Cohomological Crystallographic Defects in Cellular Automata

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Cellular Automata

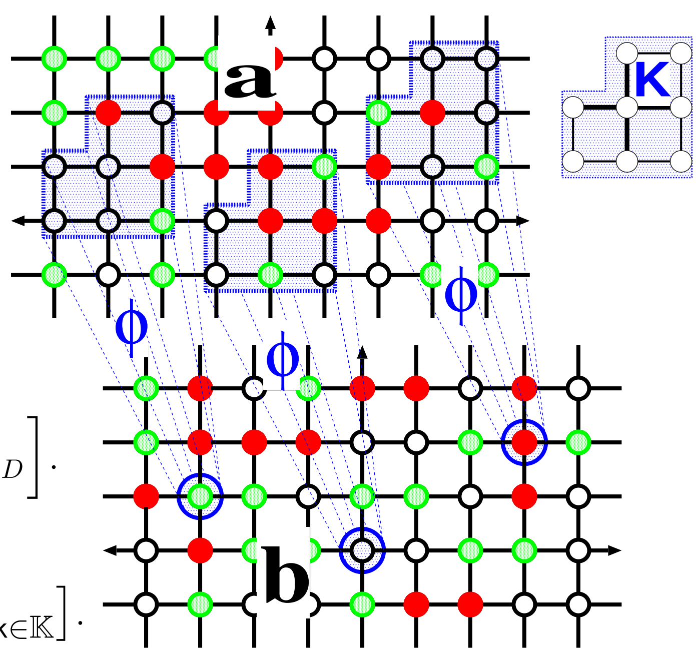
CA are the 'discrete analog' of partial differential equations. They are *spatially distributed* dynamical systems whose dynamics are driven by *local interactions* governed by *translationally equivariant* rules.

- **Space** is a lattice $\mathbb{Z}^D$ (for $D \geq 1$).
- The **local state** at each point in the lattice is an element of a finite alphabet, e.g. $\mathcal{A} := \{0, 1\}$.
- The **global state** is a $\mathbb{Z}^D$-indexed *configuration* $\mathbf{a} : \mathbb{Z}^D \longrightarrow \mathcal{A}$.
  The space of such configurations is denoted $\mathcal{A}^{\mathbb{Z}^D}$.
  A generic element of $\mathcal{A}^{\mathbb{Z}^D}$ will be denoted by $\mathbf{a} := \left[a_{\mathsf{z}}|_{\mathsf{z} \in \mathbb{Z}^D} \right]$.
- The evolution is governed by a map $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$, computed by applying a '**local rule**' ϕ at every point in space.

Neighbourhood:

$\mathbb{K} \subset \mathbb{Z}^D$ (finite set)

Local rule: ϕ: $\mathcal{A}^{\mathbb{K}} \longrightarrow \mathcal{A}$



Let $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$, $\quad \mathbf{a} := \left[a_{\mathsf{z}}|_{\mathsf{z} \in \mathbb{Z}^D} \right]$.

$\forall \mathsf{z} \in \mathbb{Z}^D$, let $b_{\mathsf{z}} := \phi\left[a_{(\mathsf{k}+\mathsf{z})}|_{\mathsf{k} \in \mathbb{K}} \right]$.

This defines new configuration $\mathbf{b} := \left[b_{\mathsf{z}}|_{\mathsf{z} \in \mathbb{Z}^D} \right]$.

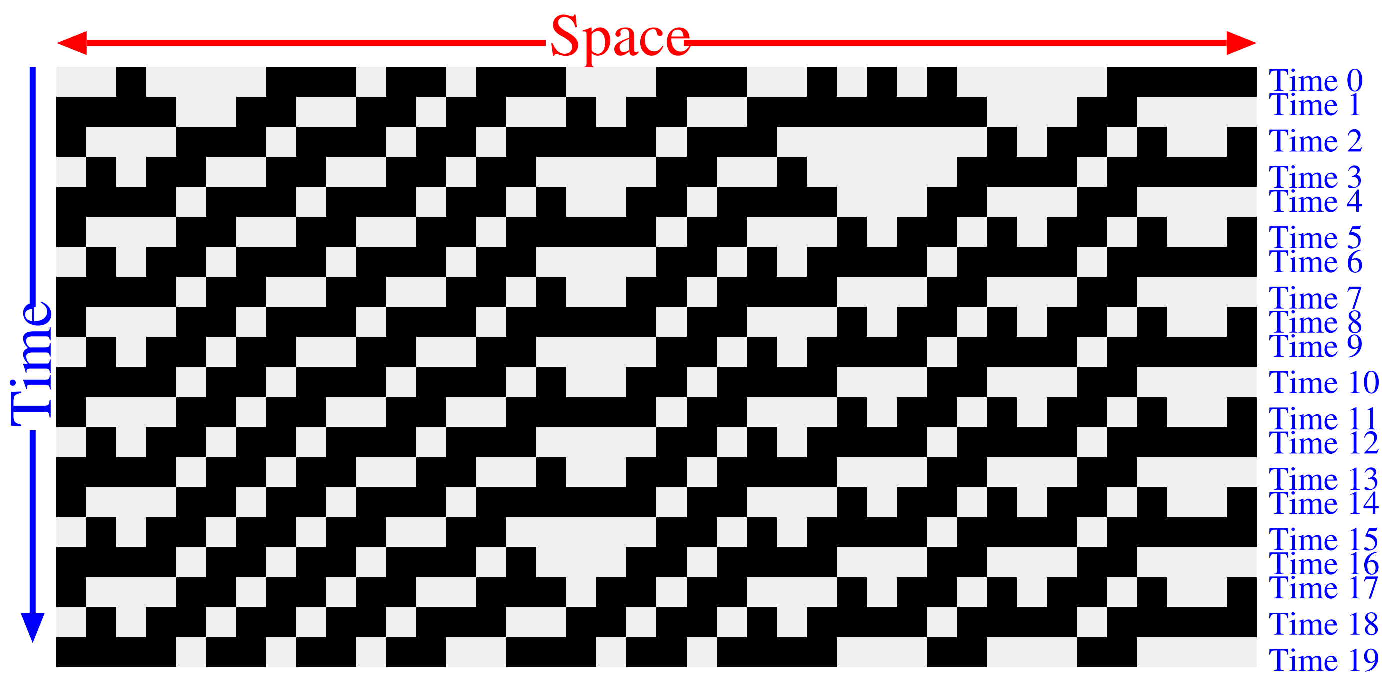
The CA **induced by** ϕ is function Φ: $\mathcal{A}^{\mathbb{Z}^D} \circlearrowleft$ defined: $\Phi(\mathbf{a}) := \mathbf{b}$.

Example: Elementary Cellular Automaton #62

Let $D := 1$, $\mathbb{K} := \{-1, 0, 1\}$, and $\mathcal{A} := \{0, 1\}$.

Define $\phi_{62} : \{0,1\}^{\{-1,0,1\}} \longrightarrow \{0,1\}$ by:

$$\begin{array}{rclrcl} \phi_{62}(0,0,1) &=& 1; & \phi_{62}(0,0,0) &=& 0; \\ \phi_{62}(0,1,0) &=& 1; & \phi_{62}(1,1,0) &=& 0; \\ \phi_{62}(0,1,1) &=& 1; & \phi_{62}(1,1,1) &=& 0; \\ \phi_{62}(1,0,0) &=& 1; & & & \\ \phi_{62}(1,0,1) &=& 1. & & & \end{array}$$



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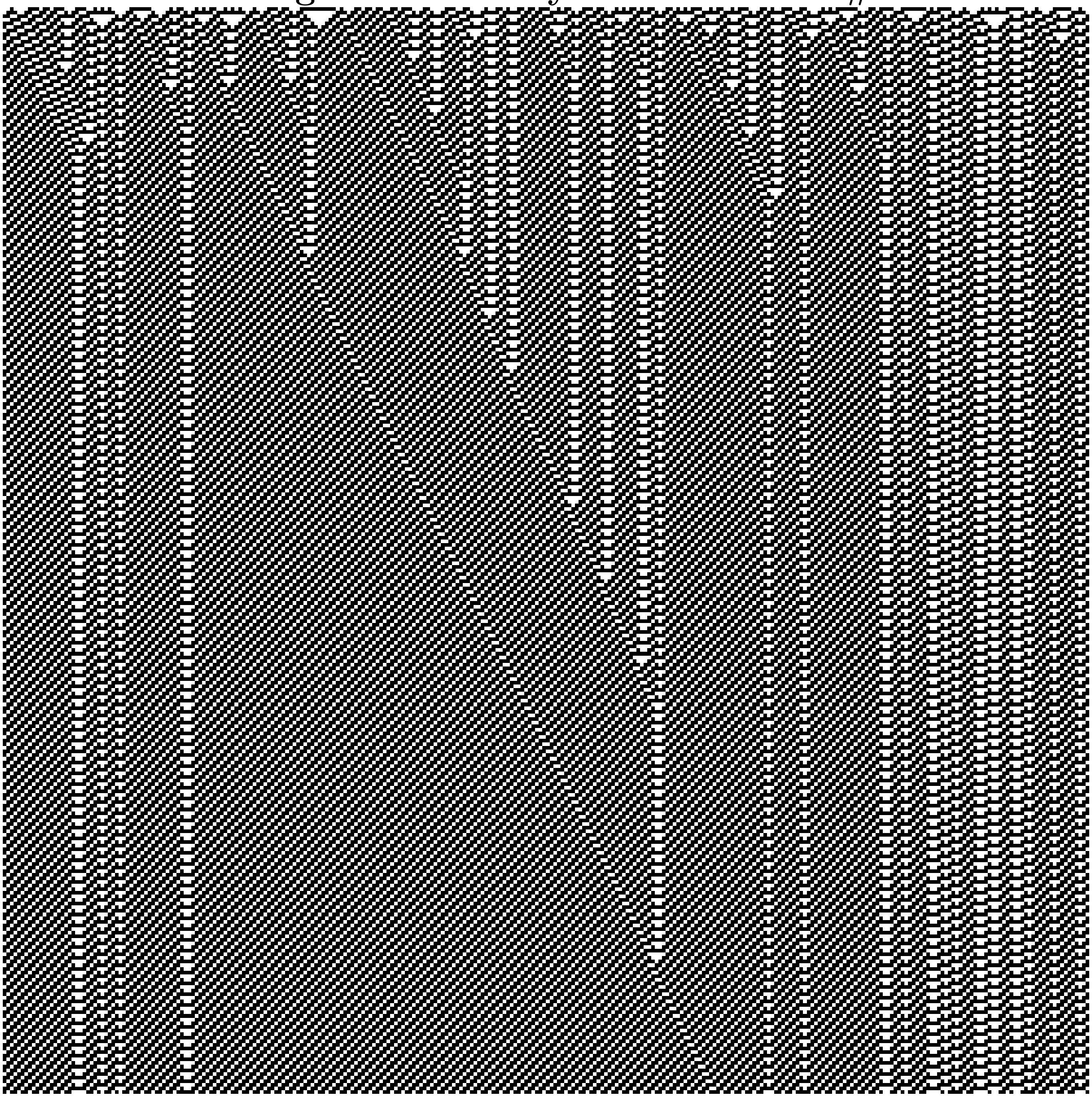
Such a nearest-neighbour CA on $\{0,1\}^{\mathbb{Z}}$ is called an **Elementary Cellular Automaton**. Each ECA is described by an 8-bit binary number (i.e. a number between 0 and 255) as follows:

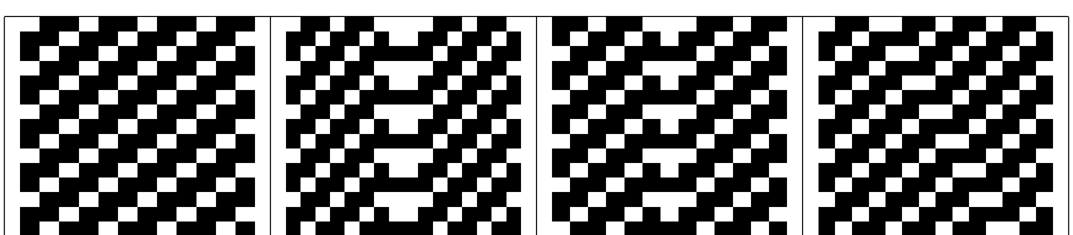
If $N = n_0 + 2n_1 + 2^2 n_2 + 2^3 n_3 + 2^4 n_4 + 2^5 n_5 + 2^6 n_6 + 2^7 n_7 \in [0...255]$

then $\phi_N(a_0, a_1, a_2) := n_k$, where $k := a_0 + 2a_1 + 4a_2 \in [0...7]$.

For example, the CA here is ECA#62, because $2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 62$.

