Bifurcations in a differentially heated rotating spherical shell

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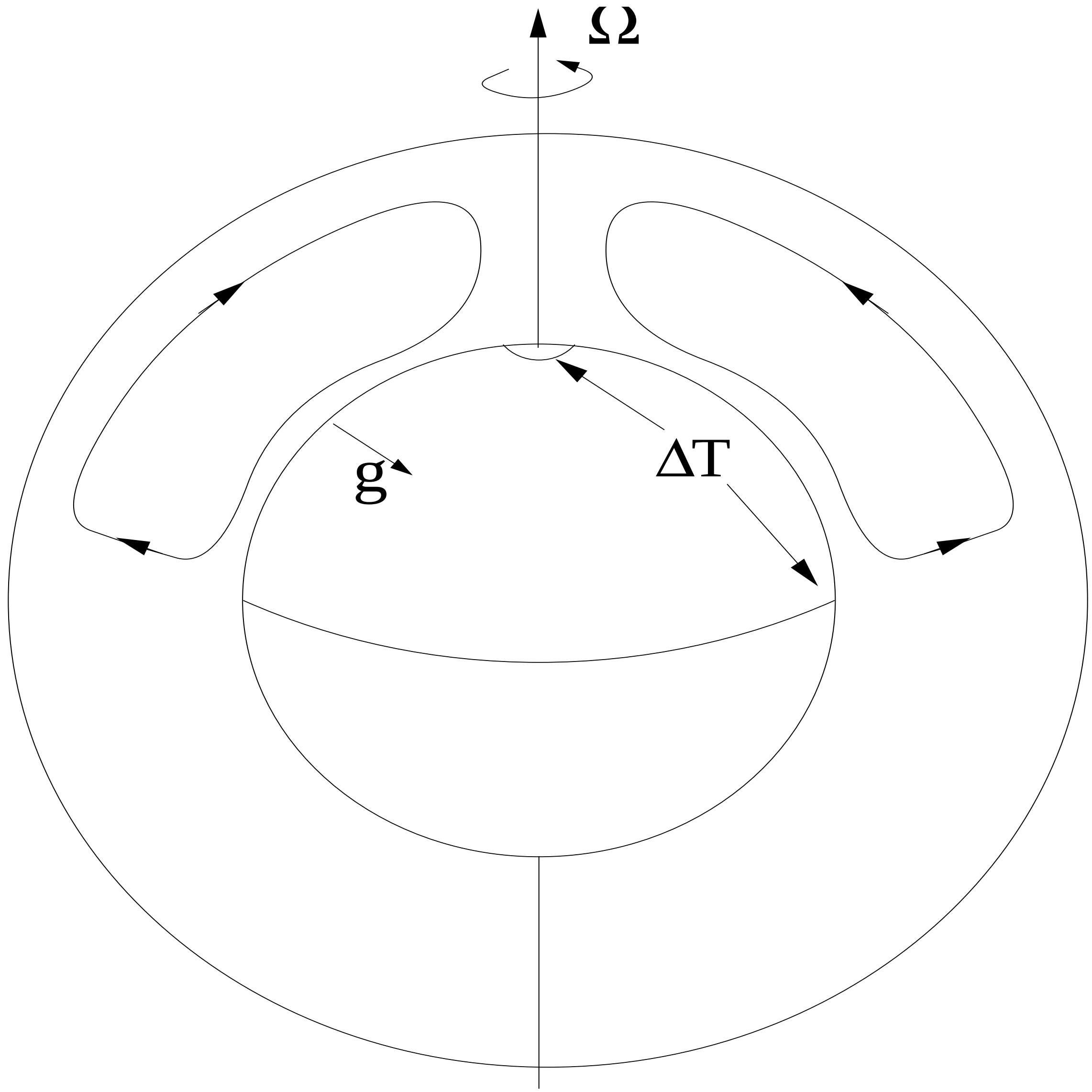
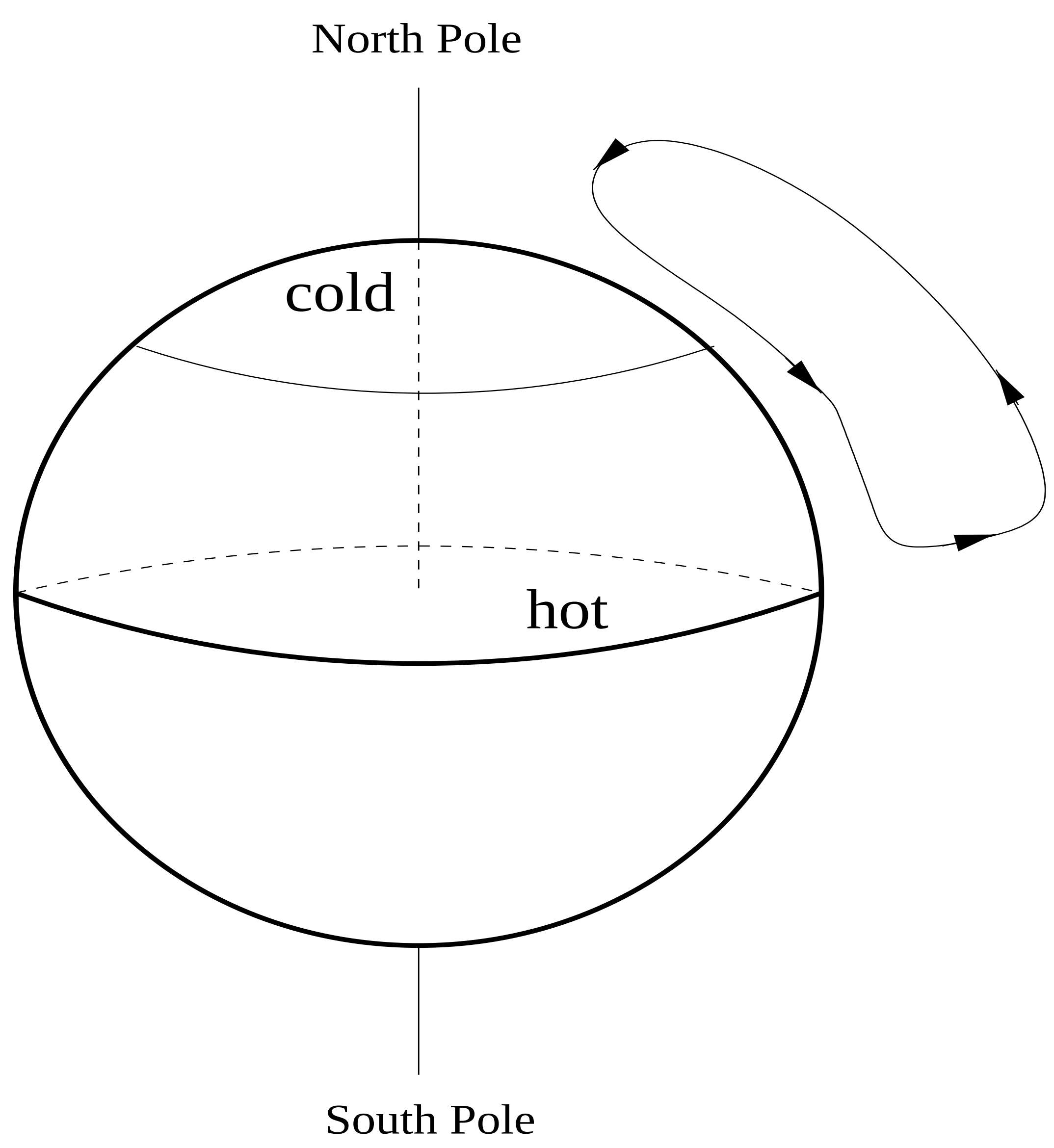
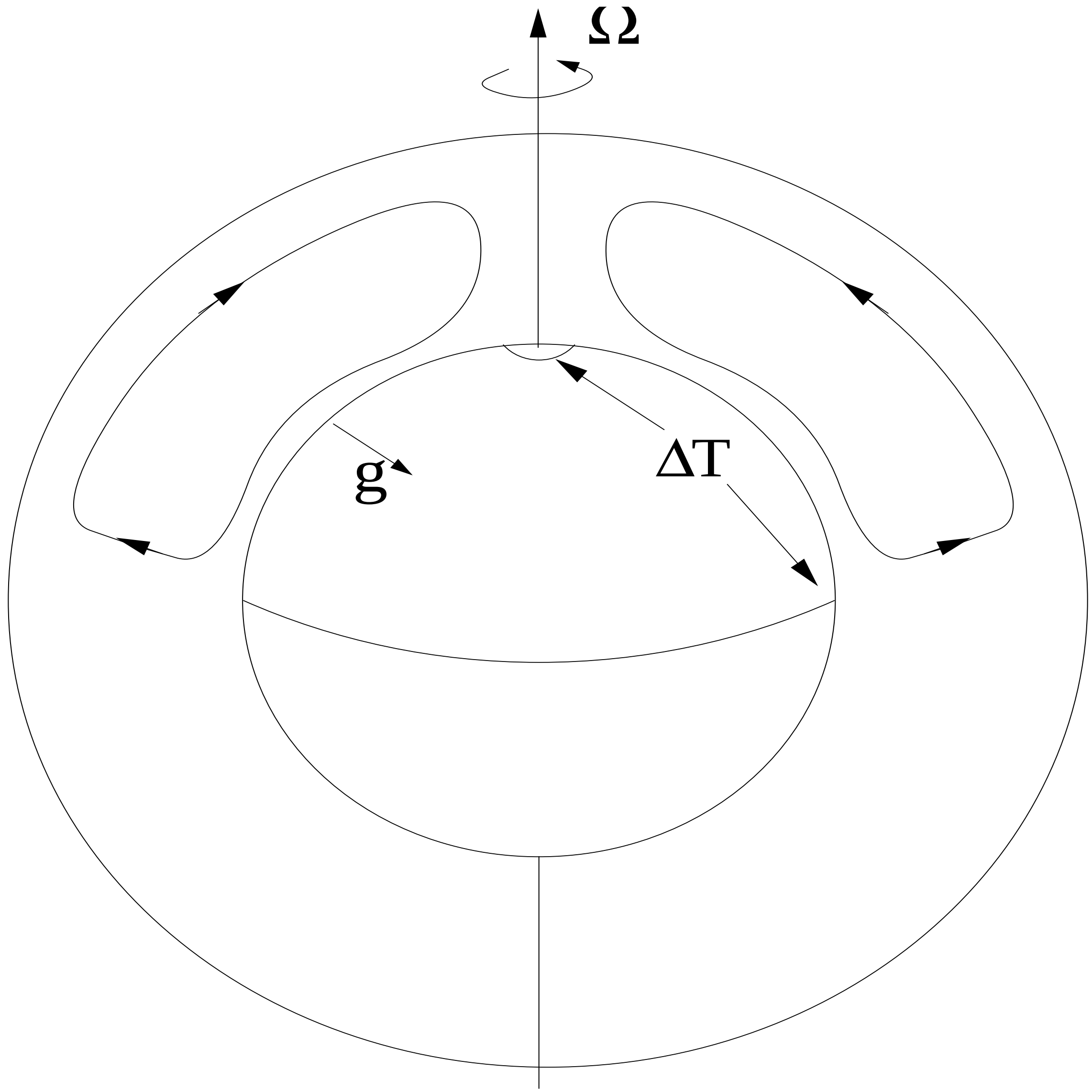
Spherical Shell



