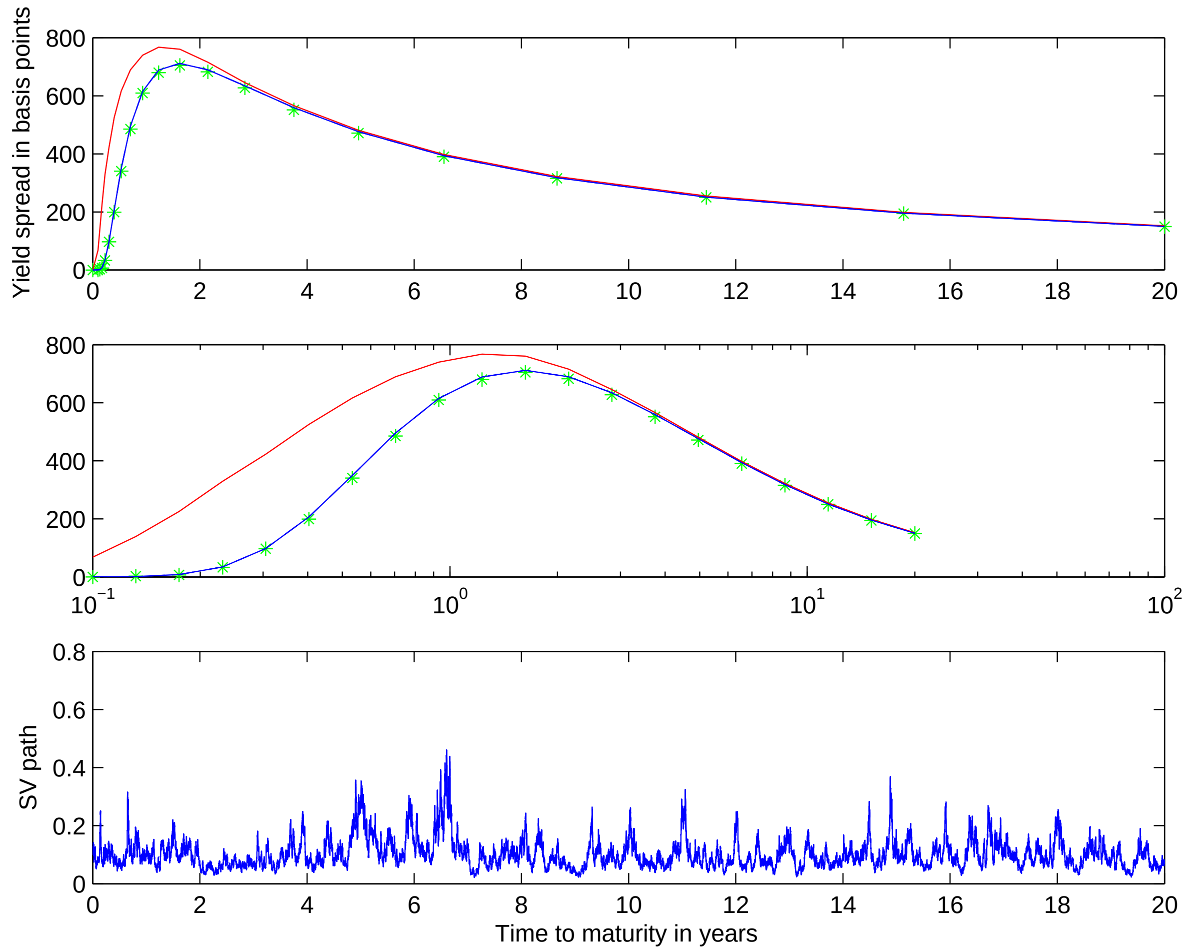


Figure 10: **High leverage correlated fast mean-reverting stochastic volatility:** $x/B = 1.2$**,** $\alpha = 10$ **and** $\rho_1 = -0.05$**.**

Barrier Options under Stochastic Volatility

$$u(t,x,y) \quad = \quad e^{-r(T-t)} \mathbb{E}^{\star} \left\{ h(X_T) \mathbf{1}_{\{\inf_{t \le s \le T} X_s > B\}} \mid X_t = x, Y_t = y \right\} ,$$

$$P^B(t,T) = \mathbf{1}_{\{\inf_{0 \le s \le t} X_s > B\}} u(t, X_t, Y_t).$$

The function $u(t,x,y)$ satisfies for $x \ge B$ the problem

$$\begin{array}{ll} \left(\frac{\partial}{\partial t} + \mathcal{L}_{X,Y} - r\right) u = 0 & \text{on } \ x > B, \, t < T \\ u(t,B) = 0 & \text{for any } \ \ t \le T \\ u(T,x) = h(x) & \text{for } \quad x > B \end{array}$$

where $\mathcal{L}_{X,Y}$ is the infinitestimal generator of the process (X,Y) under $\mathbb{P}^{\star}$.

Leading Order Term under Stochastic Volatility

In the regime α large, as in the European case, $u(t, x, y)$ is approximated by $u_0^\star(t, x)$ which solves the constant volatility problem

$$\begin{aligned}
&\mathcal{L}_{BS}(\sigma^\star)u_0^\star = 0 && \text{on } \; x > B,\, t < T \\
&u_0^\star(t, B) = 0 && \text{for any } \; t \leq T \\
&u_0^\star(T, x) = h(x) && \text{for } \; x > B
\end{aligned}$$

where $\sigma^\star$ is the corrected effective volatility.

Stochastic Volatility Correction

Define the correction $u_1^\star(t,x)$ by

$$\begin{array}{ll} \mathcal{L}_{BS}(\sigma^\star)u_1^\star = -V_3\, x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 u_0^\star}{\partial x^2}\right) & \text{on } x > B,\, t < T \\ u_1^\star(t,B) = 0 & \text{for any } \ t \le T \\ u_1^\star(T,x) = 0 & \text{for } \ x > B \end{array} .$$

Remarkably, the small parameter V_3 is the same as in the European case (calibrated to implied volatilities).