Emergent Defect Dynamics in ECA#62





$(*)$	$(\boldsymbol{\alpha})$	$(\boldsymbol{\beta})$	$(\boldsymbol{\gamma})$

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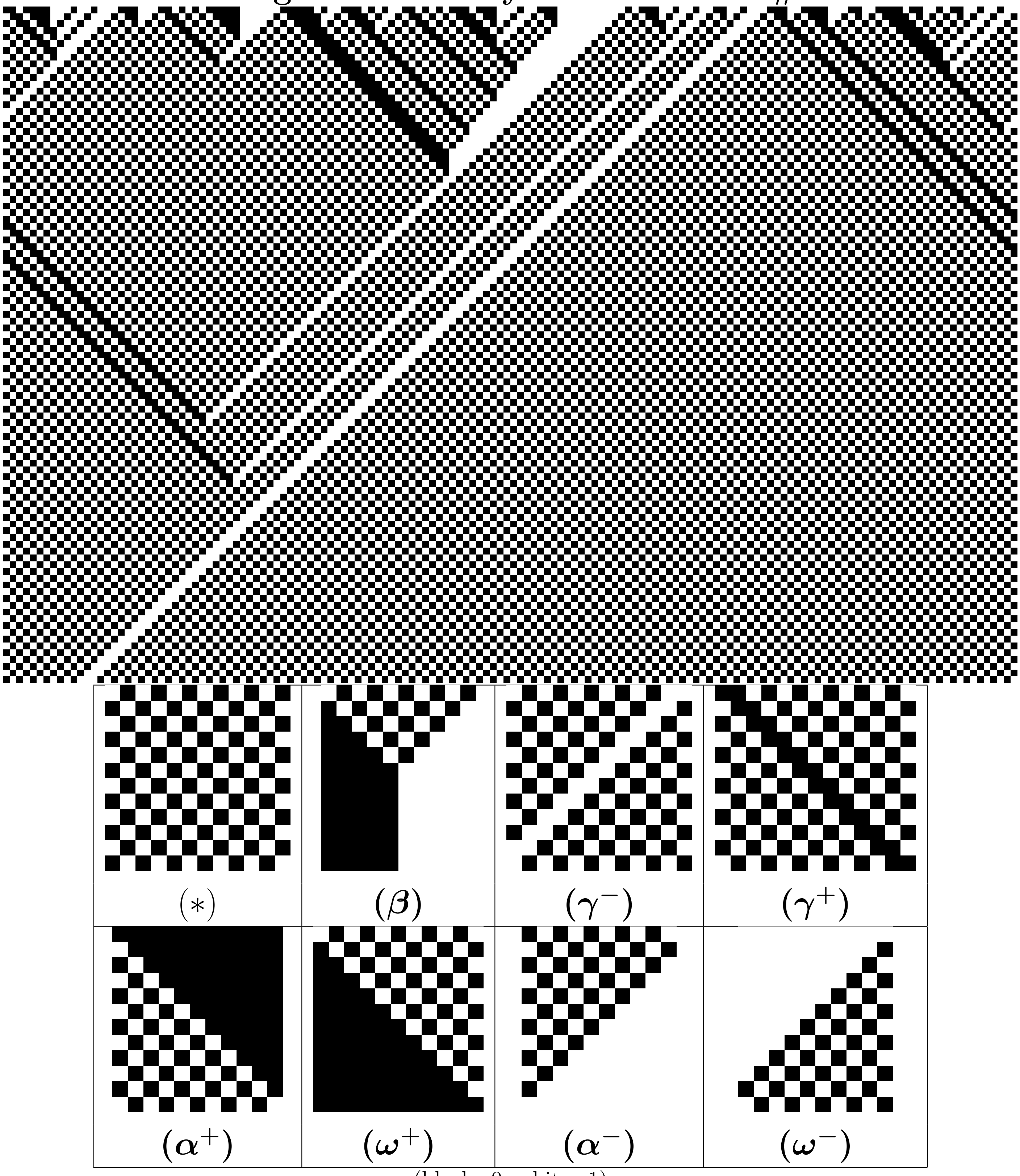
Emergent Defect Dynamics in ECA#184



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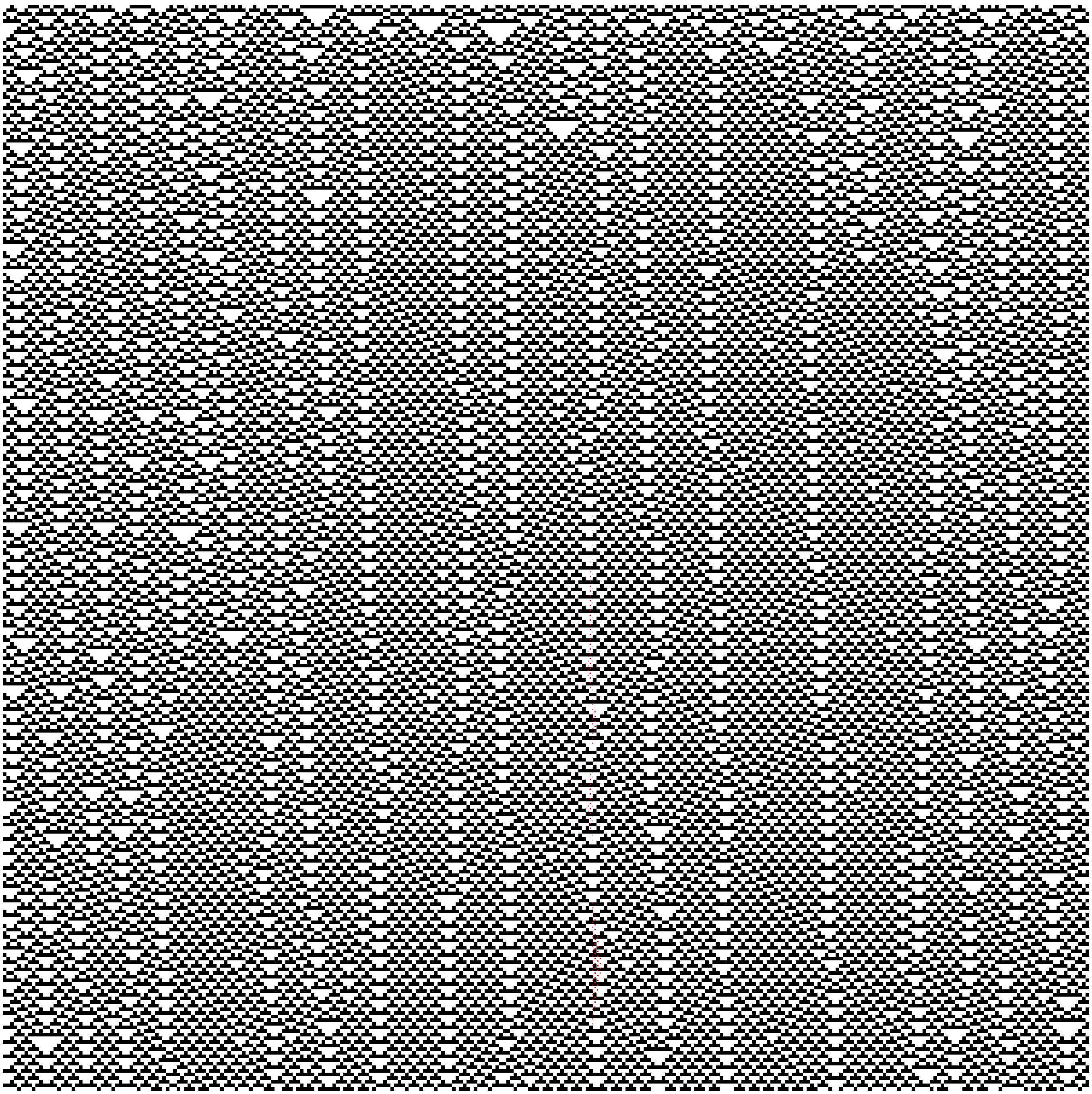
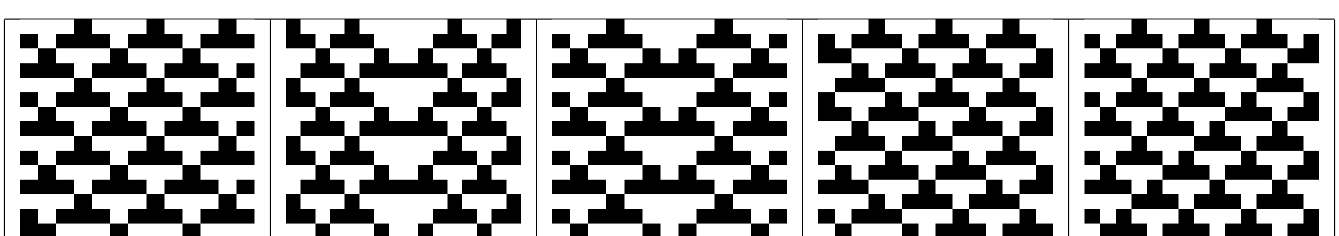
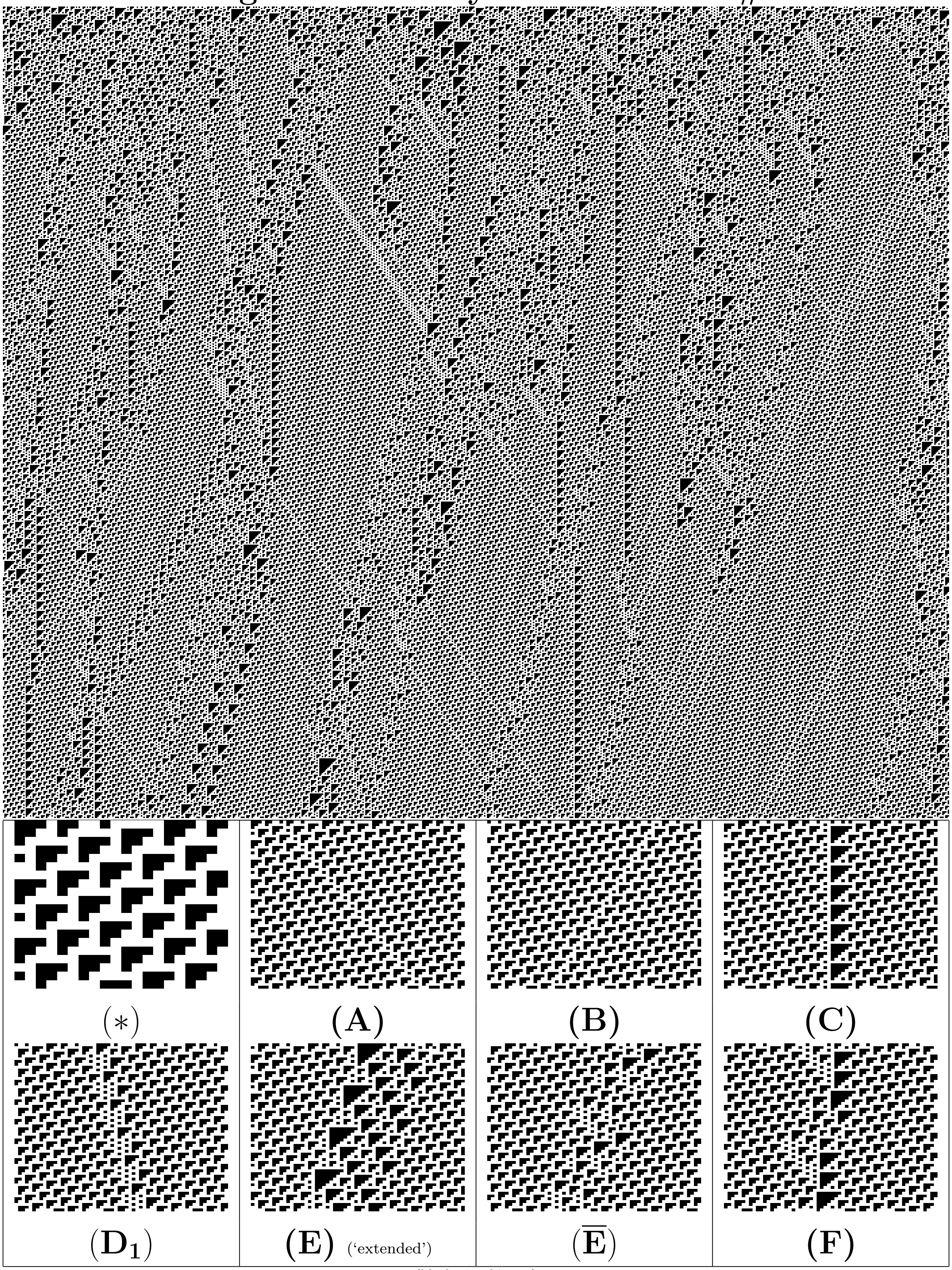
Emergent Defect Dynamics in ECA#54





$(*)$	$(\boldsymbol{\alpha})$	$(\boldsymbol{\beta})$	$(\boldsymbol{\gamma^+})$	$(\boldsymbol{\gamma^-})$

Emergent Defect Dynamics in ECA#110



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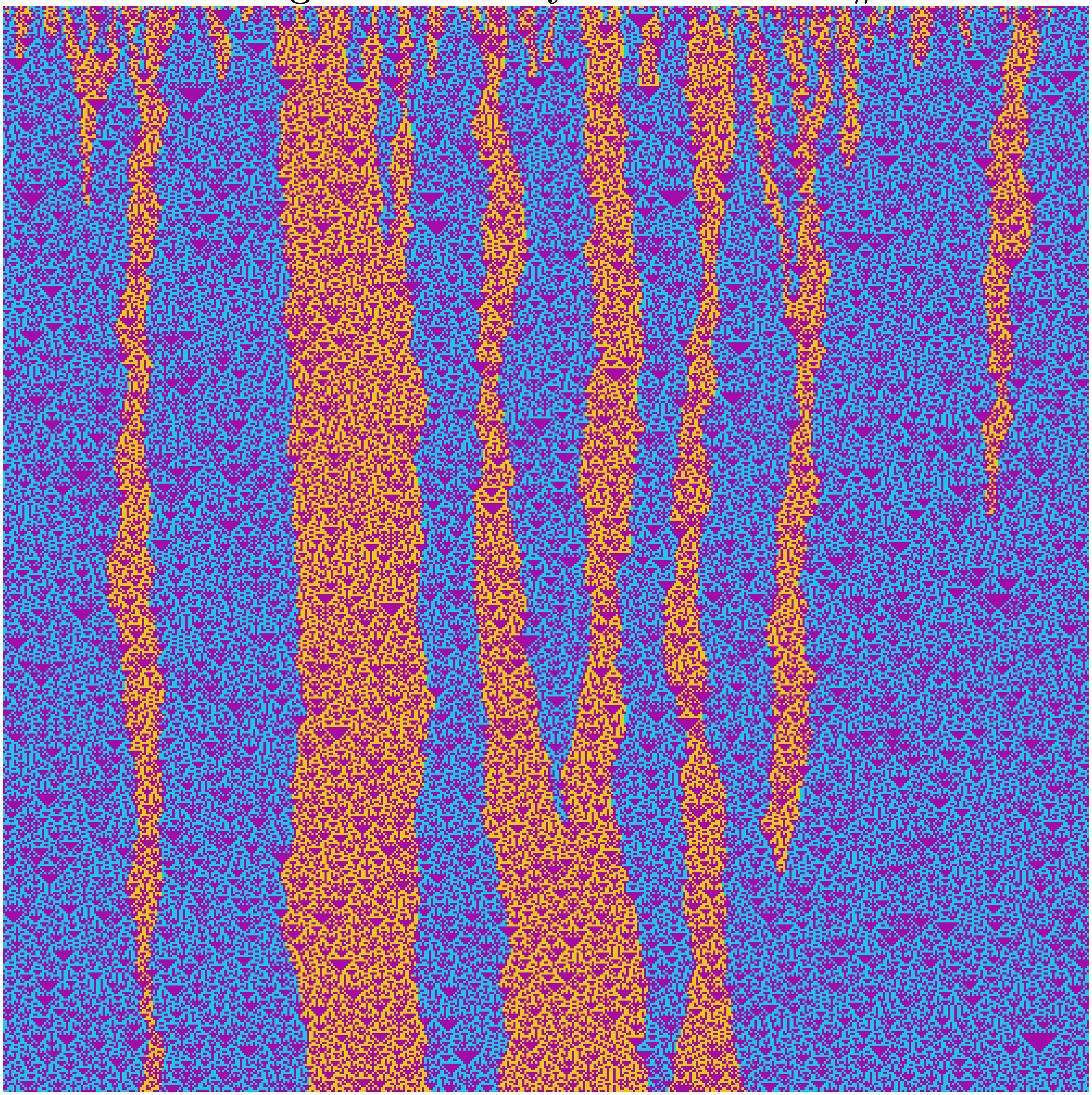
Emergent Defect Dynamics in ECA#18

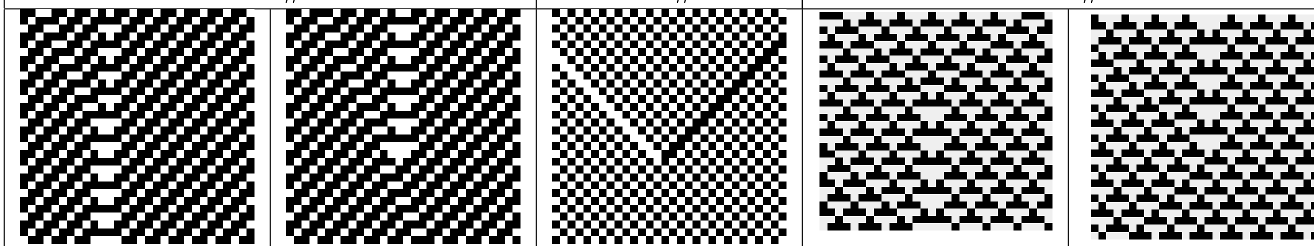


Invariant sofic subshift: (1) ⇆ (0) ⇆ (0) (the *Odd Shift*).