A differentially heated rotating planet



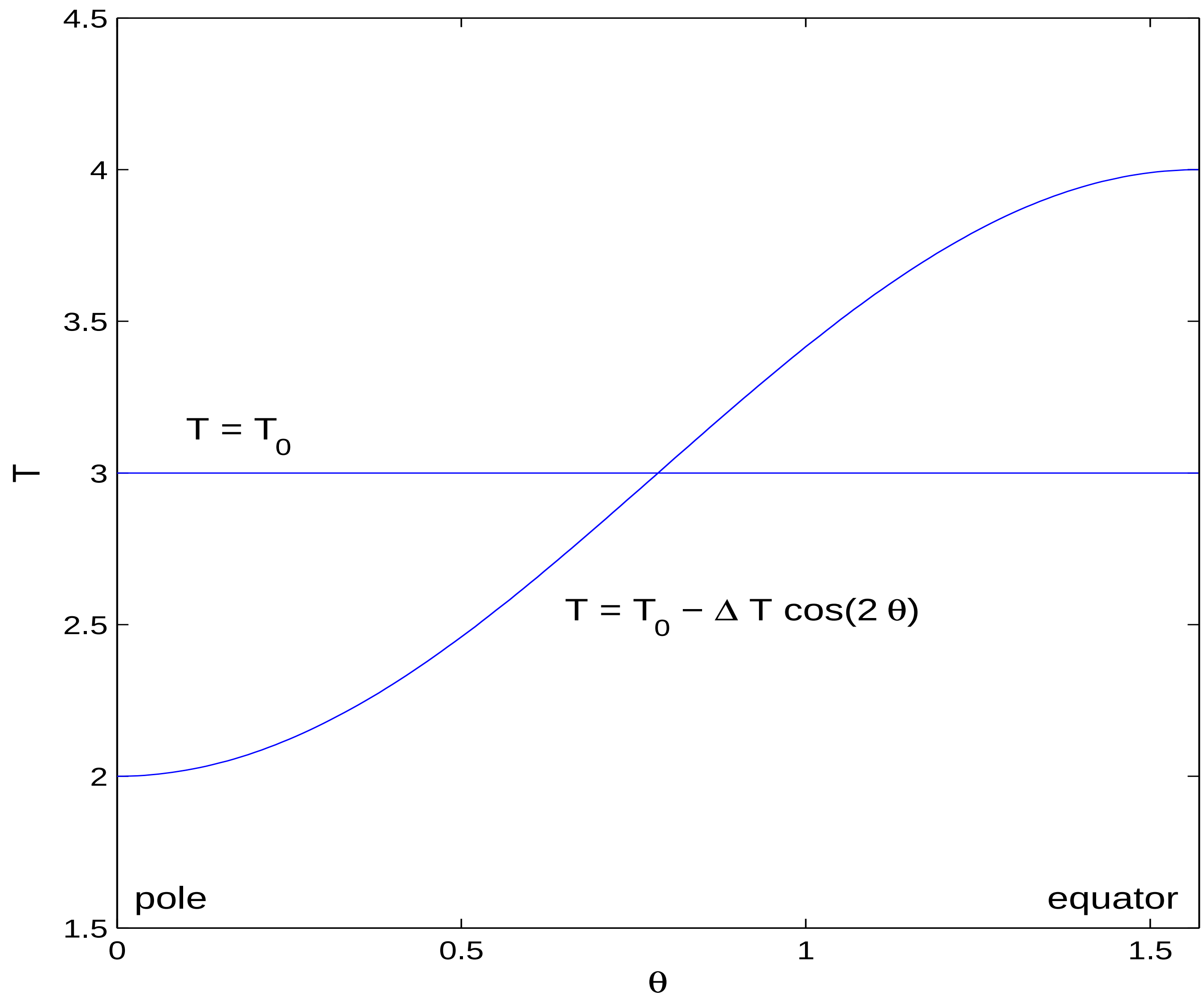
Spherical Shell



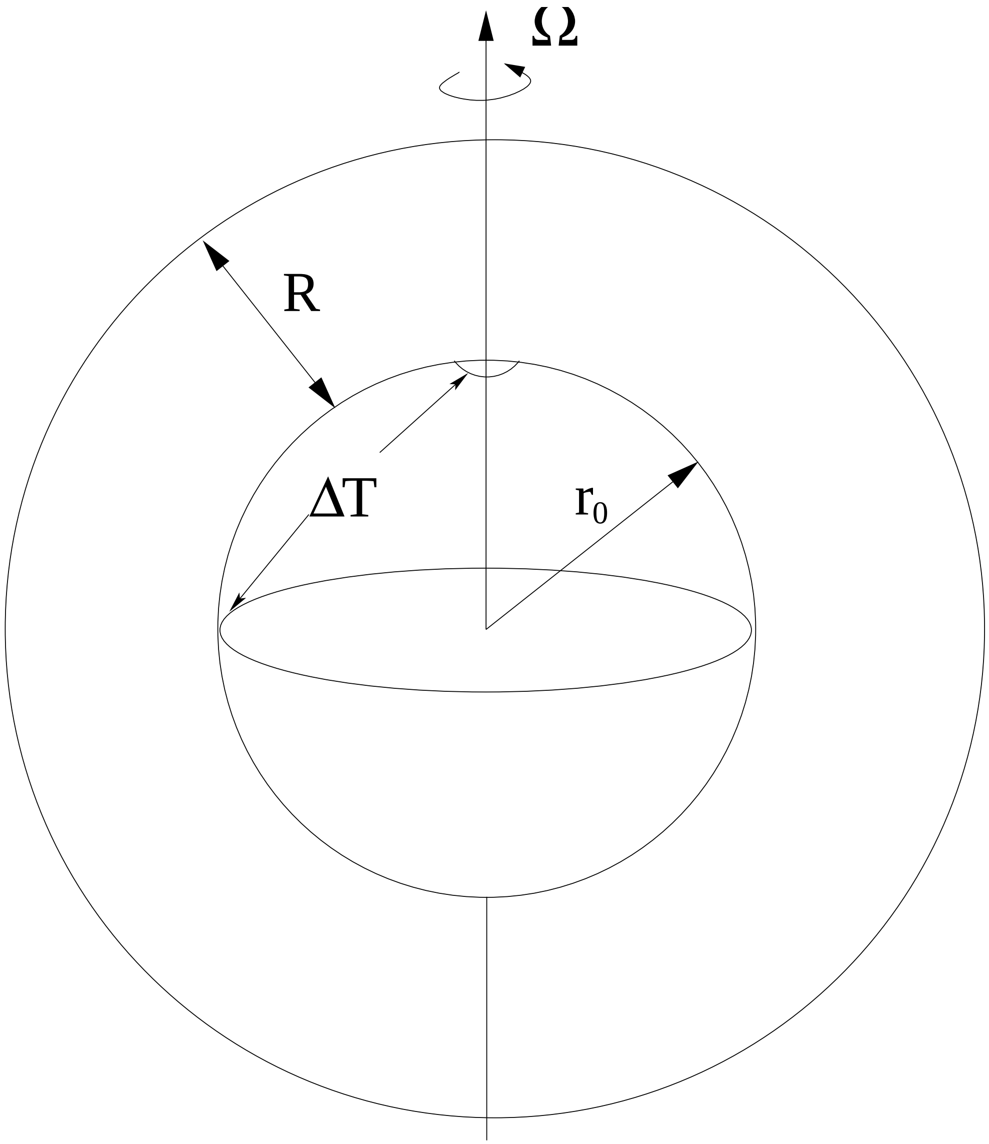
Model of fluid in a spherical shell

- Navier-Stokes equations in the Boussinesq approximation
- Spherical polar coordinates and rotating frame of reference
- No-slip boundary conditions at inner sphere
- Stress-free boundary condition at outer sphere
- Insulating outer sphere
- Differential heating imposed on inner sphere: at $r = r_0, \; T = T_0 - \Delta T \cos(2\theta)$.

Differential heating