Computation of the Correction

Define

$$v_1^\star(t,x) = u_1^\star(t,x) - (T-t)V_3\, x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 u_0^\star}{\partial x^2}\right),$$

so that $v_1^\star(t,x)$ solves the simpler problem

$$\begin{aligned}
&\mathcal{L}_{BS}(\sigma^\star)v_1^\star = 0 && \text{on } \ x > B,\, t < T \\
&v_1^\star(t,B) = g(t) && \text{for any } \ t \le T \\
&v_1^\star(T,x) = 0 && \text{for } \ x > B \\
&g(t) = -V_3\,(T-t)\lim_{x\downarrow B}\left(x\tfrac{\partial}{\partial x}\left(x^2\tfrac{\partial^2 u_0^\star}{\partial x^2}\right)\right)
\end{aligned}$$

To summarize we have

$$u(t,x,y) \quad \approx \quad u_0^\star(t,x) + (T-t)V_3\, x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 u_0^\star}{\partial x^2}\right) + v_1^\star(t,x)$$

with explicit computation in the case $h(x) = 1$.

Combined Two Scales Models

$$
\begin{aligned}
dX_t &= rX_t dt + f(Y_t, Z_t) X_t \, dW_t^{(0)\star} \\
dY_t &= \left(\frac{1}{\varepsilon}(m_1 - Y_t) - \frac{\nu_1\sqrt{2}}{\sqrt{\varepsilon}} \Lambda_1(Y_t, Z_t) \right) dt + \frac{\nu_1\sqrt{2}}{\sqrt{\varepsilon}} \, dW_t^{(1)\star} \\
dZ_t &= \left(\delta(m_2 - Z_t) - \nu_2\sqrt{2\delta} \, \Lambda_2(Y_t, Z_t) \right) dt + \nu_2\sqrt{2\delta} \, dW_t^{(2)\star}
\end{aligned}
$$

Time Scales:

$$\varepsilon \ll \mathbf{1}$$

$$\mathbf{1}/\delta \gg \mathbf{1}$$

Correlations:

$$d\langle W^{(0)\star}, W^{(1)\star} \rangle_t = \rho_1 dt, \quad d\langle W^{(0)\star}, W^{(2)\star} \rangle_t = \rho_2 dt$$

Slow Factor Correction

The first correction $u_1^{(z)}(t,x)$ solves the problem

$$\begin{aligned} &\mathcal{L}_{BS}(\bar\sigma(z))u_1^{(z)} = -2\left(V_0(z)\frac{\partial u_{BS}}{\partial \sigma} + V_1(z)x\frac{\partial}{\partial x}\left(\frac{\partial u_{BS}}{\partial \sigma}\right)\right) && \text{on } \; x>B,\, t<T,\\ &u_1^{(z)}(t,B) = 0 && \text{for } \quad t\le T,\\ &u_1^{(z)}(T,x) = 0 && \text{for } \quad x>B, \end{aligned}$$

where u_{BS} is evaluated at $(t,x,\bar\sigma(z))$, and $V_0(z)$ and $V_1(z)$ are small parameters of order $\sqrt{\delta}$, functions of the model parameters, and depending on the current level z of the slow factor.

Yield Spreads Calibration

$$\begin{aligned}
r + Y(0,T) &= -\frac{1}{T}\log(u(0,x,y,z)) \\
&\approx -\frac{1}{T}\log\left(u_0(0,x) + u_{1,\varepsilon}(0,x) + u_{1,\delta}(0,x)\right) \\
&\approx -\frac{1}{T}\log\left(u_0(0,x)\right) - \frac{1}{T}\left(\frac{u_{1,\varepsilon}(0,x)}{u_0(0,x)}\right) - \frac{1}{T}\left(\frac{u_{1,\delta}(0,x)}{u_0(0,x)}\right),
\end{aligned}$$

Four parameters:

$$\sigma^\star(z), \quad (V_0, V_1), \quad V_3$$

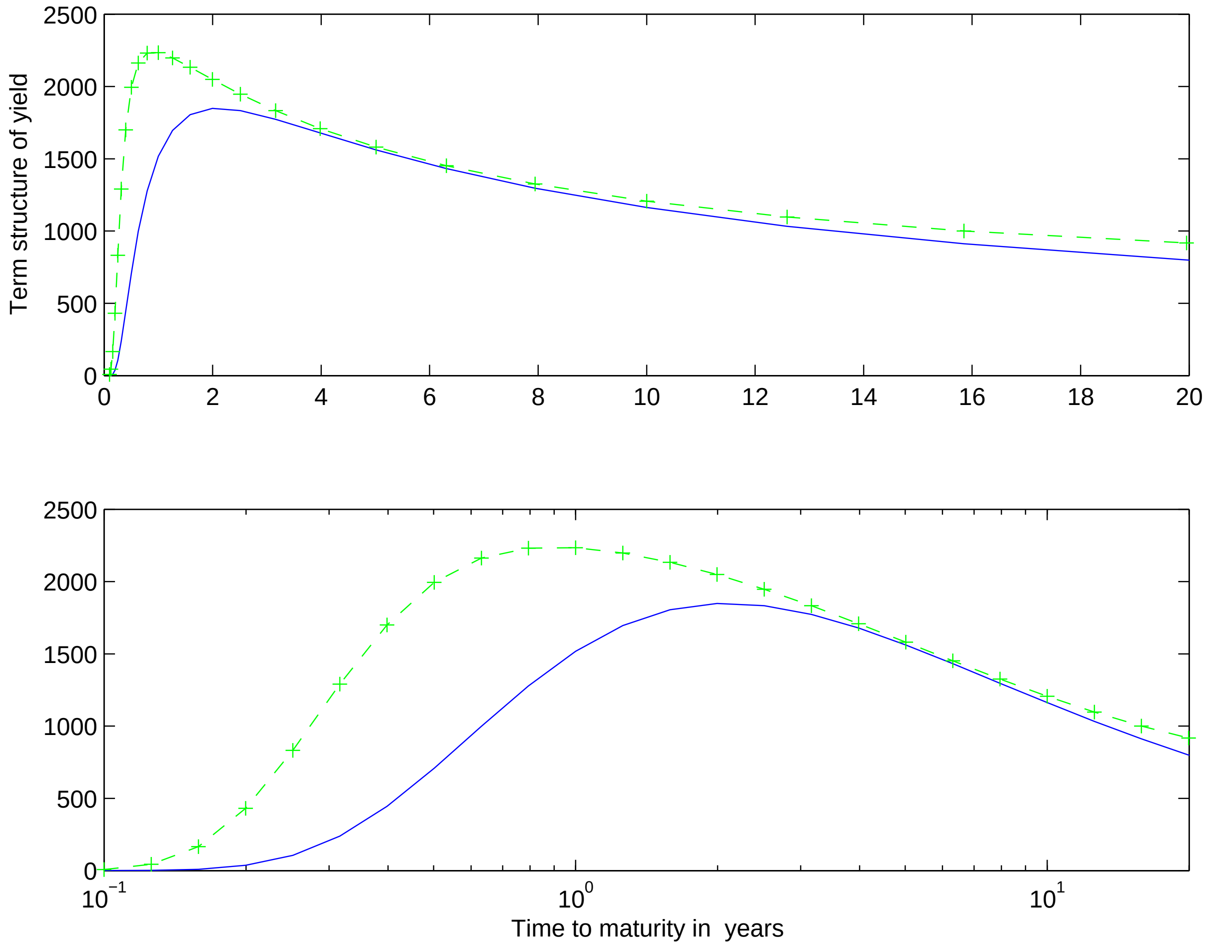


Figure 11: **The approximated yield for** $\sigma^\star = 0.12, r = 0.0$**,** $V_0 = 0.0003, V_1 = -0.0005$, $V_3 = -0.0003$, $x/B = 1.2$**.**

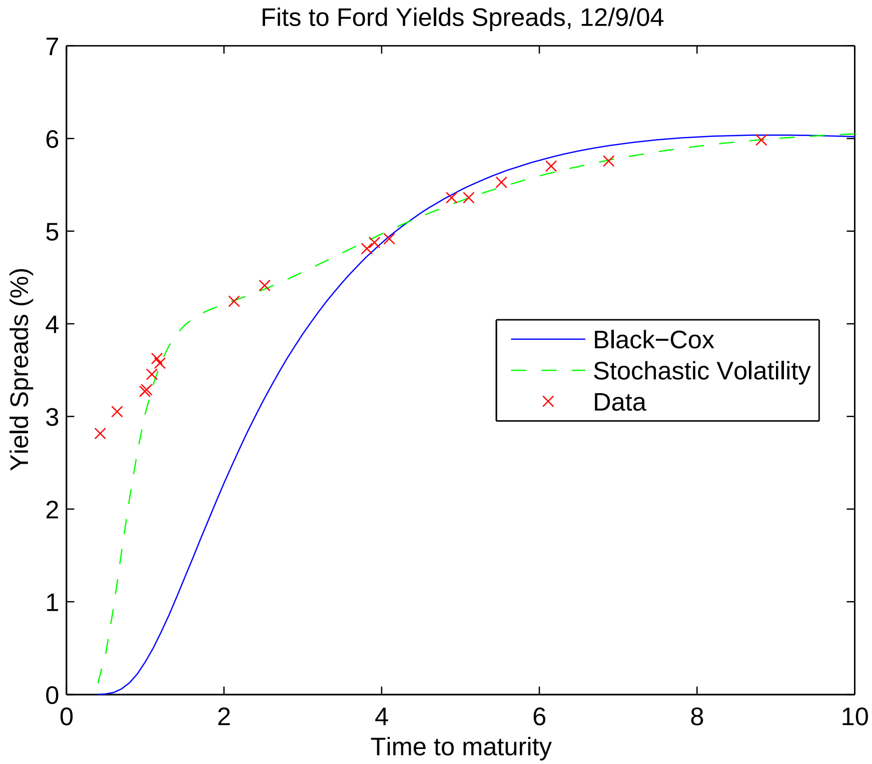
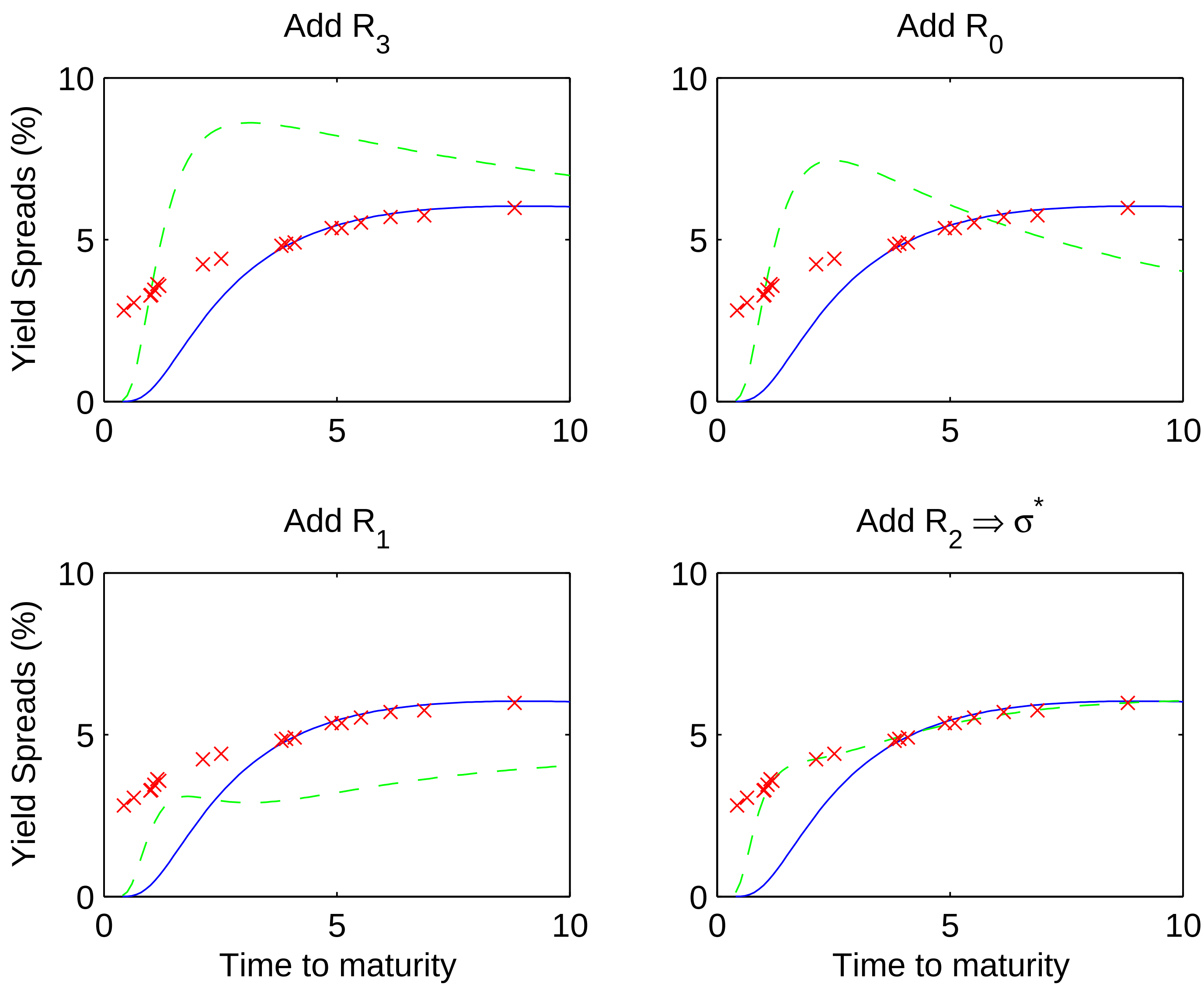