Defects are 'phase slips':

$$[\ldots \underbrace{00\ 01\ 00\ 01\ 01}_{\text{orange}}\ \ \underbrace{00\ 00\ 00\ 00\ 00\ 00\ 00\ 00\ 00}_{\text{even \# of zeroes}}\ \ \underbrace{10\ 00\ 10\ 00\ 00\ 10}_{\text{blue}} \ldots].$$

Defect Particle 'Chemistry'

ECA #62		ECA #184	ECA #54	
				
$\gamma + \beta \to \alpha$	$\gamma + \alpha \to \gamma$	$\gamma^+ + \gamma^- \to \emptyset$	$\gamma^+ + \gamma^- \to \beta$	$\gamma^+ + \beta \to \gamma^-$

Empirical Work: • P. Grassberger [1983, 1984].

• Steven Wolfram [1983-2005]. (Mainly ECA #110).

• S. Wolfram and Doug Lind [1986]. (Classified defects of ECA #110).

• N. Boccara, J. Naser, M. Rogers [1991]. (ECAs 18, 54, 62, 184).

• James Crutchfield and James Hanson's 'Computational Mechanics' [1992-2001]. (Also Cosma Shalizi, Wim Hordijk, Melanie Mitchell).

• Harold V. McIntosh [1999, 2000].

Theoretical Work: • Doug Lind [1984] conjectured:

(i) *Defects in* ECA#18 *perform random walks.*

(ii) *Defect density decays to zero through annihilations. Thus,* ECA#18 *converges 'in measure' to the 'odd' sofic shift* $\textcircled{1} \leftrightarrows \textcircled{0} \leftrightarrows \textcircled{0}$.

• Kari Eloranta [1993-1995] proved Lind's conjecture **(i)**; studied quasirandom defect motion in 'partially permutive' CA.

• Petr Kůrka and Alejandro Maass [2000, 2002] studied CA convergence to limit sets through 'defect annihilation'. Kůrka [2003] proved Lind's conjecture **(ii)**.

• S. Wolfram and Matthew Cook [2002, 2004]: ECA #110 *is computationally universal* (used 'defect physics' to engineer universal computer).

Questions:

- *Is there an 'algebraic structure' governing defect 'chemistry'?*
- *Why do defects 'persist' over time instead of disappearing? Is this related to aforementioned 'algebraic structure'?*
- *What is the 'kinematics' by which defects propagate through space?*

Formalism: Fix $D \in \mathbb{N}$. For any $r > 0$, let $\mathbb{B}(r) := [-r...r]^D \subset \mathbb{Z}^D$. Fix $r > 0$. Let $\mathfrak{A}_{(r)} \subset \mathcal{A}^{\mathbb{B}(r)}$ be a set of of **admissible r-blocks**.

The **subshift of finite type (SFT)** determined by $\mathfrak{A}_{(r)}$ is the set

$$\mathfrak{A} \quad := \quad \left\{ \mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D} \; ; \; \mathbf{a}_{\mathsf{z}+\mathbb{B}(r)} \in \mathfrak{A}_{(r)}, \; \forall \; \mathsf{z} \in \mathbb{Z}^D \right\}$$

For any $R > 0$, let $\mathfrak{A}_{(R)}$ be the projection of $\mathfrak{A}$ to $\mathcal{A}^{\mathbb{B}(R)}$.

If $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$ and $\mathsf{z} \in \mathbb{Z}^D$, then $\mathbf{a}$ is **defective** at z if $\mathbf{a}_{\mathsf{z}+\mathbb{B}(r)} \not\in \mathfrak{A}_{(r)}$. The **defect set** of $\mathbf{a}$ is the set $\mathbb{D}(\mathbf{a})$ of all such z.

Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA. We say $\mathfrak{A}$ is **Φ-invariant** if $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$.

Empirically, if $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$ has defects, then so does $\Phi(\mathbf{a})$.

We say $\mathbf{a}$ is **finitely defective** if, $\forall R > 0$, $\exists \, \mathsf{z} \in \mathbb{Z}^D$ with $\mathbf{a}_{\mathbb{B}(\mathsf{z},R)} \in \mathfrak{A}_{(R)}$.

Idea: $\mathbf{a}$ may have infinitely large defects, but $\mathbf{a}$ also has infinitely large 'nondefective' regions. Let $\widetilde{\mathfrak{A}} := \{$finitely defective $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}\}$. $(\mathfrak{A} \subset \widetilde{\mathfrak{A}})$

Lemma: *If $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$, then $\Phi(\widetilde{\mathfrak{A}}) \subseteq \widetilde{\mathfrak{A}}$.*

Also, if $\mathbf{a} \in \widetilde{\mathfrak{A}}$ and $\mathbf{a}' = \Phi(\mathbf{a})$, then the defects in $\mathbf{a}'$ are 'close' to corresponding defects in $\mathbf{a}$. □

The Fine Print: To extend the definition of 'defect' to other subshifts (not of finite type), it is necessary to introduce a 'detection range' $R > 0$. We must then talk about 'defects of range R'.

Domain Boundaries

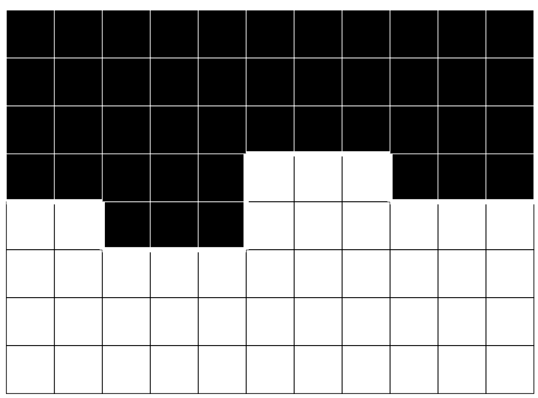
Let $\mathbb{G}(\mathbf{a}) := \{\mathsf{z} \in \mathbb{Z}^D\ ;\ \mathbf{a}$ is not defective at $\mathsf{z}\}$. Let $\mathbf{G}(\mathbf{a}) \subset \mathbb{R}^D$ be the union of all unit cubes whose corner vertices are all in $\mathbb{G}(\mathbf{a})$.

The defect in $\mathbf{a}$ is a **domain boundary*** if $\mathbf{G}(\mathbf{a})$ is disconnected.

Examples: (a) If $D = 1$, then all defects are domain boundaries.

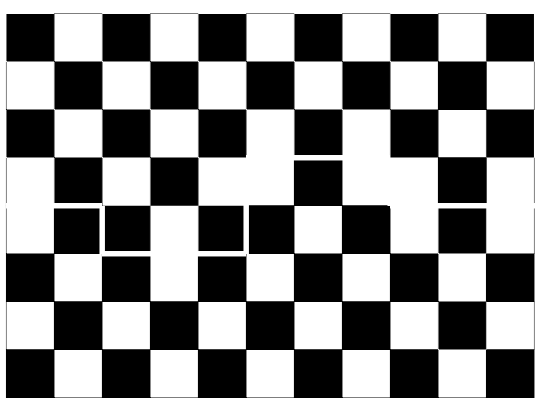
(b) (*Monochromatic*) Let $\mathcal{A} := \{\blacksquare, \square\}$. Let $\mathfrak{Mo} \subset \mathcal{A}^{\mathbb{Z}^2}$ be SFT such that no $\blacksquare$ can be adjacent to a $\square$.

The following configuration has a domain boundary defect:



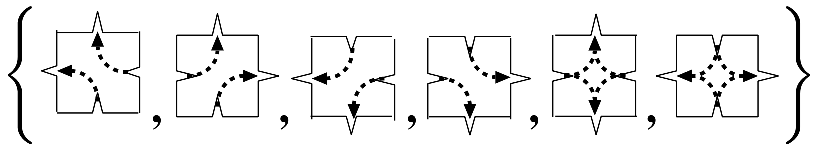
(c) (*Checkerboard*) Let $\mathcal{A} := \{\blacksquare, \square\}$. Let $\mathfrak{Ch} \subset \mathcal{A}^{\mathbb{Z}^2}$ be SFT where no $\blacksquare$ can be adjacent to a $\blacksquare$, and no $\square$ can be adjacent to a $\square$.

The following configuration has a domain boundary defect:



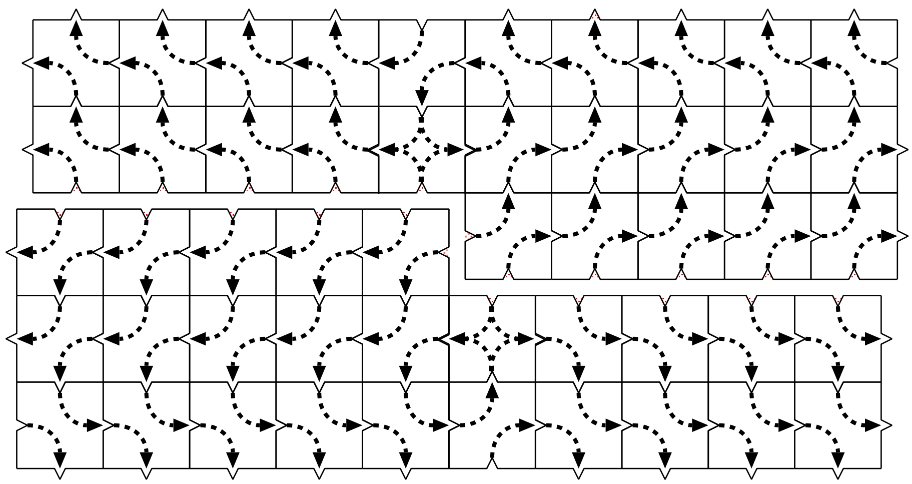
(∗) If we considering a defect of range $R > 0$, then technically this is a *domain boundary of range* R.

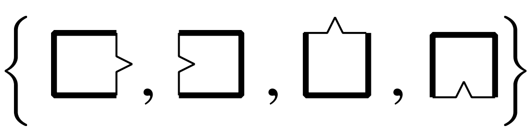
Domain Boundaries

(d) (*Square ice*) Let $\mathcal{I} :=$ .

Let $\mathfrak{Ice} \subset \mathcal{I}^{\mathbb{Z}^2}$ be the SFT defined by obvious edge-matching conditions.

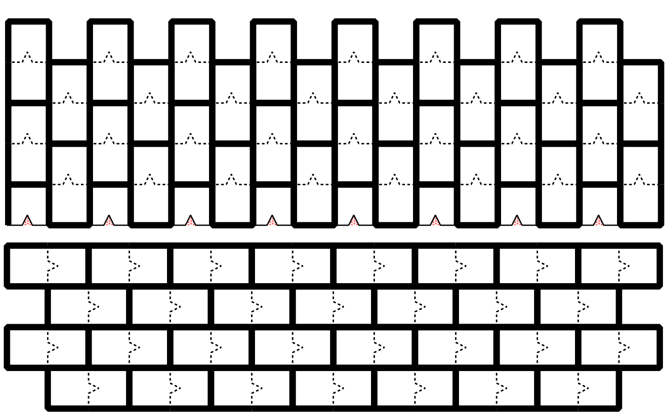
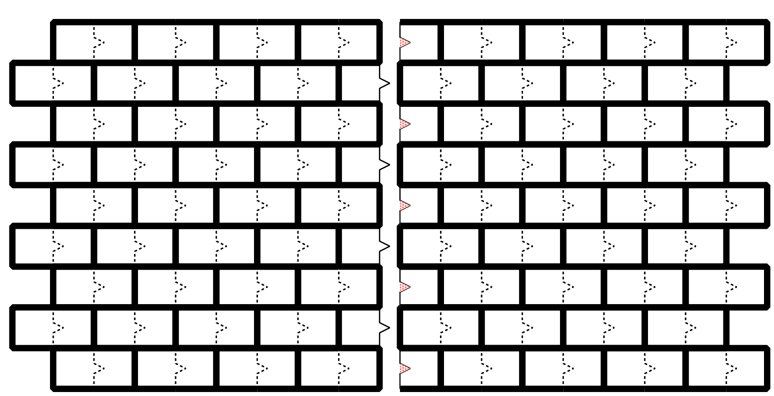
The following configuration has a domain boundary defect:



(e) (*Domino Tiling*) Let $\mathcal{D} :=$ .

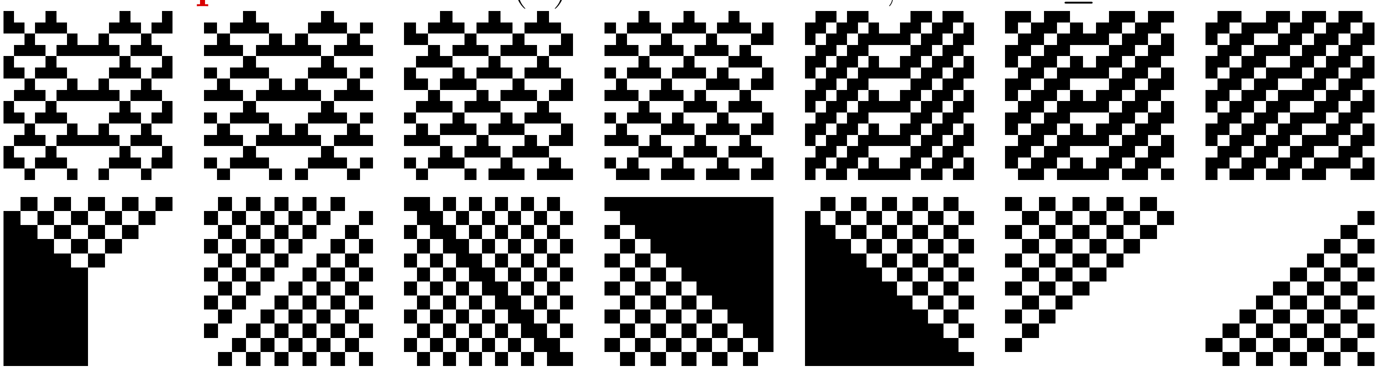
Let $\mathfrak{Dom} \subset \mathcal{D}^{\mathbb{Z}^2}$ be the SFT defined by obvious edge-matching conditions.