Spherical shell



$$\boxed{\eta = \frac{R}{r_0}}$$

Analysis

- Look for steady flows invariant under rotation and reflection about equator
  - Reduces to problem in two-spatial dimensions
  - Introduces additional boundary conditions at pole and equator
- Bifurcations of steady solutions

Numerical computations

- Steady solutions
  - use pseudo-arclength continuation
- Linear stability: eigenvalues
  - Implicitly restarted Arnoldi method
  - with Cayley transformations

Numerical computations

- Need to solve systems of 3 steady PDEs in 2 spatial dimensions
- Discretize on an $N \times N$ grid
- 2nd order centred finite differences
  - leads to large and sparse matrices
  - non-symmetric matrices

Steady solution: continuation

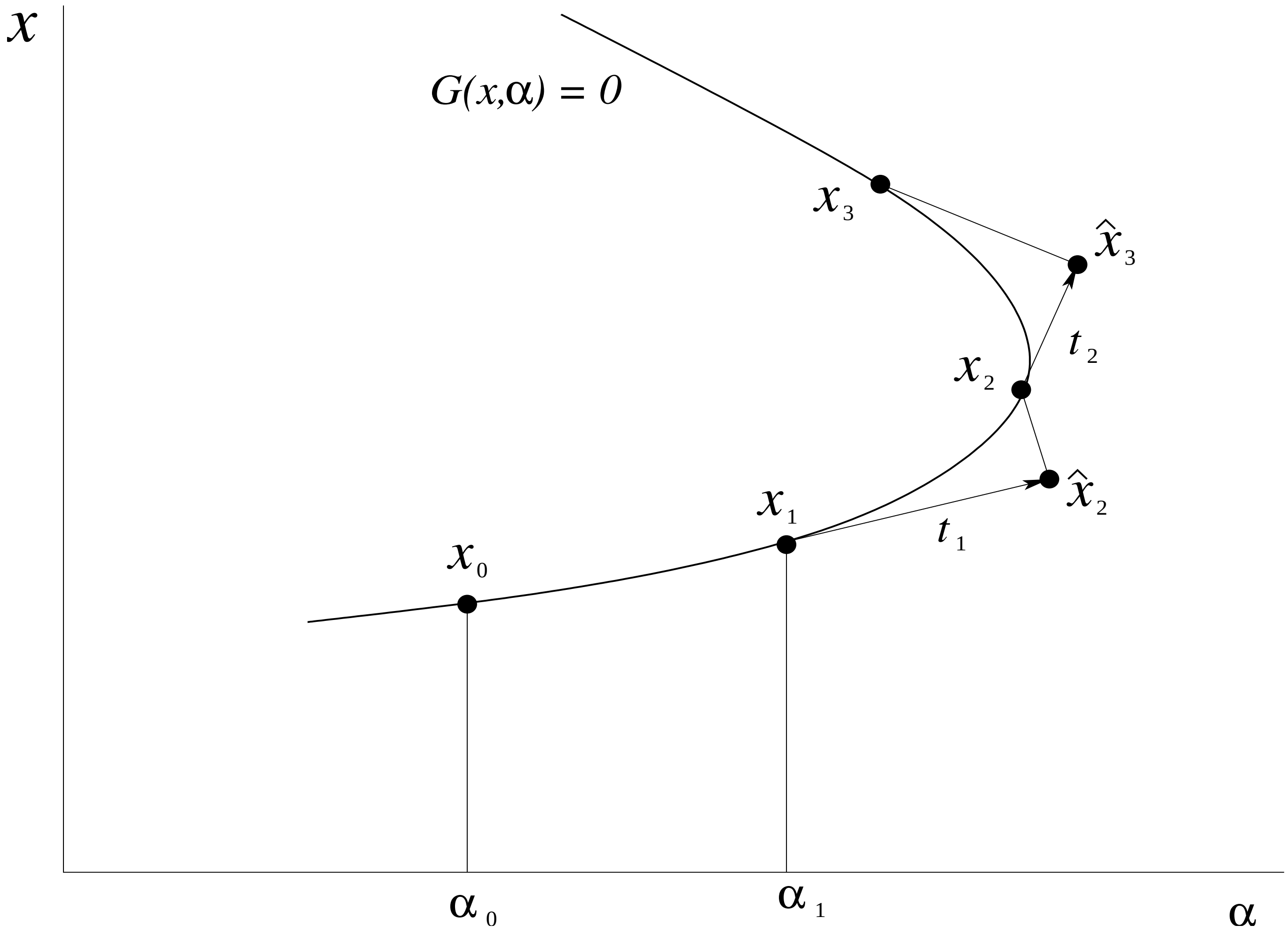
- Look for steady solutions
  - discretization reduces PDE to system of nonlinear algebraic equations
  - need to solve $G(x, \alpha) = 0$, $x \in \mathcal{R}^n$, $\alpha \in \mathcal{R}$
- Use Newton's method with continuation
  - need to have a good guess
  - assume we know x_0 at α_0 such that $G(x_0, \alpha_0) = 0$