Figure 12: *Black-Cox and two-factor stochastic volatility fits to Ford yield spread data. The short rate is fixed at* $r = 0.025$. *The fitted Black-Cox parameters are* $\bar{\sigma} = 0.35$ *and* $x/B = 2.875$. *The fitted stochastic volatility parameters are* $\sigma^\star = 0.385$, *corresponding to* $R_2 = 0.0129$, $R_3 = -0.012$, $R_1 = 0.016$ *and* $R_0 = -0.008$.



Add R_3
Add R_0
Add R_1
Add $R_2 \Rightarrow \sigma^*$
Yield Spreads (%)
Yield Spreads (%)
Time to maturity
Time to maturity
0
5
10

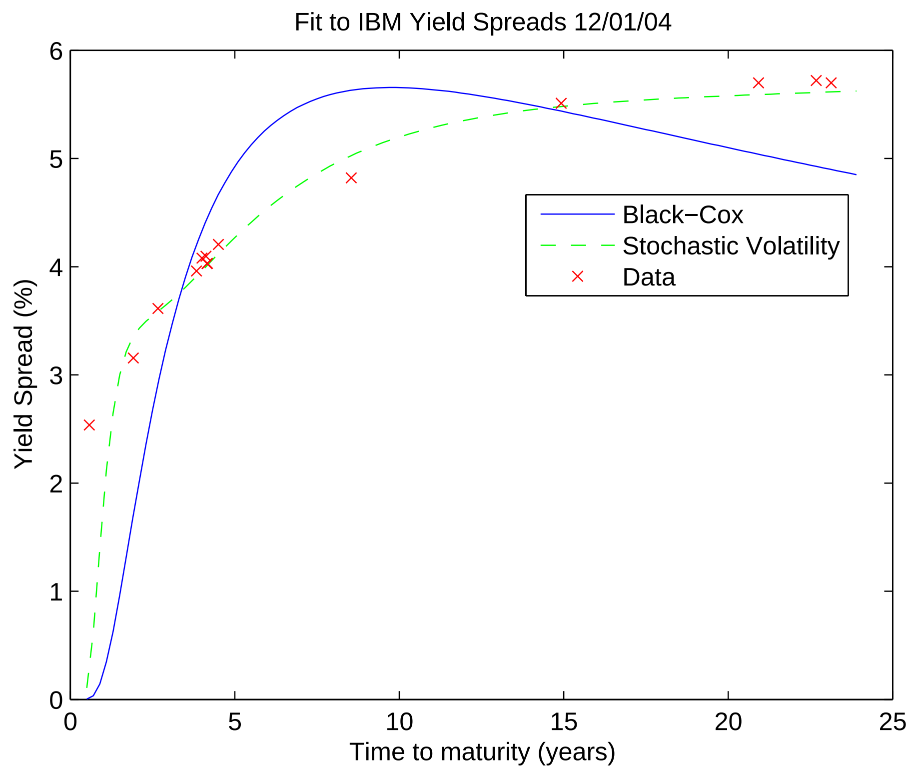


Figure 13: *Black-Cox and two-factor stochastic volatility fits to IBM yield spread data. The short rate is fixed at* $r = 0.025$*. The fitted Black-Cox parameters are* $\bar{\sigma} = 0.35$ *and* $x/B = 3$*. The fitted stochastic volatility parameters are* $\sigma^{\star} = 0.36$*, corresponding to* $R_2 = 0.00355$*,* $R_3 = -0.0112$*,* $R_1 = 0.013$ *and* $R_0 = -0.0045$*.*