The following configurations have domain boundary defects:

Persistent Defects

Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA, with $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$. Let $\mathbf{a} \in \widetilde{\mathfrak{A}}$. The defect in $\mathbf{a}$ is **Φ-persistent** if $\Phi^t(\mathbf{a})$ also has a defect, for all $t \geq 0$.

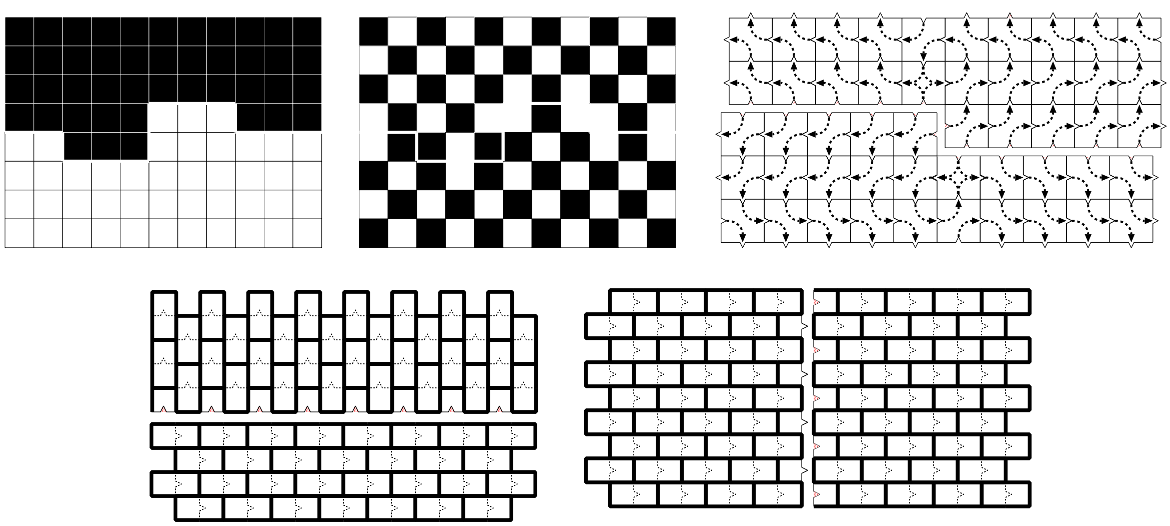


Question: These defects seem to be persistent. Are they? Why?

Essential Defects

A defect is **essential** if it can't be removed through a local change. That is, $\forall\ R > 0$, if $\mathbf{a}' \in \mathcal{A}^{\mathbb{Z}^D}$ is obtained by modifying $\mathbf{a}$ in an R-neighbourhood of defect, then $\mathbf{a}'$ is also defective.

Proposition: *If* $\Phi : \mathfrak{A} \longrightarrow \mathfrak{A}$ *is bijective* (e.g. if $\mathfrak{A} \subseteq \mathsf{Fix}[\Phi]$ or $\mathfrak{A} \subseteq \mathsf{Fix}[\Phi^p]$ or $\mathfrak{A} \subseteq \mathsf{Fix}[\Phi^p \circ \sigma^q]$), *then any essential defect is* Φ*-persistent.* □



Question: These defects to be seem essential. Are they? Why?

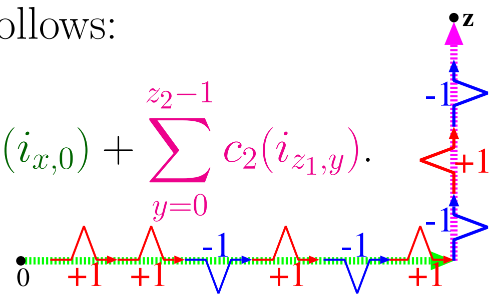
Cocycles

Let $\mathfrak{A} \subseteq \mathcal{A}^{\mathbb{Z}^D}$ be a subshift. Let $(\mathcal{G}, \cdot)$ be a (discrete) group. A $\mathcal{G}$-valued **cocycle** is continuous function $C : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ satisfying **cocycle equation:**

$$C(\mathsf{y}+\mathsf{z}, \mathbf{a}) \quad = \quad C(\mathsf{y}, \sigma^{\mathsf{z}}(\mathbf{a})) \cdot C(\mathsf{z}, \mathbf{a}), \qquad \forall\, \mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D} \text{ and } \forall\, \mathsf{y}, \mathsf{z} \in \mathbb{Z}^D.$$

Examples: (a) Let $\mathfrak{Ice} \subset \mathcal{I}^{\mathbb{Z}^2}$ be square ice. Define $c_1, c_2 : \mathcal{I} \longrightarrow \{\pm 1\}$ by $c_1(\ ) := +1 =: c_2(\ )$ and $c_1(\ ) := -1 =: c_2(\ )$ ('$*$' means 'anything'). Define cocycle $C : \mathbb{Z}^2 \times \mathfrak{Ice} \longrightarrow \mathbb{Z}$ as follows:

$$\forall\, \mathbf{i} \in \mathfrak{Ice},\ \forall\, \mathsf{z} = (z_1, z_2) \in \mathbb{Z}^2,\ C(\mathsf{z}, \mathbf{i}) \ := \ \sum_{x=0}^{z_1-1} c_1(i_{x,0}) + \sum_{y=0}^{z_2-1} c_2(i_{z_1,y}).$$

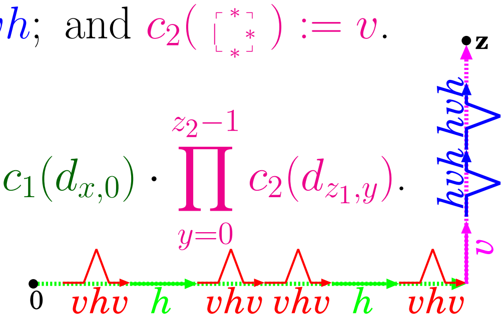


This is a **height function** (a $\mathbb{Z}$-valued cocycle). These arise in tilings [e.g. K. Eloranta 1999-2005, H.Cohn & J.Propp] and statistical mechanics [R.Baxter 1989].

(b) Let $\mathfrak{Dom} \subset \mathcal{D}^{\mathbb{Z}^2}$ be dominoes. Let $\mathcal{G} := \mathbb{Z}_{/2} * \mathbb{Z}_{/2}$ be group of finite products $vhvhv \cdots vhv$, where v and h are noncommuting generators with $v^2 = e = h^2$. Define $c_1, c_2 : \mathcal{D} \longrightarrow \mathcal{G}$ by

$c_1(\ ) := vhv$; $\quad c_1(\ ) := h$; $\quad c_2(\ ) := hvh$; and $c_2(\ ) := v$.

$$\forall\, \mathbf{d} \in \mathfrak{Dom},\ \forall\, \mathsf{z} = (z_1, z_2) \in \mathbb{Z}^2,\ C(\mathsf{z}, \mathbf{d}) \ := \ \prod_{x=0}^{z_1-1} c_1(d_{x,0}) \cdot \prod_{y=0}^{z_2-1} c_2(d_{z_1,y}).$$



(c) If $b : \mathfrak{A} \longrightarrow \mathcal{G}$ is continuous, then function $C(\mathsf{z}, \mathbf{a}) := b(\sigma^{\mathsf{z}}(\mathbf{a})) \cdot b(\mathbf{a})^{-1}$ is a cocycle, called a **coboundary**.

(d) Let $\mathbf{X}$ = topological space. Let $\mathcal{H}$ =homeo($\mathbf{X}$). Then $\mathcal{H}$-valued cocycles are the fibre-wise maps of a skew product extension of the σ-action on $\mathfrak{A}$ to a $\mathbb{Z}^D$-action on $\mathfrak{A} \times \mathbf{X}$. [R.Zimmer 1976-80, J.Kammeyer 1990-93]

Cohomology

Two cocycles C and C' are **cohomologous** ($C \approx C'$) if $\exists$ continuous **transfer function** $b : \mathfrak{A} \longrightarrow \mathcal{G}$ such that

$$C'(\mathsf{z}, \mathbf{a}) \;=\; b(\sigma^{\mathsf{z}}(\mathbf{a})) \cdot C(\mathsf{z}, \mathbf{a}) \cdot b(\mathbf{a})^{-1}, \quad \forall\, \mathsf{z} \in \mathbb{Z}^D, \text{ and } \mathbf{a} \in \mathfrak{A}.$$

Let $\underline{C}$:= cohomology equivalence class of the cocycle C.

$\mathcal{Z}^1(\mathfrak{A}, \mathcal{G})$:= $\{\mathcal{G}$-valued cocycles$\}$.

$\mathcal{H}^1(\mathfrak{A}, \mathcal{G})$:= $\{$cohomology equivalence classes in $\mathcal{Z}^1(\mathfrak{A}, \mathcal{G})\}$.

If $(\mathcal{G}, \cdot)$ is abelian, then $\mathcal{Z}^1(\mathfrak{A}, \mathcal{G})$ is a group (under pointwise multipication), and $\mathcal{H}^1(\mathfrak{A}, \mathcal{G})$ is a quotient group, called the **1st cohomology group** of $\mathfrak{A}$ (with coefficients in $\mathcal{G}$). [see e.g. K.Schmidt (1995, 1998) for discussion]

Trails and locally determined cocycles

Let $\mathbb{E} := \left\{\mathsf{z} \in \mathbb{Z}^D \,;\, \mathsf{z} = (0, ..., 0, \pm 1, 0, ..., 0)\right\}$. A **trail** is a sequence $\zeta = (\mathsf{z}_0, \mathsf{z}_1, \ldots, \mathsf{z}_N) \subset \mathbb{Z}^D$, where, $\forall n \in [1...N]$, $\mathsf{z}'_n := (\mathsf{z}_n - \mathsf{z}_{n-1}) \in \mathbb{E}$.

Let $r > 0$. Let $c : \mathbb{E} \times \mathfrak{A}_{(r)} \longrightarrow \mathcal{G}$ be such that, $\forall\, \mathsf{e}, \mathsf{e}' \in \mathbb{E}$, $\forall\, \mathbf{a} \in \mathfrak{A}$,

(a) $c(\mathsf{e}', \mathbf{a}_{\mathbb{B}(\mathsf{e},r)}) \cdot c(\mathsf{e}, \mathbf{a}_{\mathbb{B}(r)}) \;=\; c(\mathsf{e}, \mathbf{a}_{\mathbb{B}(\mathsf{e}',r)}) \cdot c(\mathsf{e}', \mathbf{a}_{\mathbb{B}(r)})$. i.e. $c\left(\uparrow\!\rightarrow\right) \;=\; c\left(\rightarrow\!\uparrow\right)$

(b) $c(-\mathsf{e}, \mathbf{a}_{\mathbb{B}(\mathsf{e},r)}) \;=\; c(\mathsf{e}, \mathbf{a}_{\mathbb{B}(r)})^{-1}$. i.e. $c\left(\downarrow\right) \;=\; c\left(\uparrow\right)^{-1}$

Then $c(\zeta, \mathbf{a}) \;:=\; \prod_{n=1}^{N} c(\mathsf{z}'_n, \mathbf{a}_{\mathbb{B}(\mathsf{z}_{n-1},r)})$ depends only on z_0 and z_N, not ζ.

Example: If ζ is **closed** (i.e. $\mathsf{z}_N = \mathsf{z}_0$) then $c(\zeta, \mathbf{a}) = e_{\mathcal{G}}$.

Define cocycle $C : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ as follows: $\forall\, \mathbf{a} \in \mathfrak{A}$, $\mathsf{z} \in \mathbb{Z}^D$, $C(\mathsf{z}, \mathbf{a}) := c(\zeta, \mathbf{a})$, (where ζ is *any* trail from 0 to z). We say C is **locally determined** with **local rule** c of **radius** r.

If $\mathcal{G}$ is discrete, then $\forall$ continuous $\mathcal{G}$-valued cocycles are locally determined. For any $r > 0$, let $\mathcal{Z}^1_r(\mathfrak{A}, \mathcal{G})$:= radius-$r$ cocycles on $\mathfrak{A}$.

Cocycles and Cellular Automata

Proposition: *Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a subshift. Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a cellular automaton with $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$. Let $\mathcal{G}$ be a group.*

(a) *Let $C \in \mathcal{Z}^1(\mathfrak{A}, \mathcal{G})$ be cocycle. Define $\Phi_* C : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ by $\Phi_* C(\mathsf{z}, \mathbf{a}) = C(\mathsf{z}, \Phi(\mathbf{a}))$. Then $\Phi_* C$ is also a cocycle on $\mathfrak{A}$.*

(b) *If Φ has radius R, and C is locally determined with radius r, then $\Phi_* C$ is locally determined with radius $r + R$.*

(c) *Let $C, C' \in \mathcal{Z}^1(\mathfrak{A}, \mathcal{G})$. If $C \approx C'$, then $\Phi^* C \approx \Phi^* C'$. Thus, Φ induces a function $\Phi_* : \mathcal{H}^1(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{H}^1(\mathfrak{A}, \mathcal{G})$.*

(d) *If $(\mathcal{G}, \cdot)$ is abelian, then Φ_* is a group endomorphism.* □

We will see that the Φ-persistence of certain kinds of defects depends critically on the surjectivity of the endomorphism Φ_*.

Question: *When is Φ_* surjective?*

Gap Defects: Definition

Some domain boundaries exhibit divergence in cocycle asymptotics.

Let $C \in \mathcal{Z}^1_r(\mathfrak{A}, \mathbb{Z})$ be a range-r cocycle (i.e. 'height function').

Let $\mathbf{a} \in \widetilde{\mathfrak{A}}$. Let $\mathbb{X}$ be an infinite, simply-connected component of $\mathbb{G}_r(\mathbf{a})$. Fix $\mathsf{x}^* \in \mathbb{X}$. For any $\mathsf{x} \in \mathbb{X}$, we define the **height difference**:

$$\mathbf{C}_{\mathbf{a}}(\mathsf{x}^*, \mathsf{x}) \quad := \quad c(\zeta, \mathbf{a}),$$

where $c : \mathfrak{A}_{(r)} \longrightarrow \mathbb{Z}$ is 'local rule', and ζ is any trail in $\mathbb{X}$ from x^* to x.