Pseudo-arclength continuation

- Consider the parameter α as an unknown
- predictor: follow tangent t_1 to get new guess $\hat{x}_2$
- for correction, add an extra condition to get new system:

$$\begin{aligned} G(x,\alpha) &= 0 \\ f(x,\alpha) &= 0 \end{aligned}$$

Pseudo-arclength continuation



Eigenvalue approximation

- Eigenvalue problem
  - Linearize about steady solution
  - get generalized eigenvalue problems

$$\lambda \mathbf{M} \Phi = \mathbf{L} \Phi$$

  - discretization leads to matrix eigenvalue problems

Eigenvalue approximation

- For eigenvalues use ‘Implicitly restarted Arnoldi method’
  - iterative
  - memory efficient
  - finds extremal eigenvalues

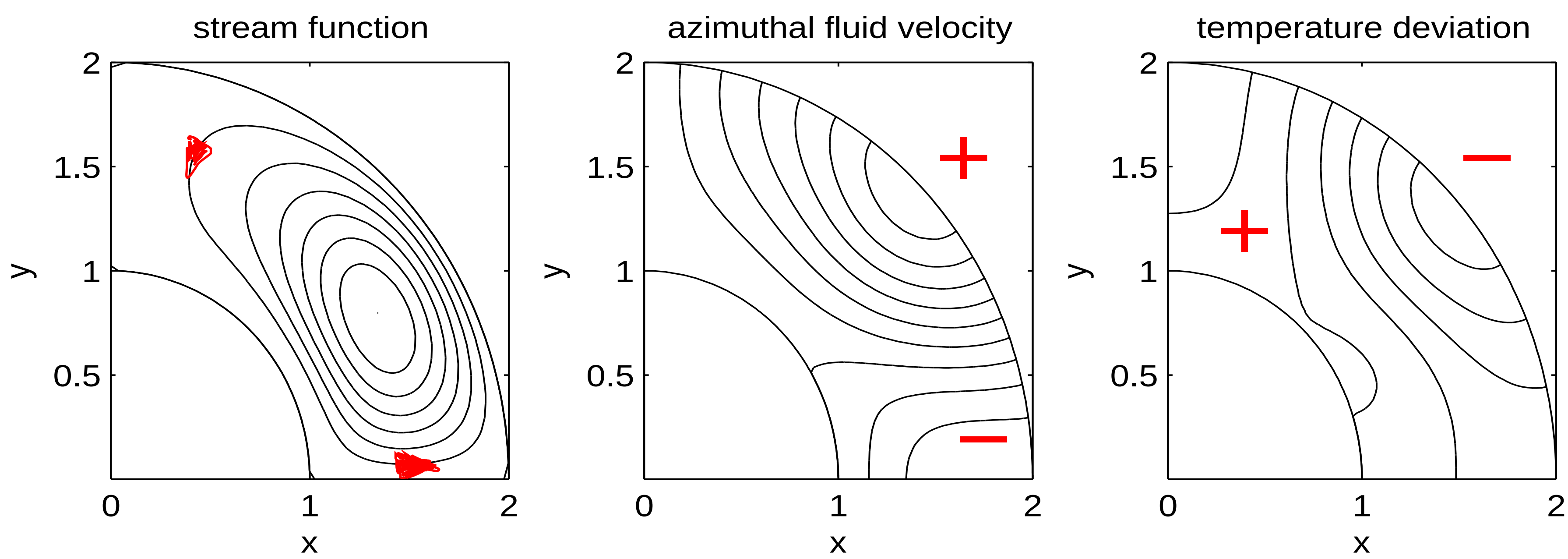
Eigenvalue approximation

- Use generalized Cayley transform

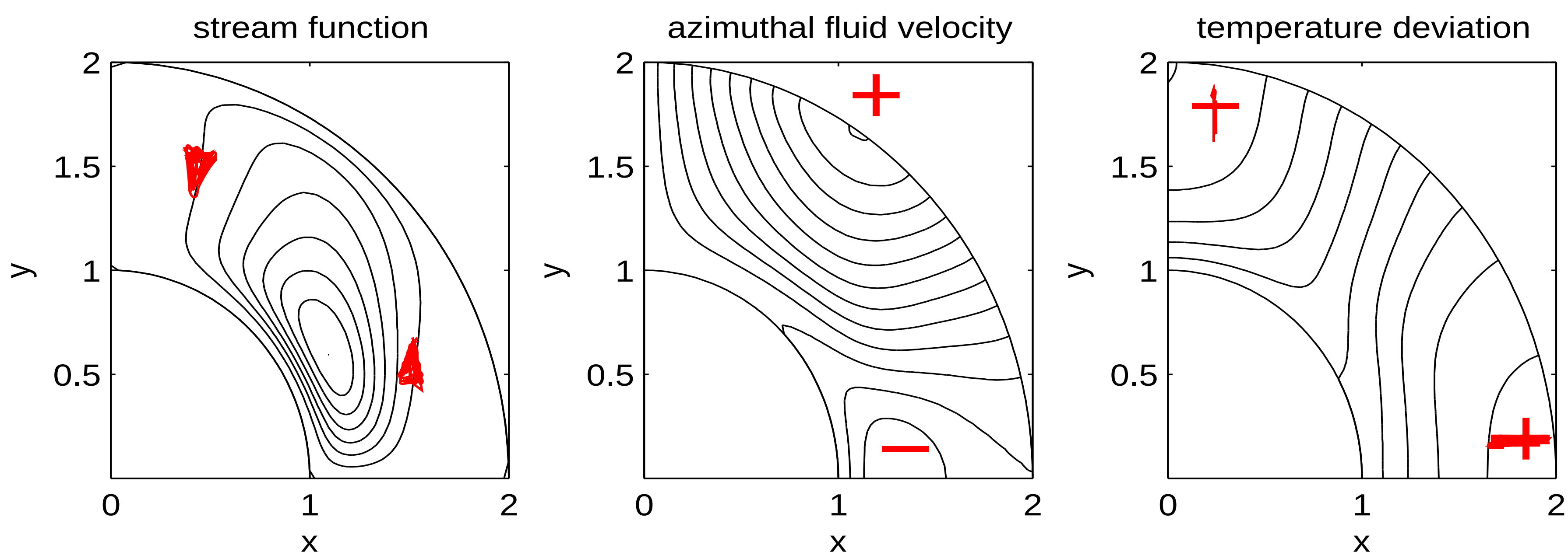
$$\mathbf{C}(\mathbf{A},\mathbf{B}) = (\mathbf{A} - \alpha_1 \mathbf{B})^{-1} (\mathbf{A} - \alpha_2 \mathbf{B})$$

  - λ are eigenvalues from $\lambda \mathbf{B} x = \mathbf{A} x$
  - μ are eigenvalues from $\mu x' = \mathbf{C} x'$
  - $Real(\lambda) > \dfrac{\alpha_1 + \alpha_2}{2} \rightarrow |\mu| > 1$