Term Structure of Default Correlation

Default times:

$$\tau_i = \inf\{t \geq 0, X_{it} \leq B_i\}$$

Default probabilities:

$$p_i(T) = \mathbb{P}^\star\{\tau_i \leq T\}$$

$$p_{12}(T) = \mathbb{P}^\star\{\tau_1 \leq T, \tau_2 \leq T\}$$

Correlation coefficients:

$$R(T) = \frac{p_{12}(T) - p_1(T)p_2(T)}{\sqrt{p_1(T)(1-p_1(T))}\sqrt{p_2(T)(1-p_2(T))}}$$

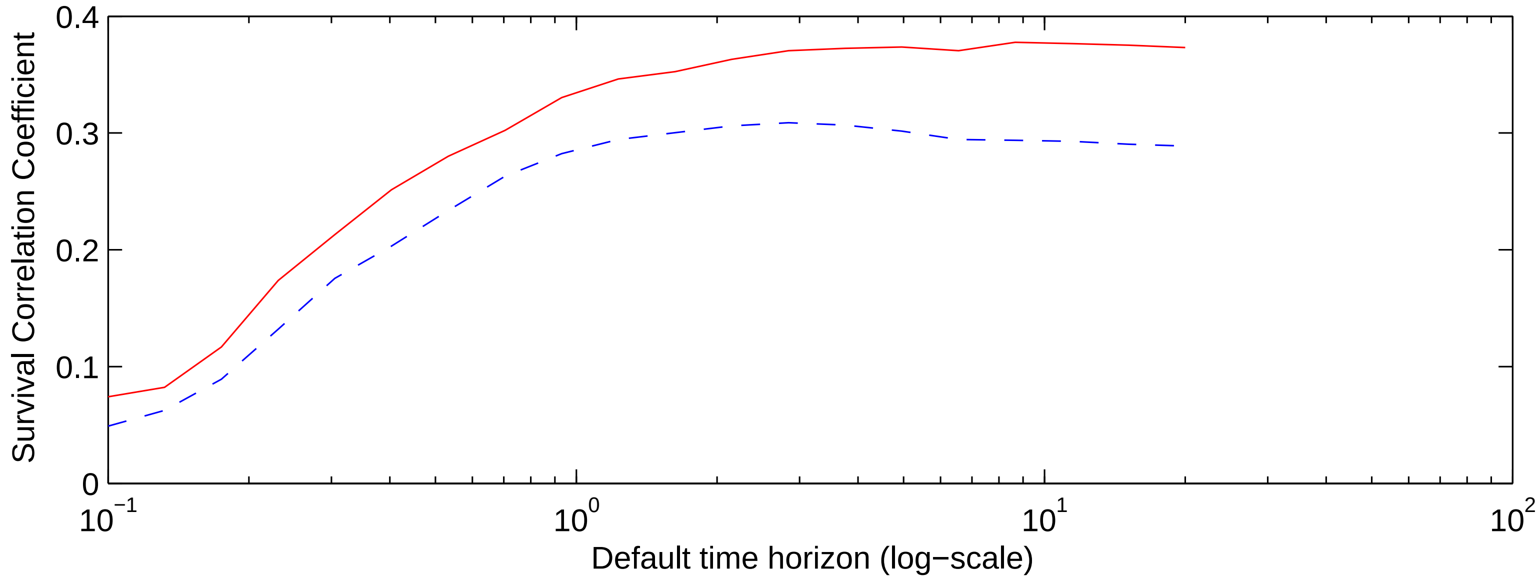


Figure 14: *The red curve in the figure shows the* **term structure of default correlation** *in the case where stochastic volatilities are driven by* **slowly mean reverting** *factors. The volatility functions f_1 and f_2 are exponentials with lower and upper cutoffs. The parameters are:* $r = 0.06, \bar{\sigma}_1 = \bar{\sigma}_1 = 0.2, m_1 = m_2 = 0, \nu_1 = \nu_2 = 1, \rho_x = 0.5, \rho_{xy} = -0.5, \rho_y = 0.5, x_1/B_1 = x_2/B_2 = 1.25,$ *and* $\alpha =$ **0.05**. *The blue curve is the corresponding correlation $R(T)$ in the constant volatility case.*