(Well-defined independent of ζ because $\mathbb{X}$ is a simply-connected.) Note:

$$|C_{\mathbf{a}}(\mathsf{x}^*, \mathsf{x})| \quad \leq \quad K \cdot d_{\mathbb{X}}(\mathsf{x}^*, \mathsf{x}),$$

where $K := \max_{\mathbf{a} \in \mathfrak{A}_{(r)}} |c(\mathbf{a})|$, and $d_{\mathbb{X}}(\mathsf{x}^*, \mathsf{x}) :=$ min length ($\mathbb{X}$-trail from x^* to x).

Let $\mathbb{Y}$ be another infinite connected component of $\mathbb{G}_r(\mathbf{a})$. Fix $\mathsf{y}^* \in \mathbb{Y}$. For any $\mathsf{y} \in \mathbb{Y}$, define $C_{\mathbf{a}}(\mathsf{y}, \mathsf{y}^*)$ in the same way as $C_{\mathbf{a}}(\mathsf{x}^*, \mathsf{x})$ above. We then define

$$\mathbf{C}(\mathsf{y}, \mathsf{x}) \quad := \quad C(\mathsf{y}, \mathsf{y}^*) + C(\mathsf{x}^*, \mathsf{x}).$$

If $\mathbb{X}$ and $\mathbb{Y}$ were the same connected component (or if we could remove the defect in $\mathbf{a}$ so that they were), then we expect

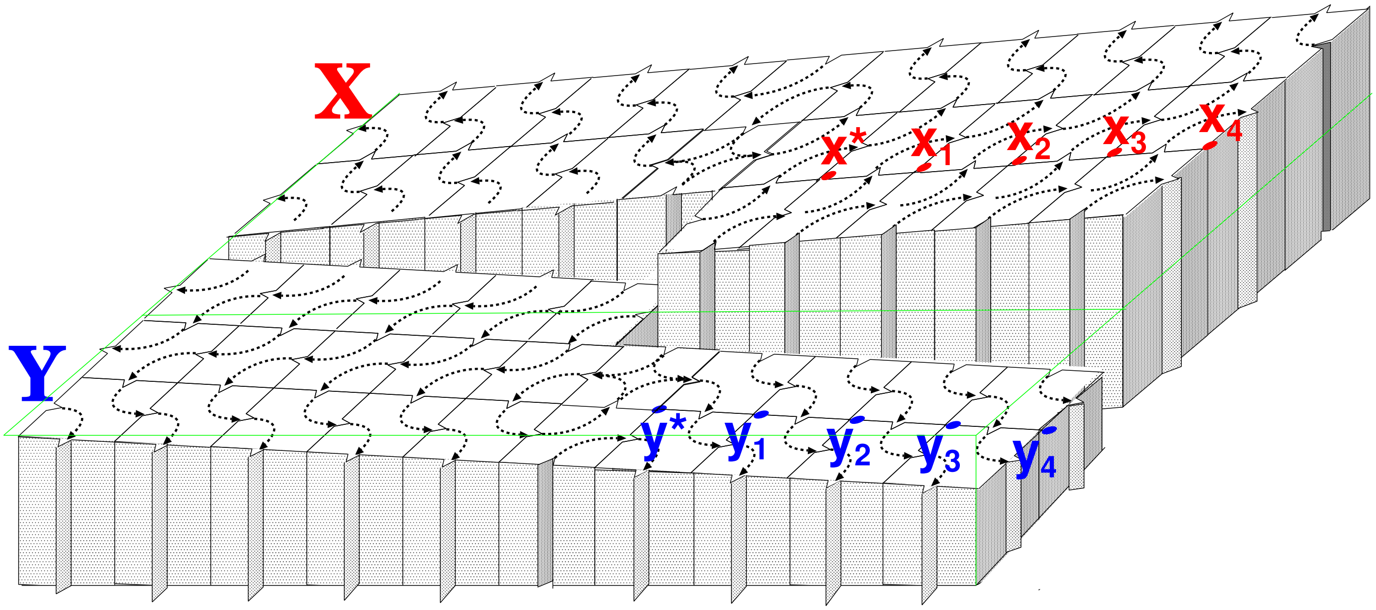
$$C(\mathsf{y}, \mathsf{x}) \quad \leq \quad K \cdot d_{\mathbb{X}}(\mathsf{y}, \mathsf{x}) + \text{const.} \quad \approx \quad K|\mathsf{y} - \mathsf{x}| + \text{const.}$$

We say there is a ***C*-gap** between $\mathbb{X}$ and $\mathbb{Y}$ if $\displaystyle\sup_{\mathsf{y} \in \mathbb{Y},\ \mathsf{x} \in \mathbb{X}} \frac{|C(\mathsf{y}, \mathsf{x})|}{|\mathsf{y} - \mathsf{x}|} \ = \ \infty.$

(This suggests that the defect separating $\mathbb{X}$ and $\mathbb{Y}$ is essential.)

Fine print: If $\mathcal{G} \neq \mathbb{Z}$, we can also define gaps for $\mathcal{G}$-valued cocycles, by first defining an appropriate *pseudonorm* $\|\bullet\| : \mathcal{G} \longrightarrow \mathbb{R}$ which satisfies the triangle inequality and is invariant under conjugation.

Gaps in the Ice



Example: Consider the defective configuration in $\widetilde{\mathfrak{Ice}}$ shown above, and let $\{\mathsf{x}^*, \mathsf{x}_1, \mathsf{x}_2, \ldots\} \subset \mathbb{X}$ and $\{\mathsf{y}^*, \mathsf{y}_1, \mathsf{y}_2, \ldots\} \subset \mathbb{Y}$ be as shown. Let $C \in \mathcal{Z}^1(\mathfrak{Ice}, \mathbb{Z})$ be the cocycle with local rule

$$c_1\left(\begin{smallmatrix} & * & \\ * & & * \\ & \wedge & \end{smallmatrix}\right) := +1 =: c_2\left(\begin{smallmatrix} & * & \\ < & & * \\ & * & \end{smallmatrix}\right) \text{ and } c_1\left(\begin{smallmatrix} & * & \\ * & & * \\ & \vee & \end{smallmatrix}\right) := -1 =: c_2\left(\begin{smallmatrix} & * & \\ > & & * \\ & * & \end{smallmatrix}\right).$$

Then $C(\mathsf{x}^*, \mathsf{x}_n) = n$ and $C(\mathsf{y}^*, \mathsf{y}_n) = -n$, so $C(\mathsf{x}_n, \mathsf{y}_n) = 2n, \ \forall\, n \in \mathbb{N}$.

But $|\mathsf{x}_n - \mathsf{y}_n| = 2$, $\forall\ n \in \mathbb{N}$, so $\displaystyle\lim_{n\to\infty} \frac{|C(\mathsf{x}_n, \mathsf{y}_n)|}{|\mathsf{x} - \mathsf{y}|} = \lim_{n\to\infty} \frac{2n}{2} = \infty$; hence there is a gap between $\mathbb{X}$ and $\mathbb{Y}$.

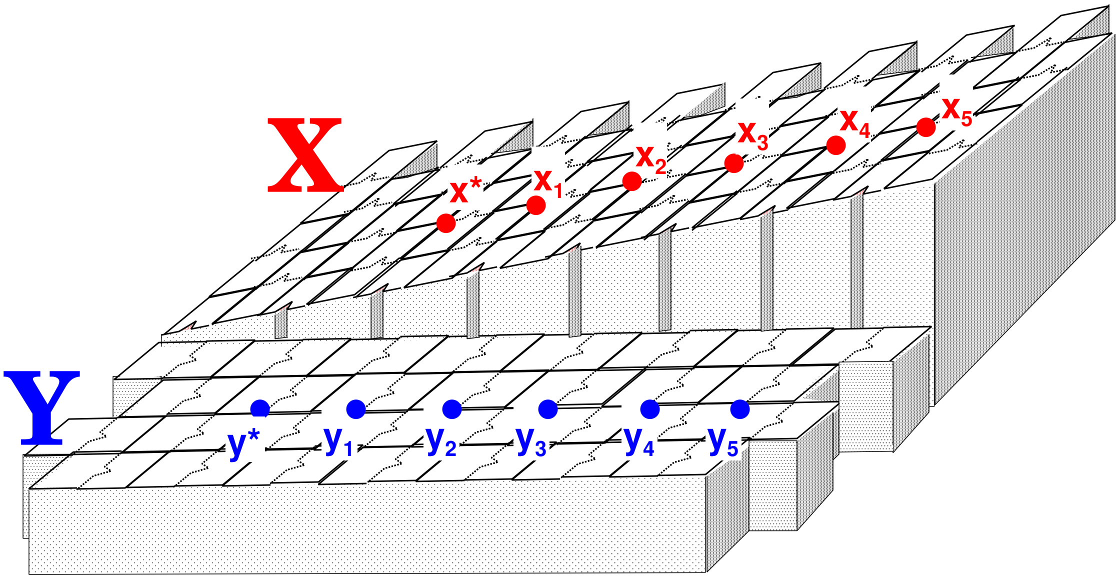
Example: Let $C : \mathbb{Z}^2 \times \mathfrak{Dom} \longrightarrow \mathcal{G} := \mathbb{Z}_{/2} * \mathbb{Z}_{/2}$ have local rule:

$$c_1\left(\begin{smallmatrix} & - & \\ & & | \\ & \wedge & \end{smallmatrix}\right) := vhv; \quad c_1\left(\begin{smallmatrix} & * & \\ * & & * \\ & & \end{smallmatrix}\right) := h; \quad c_2\left(\begin{smallmatrix} & - & \\ > & & | \\ & - & \end{smallmatrix}\right) := hvh; \text{ and } c_2\left(\begin{smallmatrix} & * & \\ & & * \\ & * & \end{smallmatrix}\right) := v.$$

Let $\mathcal{Z} := \{$cyclic subgroup generated by $vh\} \subset \mathcal{G}$. Then $(\mathcal{Z}, \cdot) \cong (\mathbb{Z}, +)$, and for all $\mathbf{d} \in \mathfrak{Dom}$ and $2\mathsf{z} \in 2\mathbb{Z}^2$, $C(2\mathsf{z}, \mathbf{d}) \in \mathcal{Z}$.

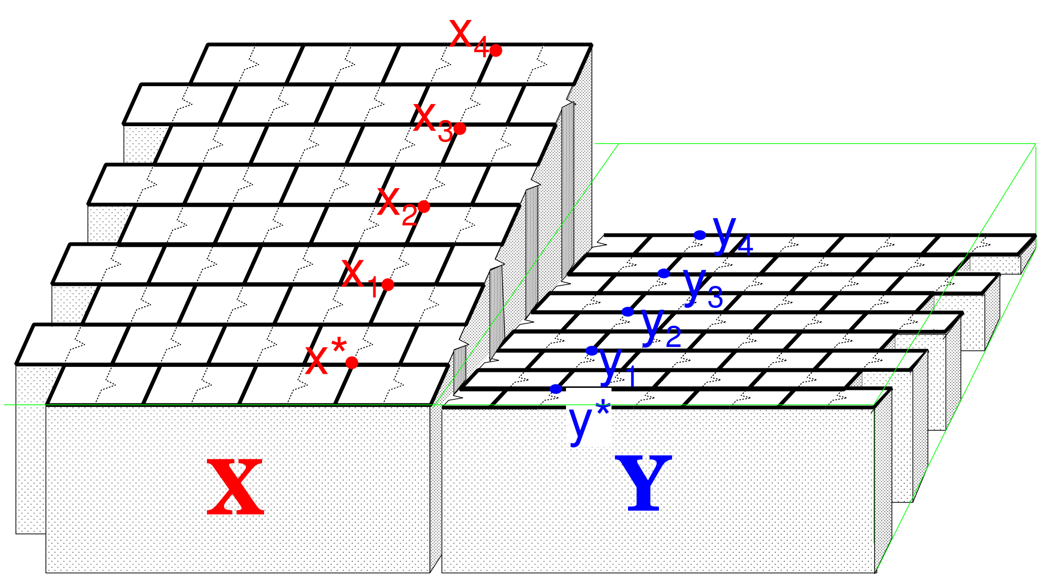
Let $\mathcal{D}_2 \subset \mathcal{D}^{2\times 2}$ be the alphabet of $\mathfrak{Dom}$-admissible 2×2 blocks. Let $\mathfrak{D}_2 \subset \mathcal{D}_2^{\mathbb{Z}^2}$ be 'recoding' of $\mathfrak{Dom}$ in this alphabet. Then $2\mathbb{Z}^2$ acts on $\mathfrak{D}_2$ in the obvious way, and C yields a cocycle $C' : 2\mathbb{Z}^2 \times \mathfrak{D}_2 \longrightarrow \mathcal{Z} \cong \mathbb{Z}$.

Gaps in Dominoes



In the $\widetilde{\mathfrak{Dom}}$-configuration shown above, $C'(\mathsf{x}^*, \mathsf{x}_n) = (vhvh)^n \cong 2n$, while $C'(\mathsf{y}^*, \mathsf{y}_n) = h^{2n} \cong 0$, so $C'(\mathsf{y}_n, \mathsf{x}_n) = n$, for all $n \in \mathbb{N}$.

But $|\mathsf{x}_n - \mathsf{y}_n| = 4$, $\forall\ n \in \mathbb{N}$, so $\displaystyle\lim_{n\to\infty} \frac{|C'(\mathsf{x}_n, \mathsf{y}_n)|}{|\mathsf{x} - \mathsf{y}|} = \lim_{n\to\infty} \frac{n}{4} = \infty.$



In the $\widetilde{\mathfrak{Dom}}$-configuration shown above, $C'(\mathsf{x}^*, \mathsf{x}_n) = (vhvh)^n \cong 2n$, while $C'(\mathsf{y}^*, \mathsf{y}_n) = (hvhv)^n \cong -2n$, so $C'(\mathsf{y}_n, \mathsf{x}_n) = -4n$, $\forall\ n \in \mathbb{N}$.

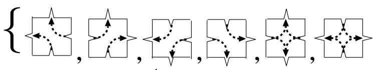
But $|\mathsf{x}_n - \mathsf{y}_n| = 4$, $\forall\ n \in \mathbb{N}$, so $\displaystyle\lim_{n\to\infty} \frac{|C'(\mathsf{x}_n, \mathsf{y}_n)|}{|\mathsf{x} - \mathsf{y}|} = \lim_{n\to\infty} \frac{-4n}{4} = -\infty.$

Persistence of Gaps

Theorem: *If $\Phi\colon \mathcal{A}^{\mathbb{Z}^D} \to \mathcal{A}^{\mathbb{Z}^D}$ is a CA, $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$, and endomorphism*

$$\Phi_* : \mathcal{H}^1(\mathfrak{A}, \mathbb{Z}) \ni C \mapsto C \circ \Phi \in \mathcal{H}^1(\mathfrak{A}, \mathbb{Z})$$

is surjective, then any gap is Φ-persistent.

Example: If $\mathcal{I} := \{$$\}$, and $\Phi : \mathcal{I}^{\mathbb{Z}^2} \longrightarrow \mathcal{I}^{\mathbb{Z}^2}$ is CA with $\Phi(\mathfrak{Ice}) \subseteq \mathfrak{Ice}$, and $\Phi_* : \mathcal{H}^1(\mathfrak{Ice}, \mathbb{Z}) \longrightarrow \mathcal{H}^1(\mathfrak{Ice}, \mathbb{Z})$ is surjective, then Φ cannot destroy the ice gap (or even change the 'difference in slope').

Proof idea: First show that C-gaps depend only on cohomology class of C, i.e.:

Lemma: *If $C \approx C'$, then any C-gap is also a C'-gap.* ♢

Now suppose $\mathbf{a}$ has C-gap. Now Φ_* is surjective, so find $C' \in \mathcal{Z}^1$ such that $\Phi_* C' \approx C$. Then $\mathbf{a}$ also has $(\Phi_* C')$-gap. But this implies that $\Phi(\mathbf{a})$ has C' gap. □

Sharp Gaps are Essential

A gap in $\mathbb{G}_r(\mathbf{a})$ is **sharp** if, for all $R \geq r \geq 0$, there exists constant $K = K(R, r) \in \mathbb{N}$ such that, for any $\mathsf{y} \in \mathbb{G}_r(\mathbf{a})$, $\exists\, \mathsf{x} \in \mathbb{G}_R(\mathbf{a})$ in same connected component $\mathbb{X}$ of $\mathbb{G}_r(\mathbf{a})$ as y, with $d_{\mathbb{X}}(\mathsf{x}, \mathsf{y}) \leq K$.

Idea: The gap does not ramify into lots of 'tributaries'.

Example: If $\mathfrak{A}$ is a subshift of finite type, and defect set $\mathbb{D}(\mathbf{a})$ is confined to a thickened hyperplane [as in previous three examples] then the gap is sharp.

Theorem: *Sharp gaps are essential defects.*

Proof idea: First show:

Lemma: *The existence of a gap does not depend on the choice of reference points $\mathsf{x}^* \in \mathbb{X}$ and $\mathsf{y}^* \in \mathbb{Y}$.* ♢

Thus, we can always move our basepoint x^* and 'gap-detection' sequence $\{\mathsf{x}_1, \mathsf{x}_2, \ldots\}$ far away from gap. Thus, a gap is 'detectable' from any distance; hence it cannot be removed by locally changing $\mathbf{a}$. □

Defect Codimension

A domain boundary is a defect of **codimension 1**.

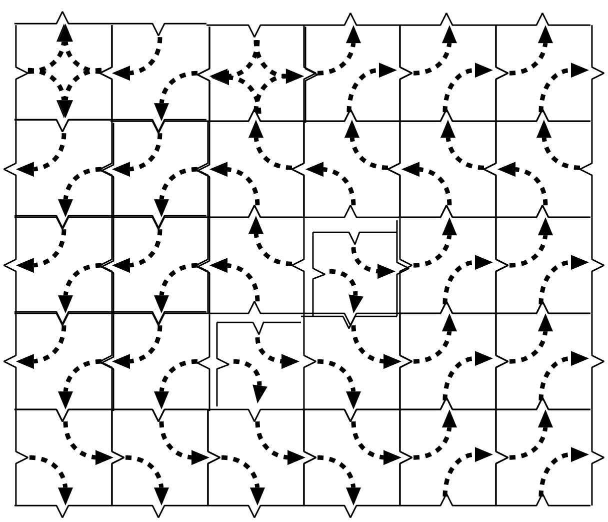
Fix $r \in \mathbb{N}$. Let $\mathbb{G}_r(\mathbf{a}) := \left\{ \mathsf{z} \in \mathbb{Z}^D \; ; \; \mathbf{a}_{\mathbb{B}(\mathsf{z},r)} \in \mathfrak{A}_{(r)} \right\}$. (Loosely, this is the complement of a radius-r neighbourhood around the defects in $\mathbf{a}$.)

Let $\mathbf{G_r}(\mathbf{a})$:= union of all unit cubes whose corners are all in $\mathbb{G}_r(\mathbf{a})$.

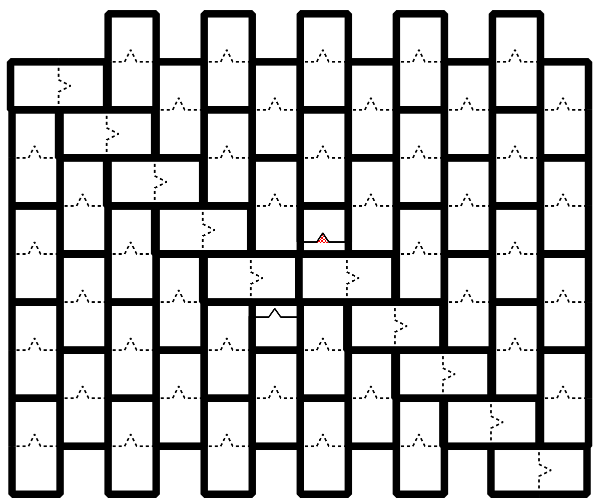
We say $\mathbf{a}$ has a (range r) **codimension** $(k+1)$ defect if the kth homotopy group $\pi_k\left[\mathbf{G}_r(\mathbf{a})\right]$ is nontrivial$^{(*)}$.

Examples of Codimension-Two Defects:

In $\mathfrak{Ice}$:



In $\mathfrak{Dom}$:



[due to S. Lightwood, via M. Einsiedler, 2001]

The sequence of inclusions $\mathbb{G}_1(\mathbf{a}) \supseteq \mathbb{G}_2(\mathbf{a}) \supseteq \mathbb{G}_3(\mathbf{a}) \supseteq \cdots$ yields sequence of homomorphisms

$$\pi_k\left[\mathbf{G}_1(\mathbf{a})\right] \longleftarrow \pi_k\left[\mathbf{G}_2(\mathbf{a})\right] \longleftarrow \pi_k\left[\mathbf{G}_3(\mathbf{a})\right] \longleftarrow \cdots$$

Define $\pi_k\left[\mathbf{G}_\infty(\mathbf{a})\right]$:= inverse limit of this sequence$^{(\dagger)}$ (detects 'extremely large scale' homotopy properties).

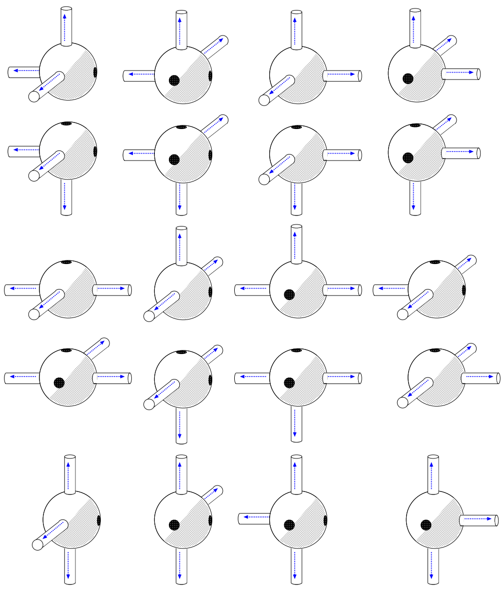
Say $\mathbf{a}$ has a **projective** codimension $(k+1)$ defect if $\pi_k\left[\mathbf{G}_\infty(\mathbf{a})\right] \neq \{0\}$.

$(*)$ Strictly speaking, we must fix a basepoint and a connected component of $\mathbf{G}_r$.

$(\dagger)$ We must fix a *proper base ray*, and assume $\mathbf{G}_r$ has unique connected component for large r.

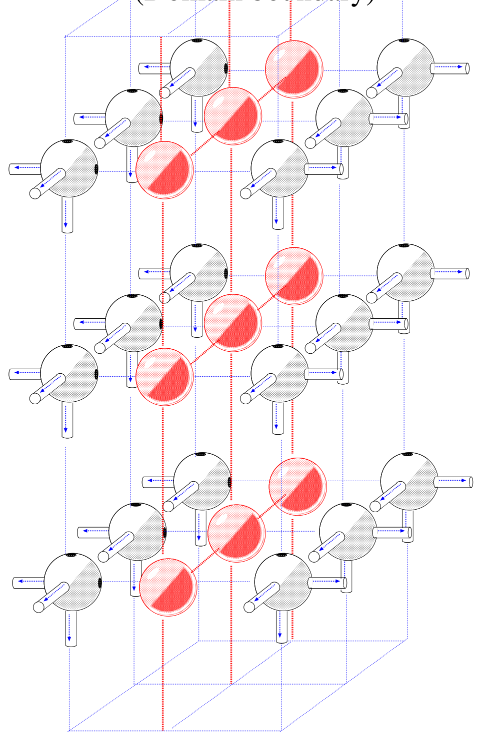
Defect Codimension in 3D

The 'Ice Cube' Shift:

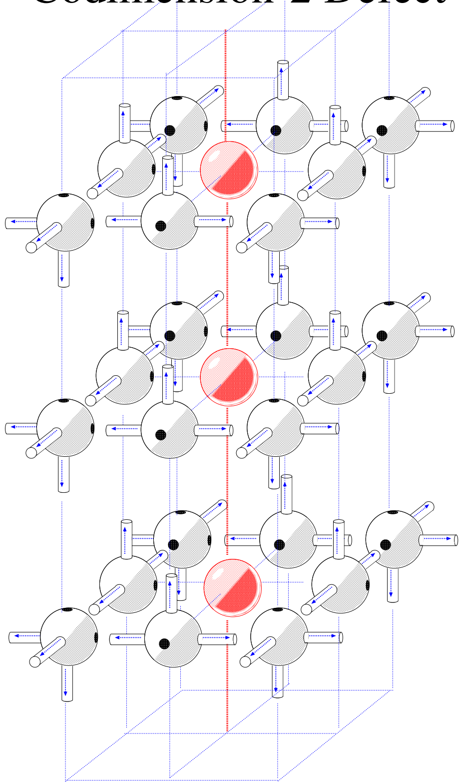


Codimension-1 Defect

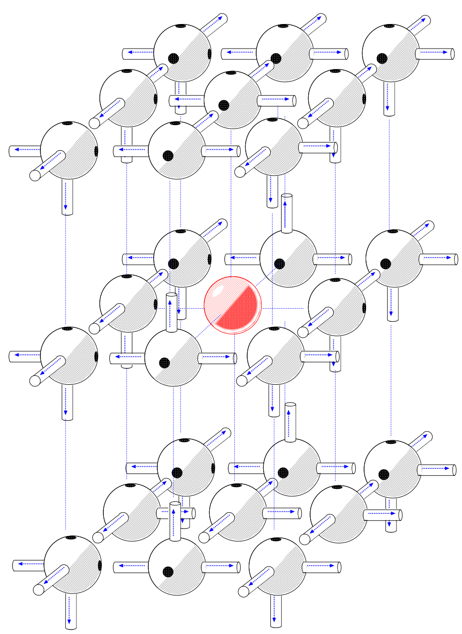
(Domain boundary)



Codimension-2 Defect



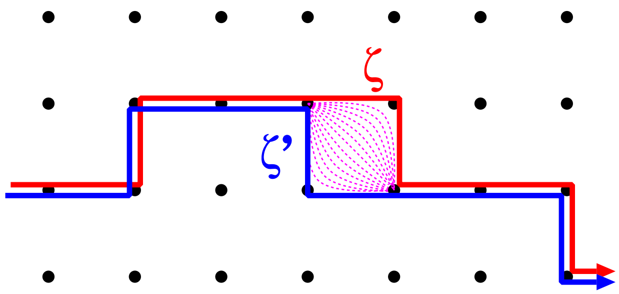
Codimension-3 Defect



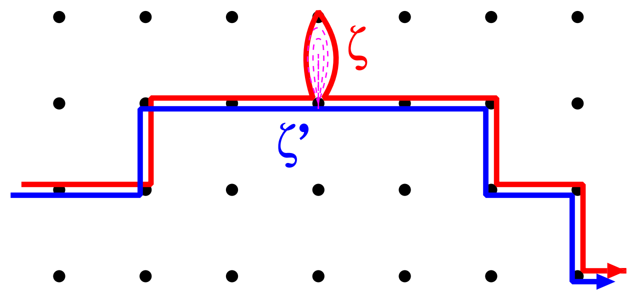
Trail Homotopy

Let $\mathbb{Y} \subseteq \mathbb{Z}^D$ and let ζ and ζ' be trails in $\mathbb{Y}$.

ζ and ζ' are **homotopic in $\mathbb{Y}$** (notation: $\zeta \approx \zeta'$) if we can move from ζ to ζ' through a sequence of transformations like:



or



If $\mathbf{Y}$ is connected, then every homotopy class of $\pi_1(\mathbf{Y})$ can be represented as a (trail) homotopy class of trails in $\mathbb{Y}$.

Hence regard $\pi_1(\mathbb{Y}) = \{$group of $\mathbb{Y}$-homotopy classes of $\mathbb{Y}$-trails$\}$.

Lemma: *Let $C \in \mathcal{Z}^1_r(\mathfrak{A}, \mathcal{G})$. Let $\mathbf{a} \in \widetilde{\mathfrak{A}}$. Let ζ be closed trail in $\mathbb{G}_r(\mathbf{a})$.*

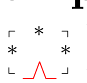
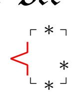
(a) *If $\zeta \approx \zeta'$ in $\mathbb{G}_r(\mathbf{a})$, then $C(\zeta, \mathbf{a}) = C(\zeta', \mathbf{a})$.*

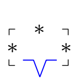
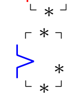
(e.g. If ζ is nullhomotopic in $\mathbb{G}_r(\mathbf{a})$, then $C(\zeta, \mathbf{a}) = e_{\mathcal{G}}$.)