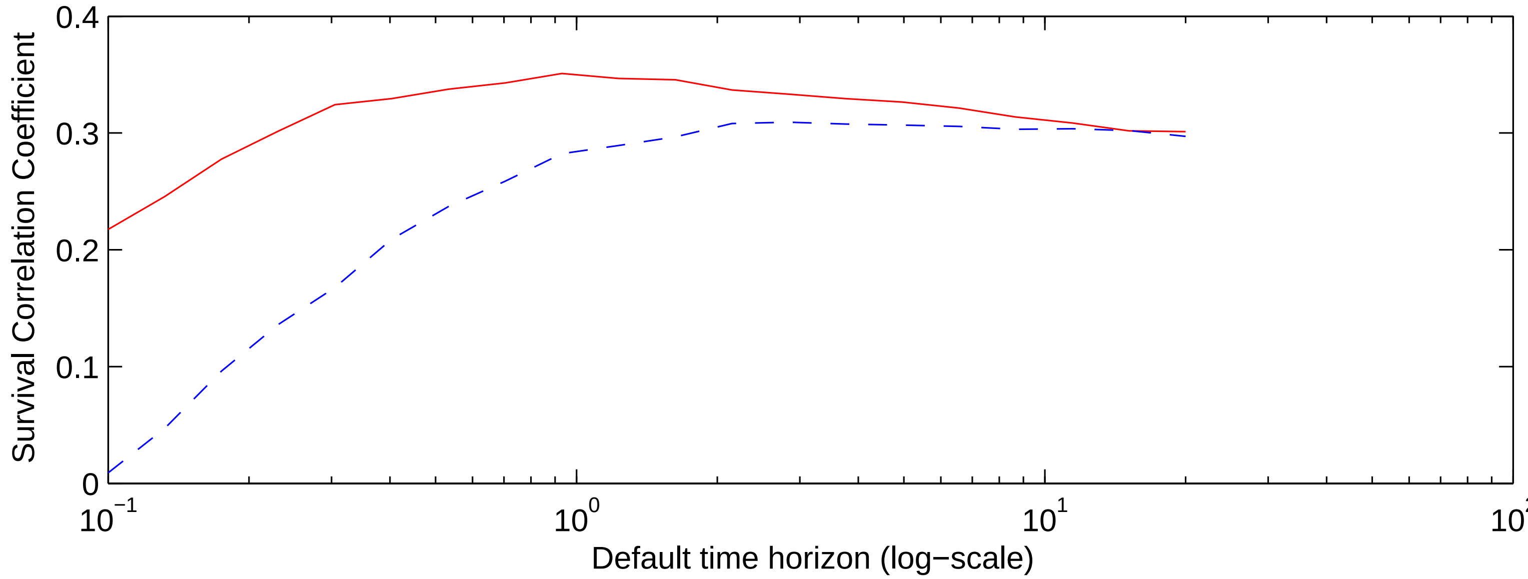


Figure 15: *The red curve in the figure shows the* **term structure of default correlation** *in the case where stochastic volatilities are driven by* **fast mean reverting** *factors. The volatility functions f_1 and f_2 are exponentials with lower and upper cutoffs. The parameters are:* $r = 0.06, \bar{\sigma}_1 = \bar{\sigma}_1 = 0.2, m_1 = m_2 = 0, \nu_1 = \nu_2 = 1, \rho_x = 0.5, \rho_{xy} = -0.5, \rho_y = 0.5, x_1/B_1 = x_2/B_2 = 1.25,$ *and* $\alpha =$ **10**. *The blue curve is the corresponding correlation $R(T)$ in the constant volatility case.*

Multiname Model Setup

Under risk neutral pricing probability:

$$
\begin{aligned}
dX_t^{(1)} &= rX_t^{(1)}dt + f_1(Y_t, Z_t)X_t^{(1)}dW_t^{(1)}, \\
dX_t^{(2)} &= rX_t^{(2)}dt + f_2(Y_t, Z_t)X_t^{(2)}dW_t^{(2)}, \\
\cdots\cdots &\cdots \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
dX_t^{(n)} &= rX_t^{(n)}dt + f_n(Y_t, Z_t)X_t^{(n)}dW_t^{(n)}, \\
dY_t &= \left[\frac{1}{\varepsilon}(m_Y - Y_t) - \frac{\nu_Y\sqrt{2}}{\sqrt{\varepsilon}}\Lambda_1(Y_t, Z_t)\right]dt + \frac{\nu_Y\sqrt{2}}{\sqrt{\varepsilon}}dW_t^{(Y)}, \\
dZ_t &= \left[\delta(m_Z - Z_t) - \nu_Z\sqrt{2\delta}\Lambda_2(Y_t, Z_t)\right]dt + \nu_Z\sqrt{2\delta}dW_t^{(Z)},
\end{aligned}
$$

where the $W_t^{(i)}$'s are **independent** standard Brownian motions and

$d\langle W^{(Y)}, W^{(i)}\rangle_t = \rho_{iY}dt$, $d\langle W^{(Z)}, W^{(i)}\rangle_t = \rho_{iZ}dt$, $d\langle W^{(Y)}, W^{(Z)}\rangle_t = \rho_{YZ}dt$.

with $\sum_{i=1}^n \rho_{iY}^2 \leq 1$ and $\sum_{i=1}^n \rho_{iZ}^2 \leq 1$.

Objective

Find the joint (risk-neutral) survival probabilities

$$\begin{aligned} u^{\varepsilon,\delta} &\equiv u^{\varepsilon,\delta}(t,\mathbf{x},y,z) \\ &\equiv I\!\!P^{\star}\left\{ \tau_t^{(1)} > T, \ldots, \tau_t^{(n)} > T \,\middle|\, \mathbf{X}_t = \mathbf{x}, Y_t = y, Z_t = z \right\}, \end{aligned}$$

where $t < T$, $\mathbf{X}_t \equiv (X_t^{(1)}, \ldots, X_t^{(n)})$, $\mathbf{x} \equiv (x_1, \ldots, x_n)$, and $\tau_t^{(i)}$ is the default time of firm i:

$$\tau_t^{(i)} = \inf\left\{ s \geq t \middle| X_s^{(i)} \leq B_i(s) \right\},$$

where $B_i(t)$ is the exogenously pre-specified default threshold at time t for firm i. Following Black and Cox (1976) we assume

$$B_i(t) = K_i \mathrm{e}^{\eta_i t},$$

with constant parameters $K_i > 0$ and $\eta_i \geq 0$.

PDE Formulation

$$\mathcal{L}^{\varepsilon,\delta} u^{\varepsilon,\delta}(t,\mathbf{x},y,z) \quad = \quad 0, \quad x_i > B_i(t), \text{ for all } i,\ t < T$$

$$\mathcal{L}^{\varepsilon,\delta} = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3$$

Boundary conditions:

$$\begin{aligned}
u^{\varepsilon,\delta}(t,B_1(t),x_2,\ldots,x_n,y,z) &= 0, \quad x_i \geq B_i(t), \text{ for } i \neq 1,\ t \leq T \\
u^{\varepsilon,\delta}(t,x_1,B_2(t),x_3,\ldots,x_n,y,z) &= 0, \quad x_i \geq B_i(t), \text{ for } i \neq 2,\ t \leq T \\
\ldots\ldots\ldots\ldots\ldots\ldots & \ldots \quad \ldots \\
u^{\varepsilon,\delta}(t,x_1,\ldots,x_{n-1},B_n(t),y,z) &= 0, \quad x_i \geq B_i(t), \text{ for } i \neq n,\ t \leq T
\end{aligned}$$

Terminal condition:

$$u^{\varepsilon,\delta}(T,x_1,x_2,\ldots,x_n,y,z) \quad = \quad 1, \quad x_i > B_i(t), \text{ for all } i$$