(b) *Suppose $(\mathcal{G}, \cdot)$ is abelian. If $C \approx C'$ then $C(\zeta, \mathbf{a}) = C'(\zeta, \mathbf{a})$.* □

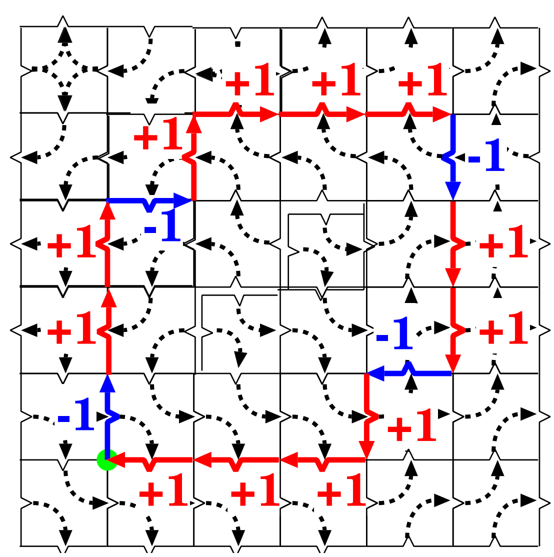
We say that $\mathbf{a}$ has a ***C*-pole** if $C(\zeta, \mathbf{a}) \neq e_{\mathcal{G}}$ for some closed trail $\zeta \in \pi_1[\mathbb{G}_r(\mathbf{a})]$.

Example: Recall $C : \mathfrak{Ice} \times \mathbb{Z}^2 \longrightarrow \mathbb{Z}$

$c_1($  $) := +1 =: c_2($  $)$

$c_1($  $) := -1 =: c_2($  $)$

If ζ is the clockwise trail around the defect, then $C(\zeta, \mathbf{a}) \;=\; 8$. Thus, $\mathbf{a}$ has a pole.



12 x (+1) + 4 x (-1) = 8

Poles and Residues

Proposition: *Let* $\mathbf{a} \in \widetilde{\mathfrak{A}}$. *Let* $C \in \mathcal{Z}^1_r(\mathfrak{A}, \mathcal{G})$.

(a) $\mathrm{Res}_{\mathbf{a}} C : \pi_1[\mathbb{G}_r(\mathbf{a})] \ni \underline{\zeta} \mapsto C(\zeta, a) \in \mathcal{G}$ *is a group homomorphism.*

(b) *If* $(\mathcal{G}, \cdot)$ *is abelian, and* $C \approx C'$ *then* $\mathrm{Res}_{\mathbf{a}} C = \mathrm{Res}_{\mathbf{a}} C'$. *Thus, we get group homomorphism*

$$\mathrm{Res}_{\mathbf{a}} : \mathcal{H}_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G}) \times \pi_1[\mathbb{G}_\infty(\mathbf{a})] \times \ni (\underline{C}, \underline{\zeta}) \mapsto C(\zeta, a) \in \mathcal{G}. \qquad \square$$

The configuration $\mathbf{a}$ has a **$\mathcal{G}$-pole** if $\mathrm{Res}_{\mathbf{a}}$ is nontrivial homomorphism. The function $\mathrm{Res}_{\mathbf{a}}$ acts as an algebraic 'signature' of the defect in $\mathbf{a}$.

Theorem: *$\mathcal{G}$-poles are essential defects.* □

Persistence of Poles

Theorem: *If the function* $\Phi_* : \mathcal{H}^1(\mathfrak{A}, \mathcal{G}) \ni C \mapsto (C \circ \Phi) \in \mathcal{H}^1(\mathfrak{A}, \mathcal{G})$ *is surjective, then all $\mathcal{G}$-poles are Φ-persistent.*

Example: If $\Phi : \mathcal{I}^{\mathbb{Z}^2} \longrightarrow \mathcal{I}^{\mathbb{Z}^2}$ was a CA with $\Phi(\mathfrak{Ice}) \subseteq \Phi(\mathfrak{Ice})$, and Φ_* was surjective, then the ice pole would persist under Φ. ♢

Proof idea: Let R :=radius(Φ). If $\mathbf{a} \in \widetilde{\mathfrak{A}}$ and $\mathbf{a}' := \Phi(\mathbf{a})$, then $\mathbb{G}_{r+R}(\mathbf{a}) \subseteq \mathbb{G}_r(\mathbf{a}')$.

This yields homomorphisms $\Phi_\dagger : \pi_1[\mathbb{G}_{r+R}(\mathbf{a})] \longrightarrow \pi_1[\mathbb{G}_r(\mathbf{b})]$, for all $r \in \mathbb{N}$.

Lemma: *For all* $\zeta \in \pi_1[\mathbb{G}_{r+R}(\mathbf{a})]$ *and* $C' \in \mathcal{Z}^1_r(\mathfrak{A}, \mathcal{G})$, *if* $\zeta' := \Phi_\dagger(\zeta)$ *and* $C \approx \Phi_*(C')$, *then* $C'(\mathbf{a}', \zeta') = C(\mathbf{a}, \zeta)$. ♢

Now, if $\mathbf{a}$ has a C-pole for some $C \in \mathcal{Z}^1(\mathfrak{A}, \mathcal{G})$, then there exists $\zeta \in \pi_1[\mathbb{G}_{r+R}(\mathbf{a})]$ with $C(\mathbf{a}, \zeta)$ nontrivial.

Φ_* is surjective, so $\exists\ C' \in \mathcal{Z}^1(\mathfrak{A}, \mathcal{G})$ with $\Phi_* C' \approx C$. Let $\zeta' := \Phi_\dagger(\zeta) \in \pi_1[\mathbb{G}_r(\mathbf{a}')]$. Then $C'(\mathbf{a}', \zeta') = C(\mathbf{a}, \zeta)$ is nontrivial. Thus $\mathbf{a}'$ has a C'-pole. □

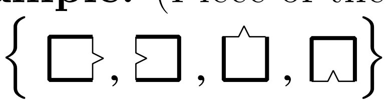
Remark: We can also characterize poles using the *fundamental cocycles* of [K.Schmidt, 1998].

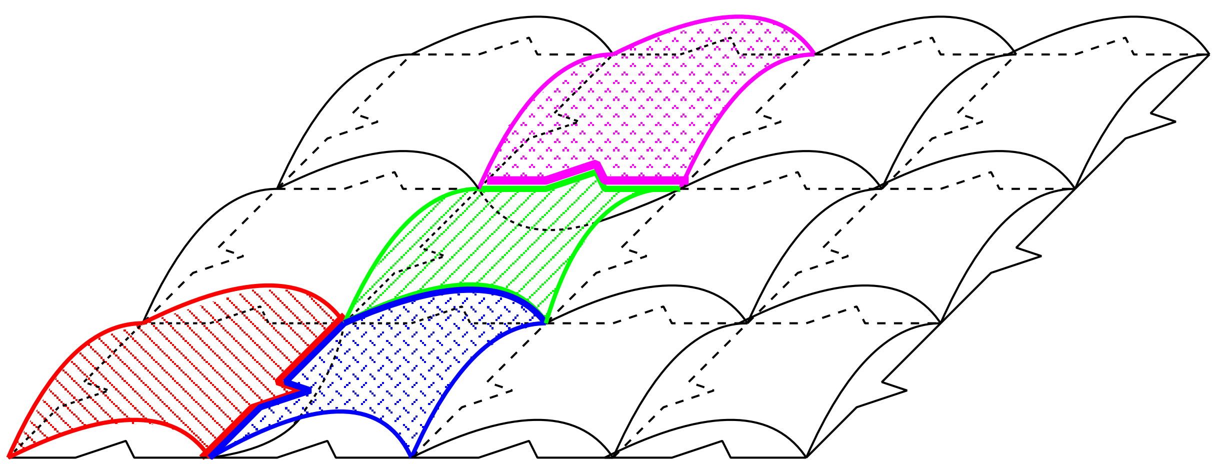
The Conway-Lagarias Tiling Group

Let $\mathcal{W}$ be a (finite) set of notched square prototiles (to tile $\mathbb{R}^2$). The **tile complex** of $\mathcal{W}$ is a 2-dimensional cell complex $\mathbf{X}$ defined as follows:

• For each $\mathsf{z} \in \mathbb{Z}^D$ and each $w \in \mathcal{W}$, there is a w-shaped 2-cell in $\mathbf{X}$, positioned in space 'over' z. Each notched edge of w is a 1-cell in $\mathbf{X}$.

• If z and z' are adjacent in $\mathbb{Z}^2$, and tiles w and w' 'match' along the corresponding edge, then glue together tiles (w, z) and (w', z') in $\mathbf{X}$.

Example: (Piece of tile-complex for $\mathfrak{Dom}$). Each square contains four 2-cells . Between each vertex-pair $\exists$ two edges {|, ⊱}.



$\exists$ natural projection $\Pi : \mathbf{X} \longrightarrow \mathbb{R}^2$ (sending the vertices of $\mathbf{X}^0$ into $\mathbb{Z}^2$).

$$\left(\text{Admissible } \mathcal{W}\text{-tiling } \mathbf{w} \text{ of } \mathbb{R}^2\right) \cong \left(\text{Continuous } \Pi\text{-section } \varsigma_{\mathbf{w}} : \mathbb{R}^2 \longrightarrow \mathbf{X}\right)$$

$$\left(\text{'Partial' } \mathcal{W}\text{-tiling } \mathbf{w} \text{ of } \mathbf{U} \subset \mathbb{R}^2\right) \cong \left(\text{'Partial' } \Pi\text{-section } \varsigma_{\mathbf{w}} : \mathbf{U} \longrightarrow \mathbf{X}\right)$$

In the second case, $\varsigma_{\mathbf{w}}$ defines homomorphism $\varsigma^*_{\mathbf{w}} : \pi_1(\mathbf{U}) \longrightarrow \pi_1(\mathbf{X})$. Then:

$$\left(\mathbf{U}^{\complement}\text{-hole in } \mathbf{w} \text{ can be admissibly filled}\right) \Longrightarrow$$

$$\left(\varsigma^*_{\mathbf{w}}\text{-image of any loop in } \mathbf{U} \text{ is nullhomotopic}\right) \iff \left(\varsigma^*_{\mathbf{w}} \text{ is trivial}\right).$$

$\pi_1(\mathbf{X})$ = '**tile homotopy group**' [J.H.Conway & J.C.Lagarias, 1990; W.Thurston, 1990]

Higher homotopy/homology groups for Wang tiles

Let $\mathcal{W}$ be a (finite) set of D-dimensional notched hypercubic **Wang tiles** (to tile $\mathbb{R}^D$). Build a D-dimensional cell complex $\mathbf{X}$ analogous to before. Get projection $\Pi : \mathbf{X} \longrightarrow \mathbb{R}^D$ such that $\Pi(\mathbf{X}^0) = \mathbb{Z}^D$.

$$\Big(\text{Admissible } \mathcal{W}\text{-tiling } \mathbf{w} \text{ of } \mathbb{R}^D\Big) \cong \Big(\text{Continuous } \Pi\text{-section } \varsigma_{\mathbf{w}} : \mathbb{R}^D \longrightarrow \mathbf{X}\Big).$$

$$\Big(\text{`Partial' } \mathcal{W}\text{-tiling } \mathbf{w} \text{ of } \mathbf{U} \subset \mathbb{R}^D\Big) \cong \Big(\text{`Partial' } \Pi\text{-section } \varsigma_{\mathbf{w}} : \mathbf{U} \longrightarrow \mathbf{X}\Big).$$

In this case, for all $k \in \mathbb{N}$, the section $\varsigma_{\mathbf{w}}$ defines homomorphisms:

$$\begin{array}{rcll} \pi_{\mathbf{k}}\varsigma_{\mathbf{w}} : \pi_k(\mathbf{U}, u) & \longrightarrow & \pi_k(\mathbf{X}, x); & \quad (x, u = \text{suitable basepoints}) \\ \mathcal{H}_{\mathbf{k}}\varsigma_{\mathbf{w}} : \mathcal{H}_k(\mathbf{U}, \mathcal{G}) & \longrightarrow & \mathcal{H}_k(\mathbf{X}, \mathcal{G}); & \quad ((\mathcal{G}, +) = \text{some coefficient group, e.g. } \mathcal{G} = \mathbb{Z}) \\ \mathcal{H}^{\mathbf{k}}\varsigma_{\mathbf{w}} : \mathcal{H}^k(\mathbf{U}, \mathcal{G}) & \longrightarrow & \mathcal{H}^k(\mathbf{X}, \mathcal{G}) & \end{array}$$

$$\Big(\text{Hole in } \mathbf{w} \text{ is fillable}\Big) \Longrightarrow \Big(\pi_k\varsigma_{\mathbf{w}}, \mathcal{H}_k\varsigma_{\mathbf{w}} \text{ and } \mathcal{H}^k\varsigma_{\mathbf{w}} \text{ are trivial, } \ \forall\, k \in \mathbb{N}\Big).$$

Homotopy/homology groups for subshifts of finite type

Let $\mathcal{A}$ be a finite alphabet. Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a subshift of finite type of radius $r > 0$. Fix $R \geq r$. Treat $\mathcal{W} := \mathfrak{A}_{(R)}$ as Wang tiles with obvious edge-matching conditions. Get tile complex $\mathbf{X}_R$. Then:

$$\Big(\mathbf{a} \in \mathfrak{A}\Big) \cong \Big(\mathcal{W}\text{-admissible tiling of } \mathbb{R}^D\Big) \cong \Big(\Pi\text{-section } \varsigma_{\mathbf{a}} : \mathbb{R}^D \longrightarrow \mathbf{X}_R\Big).$$

Idea: Use homotopy/(co)homology groups of $\mathbf{X}_R$ as invariant for $\mathfrak{A}$ (and get algebraic invariants for codimension-$(k+1)$ defects in $\widetilde{\mathfrak{A}}$).

Problems:

[i] There $\exists$ many different Wang representations for $\mathfrak{A}$. None is 'canonical'. Different Wang representations may yield non-isomorphic groups.

[ii] Wang representations (and hence, their homotopy/homology groups) do not behave well under subshift homomorphisms (i.e. CA).

The Geller-Propp Projective Fundamental Group

Solution: There are natural surjections $\mathbf{X}_r \leftarrow \mathbf{X}_{r+1} \leftarrow \mathbf{X}_{r+2} \leftarrow \cdots$

Get homomorphisms $\pi_k(\mathbf{X}_r, x_r) \leftarrow \pi_k(\mathbf{X}_{r+1}, x_{r+1}) \leftarrow \pi_k(\mathbf{X}_{r+2}, x_{r+2}) \leftarrow \cdots$

(Here, $\{x_k\}$ are basepoints determined by some fixed $\mathbf{a} \in \mathfrak{A}$.)

Define kth **projective homotopy group** $\pi_k(\mathfrak{A}, \mathbf{a})$:= inverse limit of this sequence. (If $k = 1$ this is the *projective fundamental group* of W.Geller & J.Propp, 1995).