Expansion and Approximation

$$u^{\varepsilon,\delta} = \underbrace{\mathbf{u_0} + \sqrt{\varepsilon}\mathbf{u_{1,0}} + \sqrt{\delta}\mathbf{u_{0,1}}} + \varepsilon u_{2,0} + \sqrt{\varepsilon\delta}u_{1,1} + \delta u_{0,2} + \cdots$$

Leading Order Term $\mathbf{u_0}$:

$$\begin{aligned} \langle \mathcal{L}_2 \rangle u_0 &= 0, \quad x_i > B_i(t), \text{ for all } i,\, t < T \\ u_0(t, x_1, \cdots, B_i(t), \cdots, x_n) &= 0, \quad x_j \geq B_j(t), \text{ for } j \neq i,\, t \leq T \\ u_0(T, x_1, x_2, \ldots, x_n) &= 1, \quad x_i > B_i(t), \text{ for all } i \end{aligned}$$

$$\begin{aligned} \langle \mathcal{L}_2 \rangle &= \frac{\partial}{\partial t} + \sum_{i=1}^{n} \left(\frac{1}{2} \sigma_i(z)^2 x_i^2 \frac{\partial^2}{\partial x_i^2} + r x_i \frac{\partial}{\partial x_i} \right) \\ \sigma_i(z) &= \sqrt{\langle f_i^2(\cdot, z) \rangle}, \quad \langle \cdot \rangle : \textit{average w.r.t. } \mathcal{N}(m_Y, \nu_Y^2) \end{aligned}$$

A Formula for $\mathbf{u_0}$

$$u_0 = \prod_{i=1}^{n} Q_i \equiv \prod_{i=1}^{n} \left[\mathrm{N}\left(d_{2(i)}^{+}\right) - \left(\frac{x_i}{B_i(t)}\right)^{p_i} \mathrm{N}\left(d_{2(i)}^{-}\right) \right],$$

where $\mathrm{N}(\cdot)$ is the standard normal distribution function,

$$\begin{aligned} d_{2(i)}^{\pm} &= \frac{\pm \ln \frac{x_i}{B_i(t)} + \left(r - \eta_i - \frac{\sigma_i^2(z)}{2}\right)(T-t)}{\sigma_i(z)\sqrt{T-t}}, \\ \sigma_i(z) &= \sqrt{\langle f_i^2(\cdot, z)\rangle}, \\ p_i &= 1 - \frac{2(r-\eta_i)}{\sigma_i^2(z)}. \end{aligned}$$

Correction Term $\sqrt{\varepsilon}\,\mathbf{u_{1,0}}$

$$\begin{aligned}
\langle \mathcal{L}_2 \rangle u_{1,0} &= \mathcal{A} u_0, \quad x_i > B_i(t), \ \text{for all } i, \ t < T \\
u_{1,0}(t, x_1, \cdots, B_i(t), \ldots, x_n) &= 0, \quad x_j \geq B_j(t), \ \text{for } j \neq i, \ t \leq T \\
u_{1,0}(T, x_1, x_2, \ldots, x_n) &= 0, \quad x_i > B_i(t), \ \text{for all } i
\end{aligned}$$

$$\mathcal{A} = \langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle =$$

$$\frac{\nu_Y}{\sqrt{2}} \left[\sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{iY} \left\langle f_i(\cdot, z) \frac{\partial \phi_j}{\partial y} \right\rangle x_i \frac{\partial}{\partial x_i} \left(x_j^2 \frac{\partial^2}{\partial x_j^2} \right) - \sum_{j=1}^{n} \left\langle \Lambda_1(\cdot, z) \frac{\partial \phi_j}{\partial y} \right\rangle x_j^2 \frac{\partial^2}{\partial x_j^2} \right]$$

where the ϕ_i's are given by the Poisson equations w.r.t. y:

$$\mathcal{L}_0 \phi_i(y, z) = f_i^2(y, z) - \langle f_i^2(\cdot, z) \rangle.$$

Then use $u_0(t, x_1, \cdots, x_n) = \prod_{i=1}^{n} Q_i(t, x_i)$.

Correction Term $\sqrt{\delta}\, \mathbf{u_{0,1}}$

$$
\begin{aligned}
\langle \mathcal{L}_2 \rangle u_{0,1} &= -\langle \mathcal{M}_1 \rangle u_0, \quad x_i > B_i(t), \text{for all } i,\ t < T \\
u_{0,1}(t, x_1, \cdots, B_i(t), \ldots, x_n) &= 0, \quad x_j \geq B_j(t),\ \text{for } j \neq i,\ t \leq T \\
u_{0,1}(T, x_1, x_2, \ldots, x_n) &= 0, \quad x_i > B_i(t),\ \text{for all } i.
\end{aligned}
$$

$$
\langle \mathcal{M}_1 \rangle = \nu_Z \sqrt{2} \left[\sum_{i=1}^{n} \rho_{iZ} \langle f(\cdot, z) \rangle x_i \frac{\partial^2}{\partial x_i \partial z} - \langle \Lambda_2(\cdot, z) \rangle \frac{\partial}{\partial z} \right] =
$$

$$
\nu_Z \sqrt{2} \left[\sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{iZ} \langle f(\cdot, z) \rangle \sigma_j'(z) x_i \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial \sigma_j} \right) - \langle \Lambda_2(\cdot, z) \rangle \sum_{i=1}^{n} \sigma_i'(z) \frac{\partial}{\partial \sigma_i} \right]
$$

Then use $u_0(t, x_1, \cdots, x_n) = \prod_{i=1}^{n} Q_i(t, x_i)$.

Homogeneous Portfolio Case

$$u_0(t, x, \cdots, x) = \prod_{i=1}^{n} Q_i(t, x) = Q(t, x)^n \equiv q^n$$

$$\begin{aligned}
\sqrt{\varepsilon}\, u_{1,0} &= n\left(R_1^{(2)} w_1^{(2)}(t, x) + R_1^{(3)} w_1^{(3)}(t, x)\right) q^{n-1} \\
&\quad + n(n-1) R_{12}^{(3)} w_{12}^{(3)}(t, x, x) q^{n-2} \\
\sqrt{\delta}\, u_{0,1} &= n\left(R_1^{(0)} w_1^{(0)}(t, x) + R_1^{(1)} w_1^{(1)}(t, x)\right) q^{n-1} \\
&\quad + n(n-1) R_{12}^{(1)} w_{12}^{(1)}(t, x, x) q^{n-2}
\end{aligned}$$

Joint survival probabilities:

$$S_n \approx \tilde{u} \equiv u_0 + \sqrt{\epsilon}\, u_{1,0} + \sqrt{\delta}\, u_{0,1} = q^n + Anq^{n-1} + Bn(n-1)q^{n-2}$$

Loss Distribution

For N names perfectly symmetric, if L is the number of defaults by time T, then

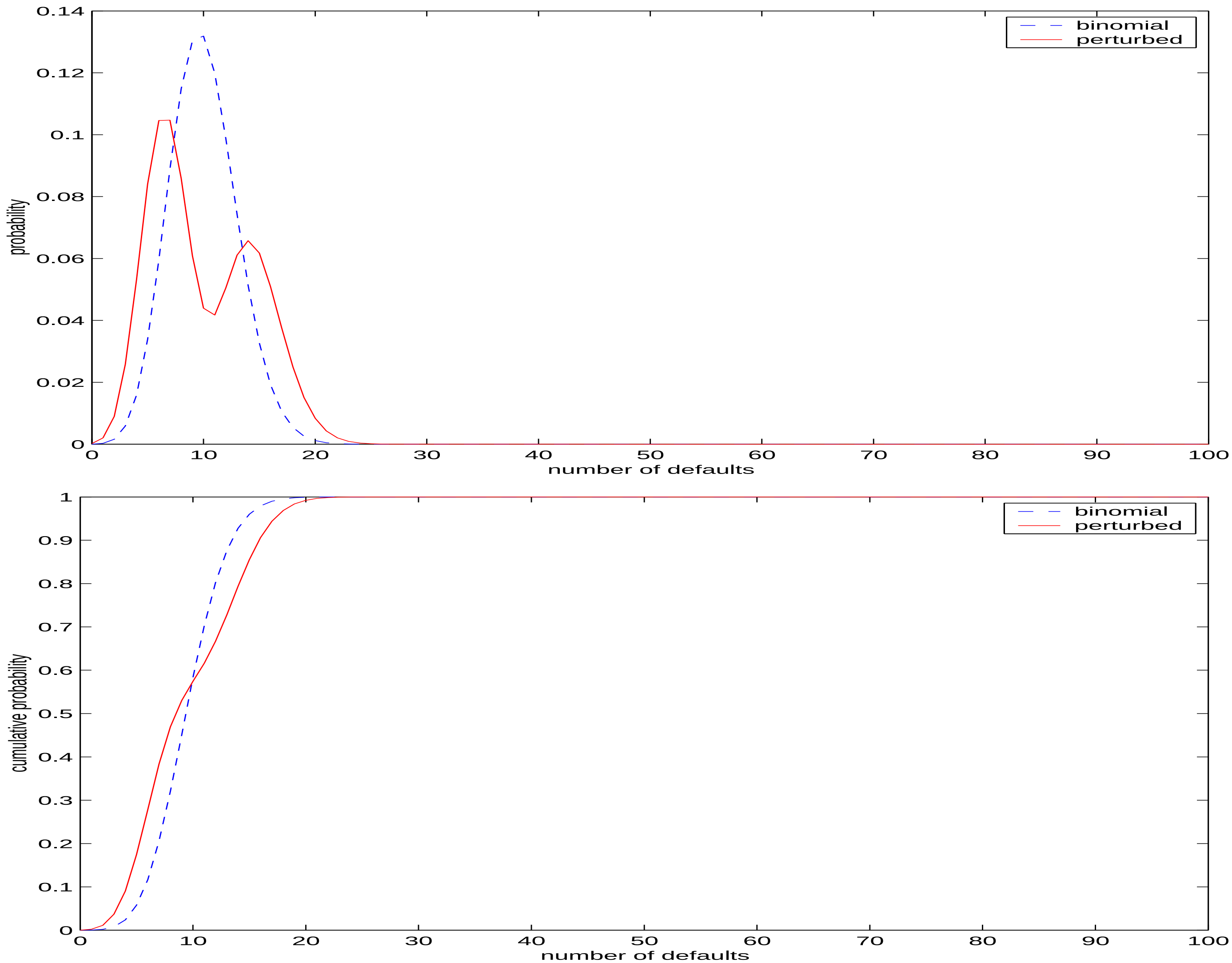
$$
\begin{aligned}
\mathbb{P}^\star(L = k) &= \binom{N}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j S_{N+j-k} \\
&\approx \binom{N}{k} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} q^{N-i} \\
&\quad + A \binom{N}{k} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} (N-i) q^{N-i-1} \\
&\quad + B \binom{N}{k} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} (N-i)(N-i-1) q^{N-i-2} \\
&\equiv I_0 + I_1 + I_2
\end{aligned}
$$

Loss Distribution Formulas

$$\mathbb{P}^{\star}(L = k) \quad \approx \quad I_0 + I_1 + I_2$$

with

$$\begin{aligned} I_0 &= \binom{N}{k}(1-q)^k q^{N-k} \\ I_1 &= A\left[\frac{N-k}{q} - \frac{k}{1-q}\right] I_0 \\ I_2 &= B\left[\frac{(N-k)(N-k-1)}{q^2} - \frac{2k(N-k)}{q(1-q)} + \frac{k(k-1)}{(1-q)^2}\right] I_0 \end{aligned}$$



$$N = 100, \quad q = 0.9, \quad A = 0.00, \quad B = 0.0006$$