Likewise, we define kth **projective (co)homology groups**

$$\mathcal{H}_{\mathbf{k}}(\mathfrak{A}, \mathcal{G}) := \lim_{\longleftarrow} (\mathcal{H}_k(\mathbf{X}_r, \mathcal{G}) \leftarrow \mathcal{H}_k(\mathbf{X}_{r+1}, \mathcal{G}) \leftarrow \mathcal{H}_k(\mathbf{X}_{r+2}, \mathcal{G}) \leftarrow \cdots)$$
$$\mathcal{H}^{\mathbf{k}}(\mathfrak{A}, \mathcal{G}) := \lim_{\longrightarrow} \left(\mathcal{H}^k(\mathbf{X}_r, \mathcal{G}) \rightarrow \mathcal{H}^k(\mathbf{X}_{r+1}, \mathcal{G}) \rightarrow \mathcal{H}^k(\mathbf{X}_{r+2}, \mathcal{G}) \rightarrow \cdots\right)$$

- Isomorphism invariants of $\mathfrak{A}$.
- Detects codimension $(k+1)$ defects.

Basepoint Freedom

The definition of $\pi_k(\mathfrak{A})$ depends upon a chosen 'basepoint' $\mathbf{a} \in \mathfrak{A}$.

We say $\mathfrak{A}$ is **basepoint free** in dimension k if, for any $\mathbf{a}, \mathbf{a}' \in \mathfrak{A}$, there is a canonical isomorphism $\pi_k(\mathfrak{A}, \mathbf{a}) \cong \pi_k(\mathfrak{A}, \mathbf{a}')$.

Proposition:

(a) *Suppose* $\Pi_r^0 : \mathbf{X}_r^0 \longrightarrow \mathbb{Z}^D$ *is injective for all large enough* $r \in \mathbb{N}$. *Then* $\mathfrak{A}$ *is basepoint-free in all dimensions.*

Suppose $(\mathfrak{A}, \sigma)$ *is topologically weakly mixing* [i.e. the Cartesian product $(\mathfrak{A} \times \mathfrak{A}, \sigma \times \sigma)$ is topologically transitive]. *Then:*

(b) *If* $\pi_1(\mathfrak{A}, \mathbf{a})$ *is abelian, then* $\mathfrak{A}$ *is basepoint free in dimension* 1.

(c) *If* $\pi_1(\mathfrak{A}, \mathbf{a})$ *is trivial, then* $\mathfrak{A}$ *is basepoint free in all dimensions.* □

Projective Groups and Cellular Automata

Proposition: *Let* $\Phi\colon \mathcal{A}^{\mathbb{Z}^D}\longrightarrow\mathcal{A}^{\mathbb{Z}^D}$ *be a CA with* $\Phi(\mathfrak{A})\subseteq\mathfrak{A}$*. Then* Φ *induces group endomorphisms:*

$$\begin{aligned} \boldsymbol{\pi_d\Phi}\colon \pi_d(\mathfrak{A},\mathbf{a}) &\longrightarrow \pi_d(\mathfrak{A},\mathbf{a}') \quad (\cong \pi_d(\mathfrak{A},\mathbf{a}) \text{ if basepoint free})\\ \boldsymbol{\mathcal{H}_d\Phi}\colon \mathcal{H}_d(\mathfrak{A},\mathcal{G}) &\longrightarrow \mathcal{H}_d(\mathfrak{A},\mathcal{G})\\ \boldsymbol{\mathcal{H}^d\Phi}\colon \mathcal{H}^d(\mathfrak{A},\mathcal{G}) &\longrightarrow \mathcal{H}^d(\mathfrak{A},\mathcal{G}). \end{aligned}$$

Proof: (Idea) If Φ has radius q, then Φ induces a cellular map $\Phi_* : \mathbf{X}_{R+q}\longrightarrow\mathbf{X}_R$ for all $R\geq r$, which yields corresponding homotopy/(co)homology homomorphisms. The resulting infinite commuting ladder of homomorphisms defines a homomorphism of the inverse/direct limit groups. □

Recall that $\boldsymbol{\pi_k}[\mathbb{G}_\infty(\mathbf{a})]$:= inverse limit of $\pi_k[\mathbb{G}_r(\mathbf{a})]$ as $r\to\infty$.

Likewise define $\mathcal{H}^k[\mathbb{G}_\infty(\mathbf{a})]$ (direct limit) and $\mathcal{H}_k[\mathbb{G}_\infty(\mathbf{a})]$ (inverse limit), $\forall\, k\in\mathbb{N}$.

If $\mathbf{a}\in\widetilde{\mathfrak{A}}$, then $\mathbf{a}$ defines 'partial' Π-section $\varsigma_{\mathbf{a}} : \mathbf{G}_R(\mathbf{a})\longrightarrow\mathbf{X}_R$ for all $R\geq r$. This induces group homomorphisms:

$$\begin{aligned} \mathcal{H}_k\mathbf{a}\colon \mathcal{H}_k[\mathbb{G}_R(\mathbf{a}),\mathcal{G}] &\longrightarrow \mathcal{H}_k(\mathbf{X}_R,\mathcal{G});\\ \mathcal{H}^k\mathbf{a}\colon \mathcal{H}^k(\mathbf{X}_R,\mathcal{G}) &\longrightarrow \mathcal{H}^k[\mathbb{G}_R(\mathbf{a}),\mathcal{G}];\\ \pi_k\mathbf{a}\colon \pi_k[\mathbb{G}_R(\mathbf{a})] &\longrightarrow \pi_k(\mathbf{X}_R). \end{aligned}$$

The resulting infinite commuting ladders of homomorphisms define homomorphisms of the inverse/direct limit groups. Thus, we have:

Theorem: (a) *Any* $\mathbf{a}\in\widetilde{\mathfrak{A}}$ *induces group homomorphisms:*
$\boldsymbol{\mathcal{H}_k\mathbf{a}}\colon \mathcal{H}_k[\mathbb{G}_\infty(\mathbf{a}),\mathcal{G}]\longrightarrow\mathcal{H}_k(\mathfrak{A},\mathcal{G})$ *and* $\boldsymbol{\mathcal{H}^k\mathbf{a}}\colon \mathcal{H}^k(\mathfrak{A},\mathcal{G})\longrightarrow\mathcal{H}^k[\mathbb{G}_\infty(\mathbf{a}),\mathcal{G}]$.
(b) *If* $\mathfrak{A}$ *is basepoint-free in dimension* k*, then* $\mathbf{a}$ *also induces a group homomorphism* $\boldsymbol{\pi_k\mathbf{a}} : \pi_k[\mathbb{G}_\infty(\mathbf{a})]\longrightarrow\pi_k(\mathfrak{A})$.

We call $\pi_k\mathbf{a}$ (resp. $\mathcal{H}_k\mathbf{a}$ or $\mathcal{H}^k\mathbf{a}$) the **kth homotopy** (resp. **(co)homology**) **signature** of $\mathbf{a}$; if it is nontrivial, we say $\mathbf{a}$ has a **homotopy** (resp. (**co**)**homology**) **defect** of codimension $(k+1)$.

Persistence of Homotopy/(co)homology Defects

Theorem: *Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be SFT. Let $\Phi\colon \mathcal{A}^{\mathbb{Z}^D} \to \mathcal{A}^{\mathbb{Z}^D}$ be CA with $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$.*

(a) *Suppose $\mathfrak{A}$ is basepoint-free in dimension k. If $\pi_k\Phi : \pi_k(\mathfrak{A}) \longrightarrow \pi_k(\mathfrak{A})$ is injective, then every homotopy defect of codimension $(k+1)$ is Φ-persistent.*

(b) *If $\mathcal{H}_k\Phi : \mathcal{H}_k(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{H}_k(\mathfrak{A}, \mathcal{G})$ is injective, then every homology defect of codimension $(k+1)$ is Φ-persistent.*

(c) *If $\mathcal{H}^k\Phi : \mathcal{H}^k(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{H}^k(\mathfrak{A}, \mathcal{G})$ is surjective, then every cohomology defect of codimension $(k+1)$ is Φ-persistent.* □

This follows from:

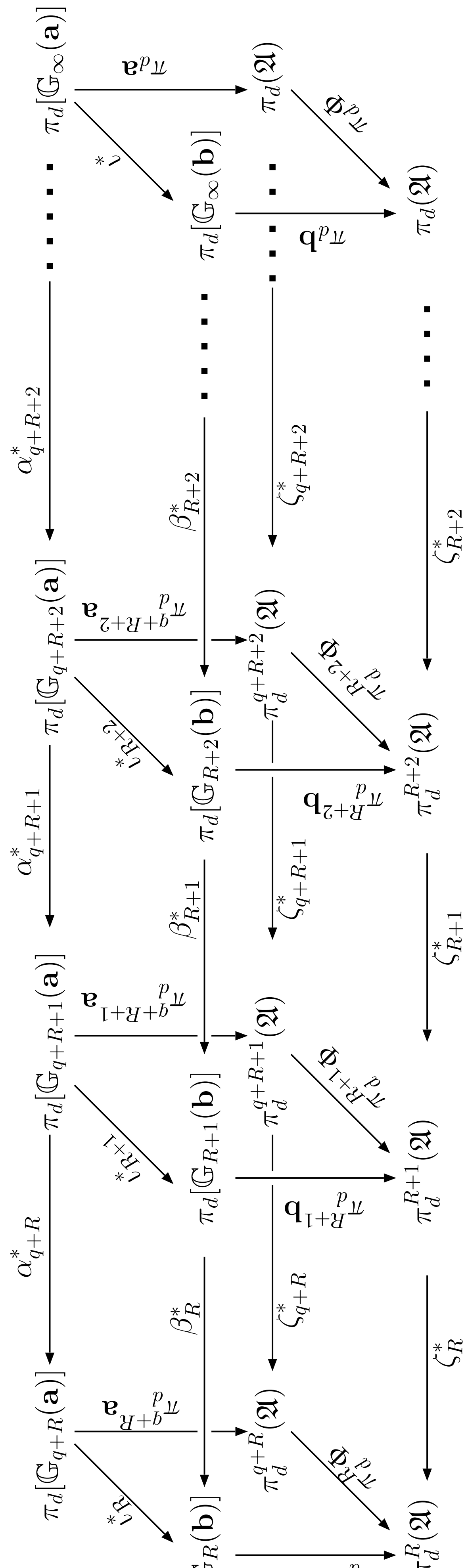
Theorem: *Let $\Phi\colon \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA with $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$. Let $\mathbf{a} \in \widetilde{\mathfrak{A}}$ and let $\Phi(\mathbf{a}) = \mathbf{b}$. Then we have commuting diagrams:*

$$
\begin{array}{ccc}
\mathcal{H}_k[\mathbb{G}_\infty(\mathbf{a}), \mathcal{G}] & \xrightarrow{\mathcal{H}_k\iota} & \mathcal{H}_k[\mathbb{G}_\infty(\mathbf{b}), \mathcal{G}] \\
{\scriptstyle \mathcal{H}_k\mathbf{a}}\downarrow & & \downarrow{\scriptstyle \mathcal{H}_k\mathbf{b}} \\
\mathcal{H}_k(\mathfrak{A}, \mathcal{G}) & \xrightarrow{\mathcal{H}_k\Phi} & \mathcal{H}_k(\mathfrak{A}, \mathcal{G})
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{H}^k[\mathbb{G}_\infty(\mathbf{a}), \mathcal{G}] & \xleftarrow{\mathcal{H}^k\iota} & \mathcal{H}^k[\mathbb{G}_\infty(\mathbf{b}), \mathcal{G}] \\
{\scriptstyle \mathcal{H}^k\mathbf{a}}\uparrow & & \uparrow{\scriptstyle \mathcal{H}^k\mathbf{b}} \\
\mathcal{H}^k(\mathfrak{A}, \mathcal{G}) & \xleftarrow{\mathcal{H}^k\Phi} & \mathcal{H}^k(\mathfrak{A}, \mathcal{G})
\end{array}
$$

If $\mathfrak{A}$ is basepoint-free, we also get a commuting diagram:

$$
\begin{array}{ccc}
\pi_k[\mathbb{G}_\infty(\mathbf{a}), \omega] & \xrightarrow{\pi_k\iota} & \pi_k[\mathbb{G}_\infty(\mathbf{b}), \omega] \\
{\scriptstyle \pi_k\mathbf{a}}\downarrow & & \downarrow{\scriptstyle \pi_k\mathbf{b}} \\
\pi_k(\mathfrak{A}) & \xrightarrow{\pi_k\Phi} & \pi_k(\mathfrak{A})
\end{array}
$$

Proof: (Idea) Stick together all the aforementioned infinite commuting ladders to get infinite commuting 'girder', which yields commuting square of inverse limit homomorphisms. □


$\pi_d[\mathbb{G}_\infty(\mathbf{a})]$
$\pi_d\mathbf{a}$
$\pi_d(\mathfrak{A})$
$\pi_d\Phi$
ι^*
$\pi_d[\mathbb{G}_\infty(\mathbf{b})]$
$\pi_d\mathbf{b}$
$\pi_d(\mathfrak{A})$
α^*_{q+R+2}
β^*_{R+2}
ζ^*_{q+R+2}
ζ^*_{R+2}
$\pi_d[\mathbb{G}_{q+R+2}(\mathbf{a})]$
$\pi^d_{q+R+2}\mathbf{a}$
$\pi_d^{q+R+2}(\mathfrak{A})$
$\pi^d_{R+2}\Phi$
ι^*_{R+2}
$\pi_d[\mathbb{G}_{R+2}(\mathbf{b})]$
$\pi^d_{R+2}\mathbf{b}$
$\pi_d^{R+2}(\mathfrak{A})$
α^*_{q+R+1}
β^*_{R+1}
ζ^*_{q+R+1}
ζ^*_{R+1}
$\pi_d[\mathbb{G}_{q+R+1}(\mathbf{a})]$
$\pi^d_{q+R+1}\mathbf{a}$
$\pi_d^{q+R+1}(\mathfrak{A})$
$\pi^d_{R+1}\Phi$
ι^*_{R+1}
$\pi_d[\mathbb{G}_{R+1}(\mathbf{b})]$
$\pi^d_{R+1}\mathbf{b}$
$\pi_d^{R+1}(\mathfrak{A})$
α^*_{q+R}
β^*_R
ζ^*_{q+R}
ζ^*_R
$\pi_d[\mathbb{G}_{q+R}(\mathbf{a})]$
$\pi^d_{q+R}\mathbf{a}$
$\pi_d^{q+R}(\mathfrak{A})$
$\pi^d_R\Phi$
ι^*_R
$\pi_d^R(\mathfrak{A})$