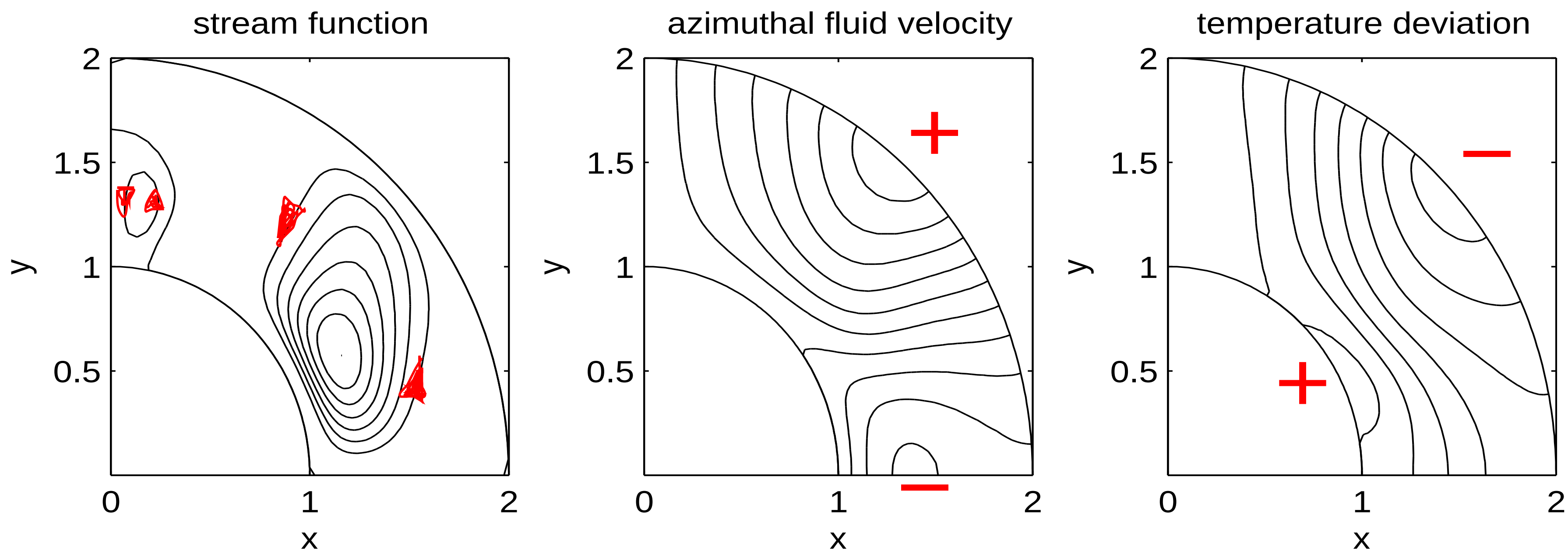
Steady Solution: ‘Hadley Cell’



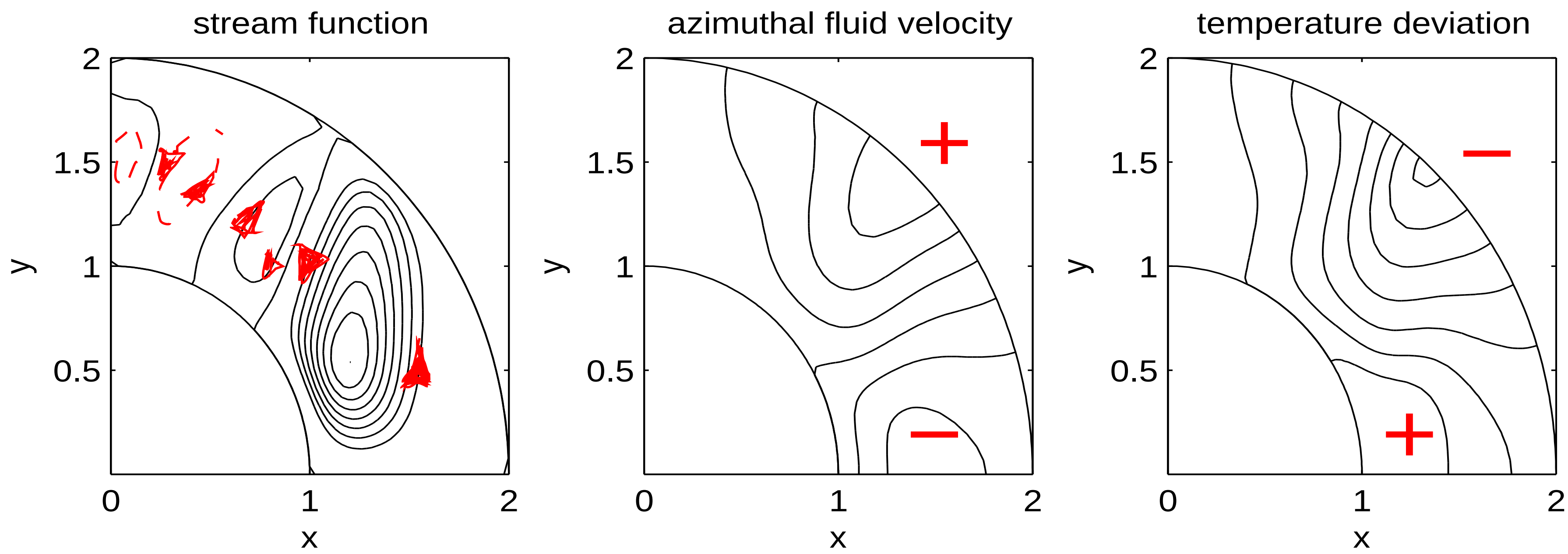
Steady Solution: $\eta = R/r_0 = 1/2$, $\Delta T = 0.004$



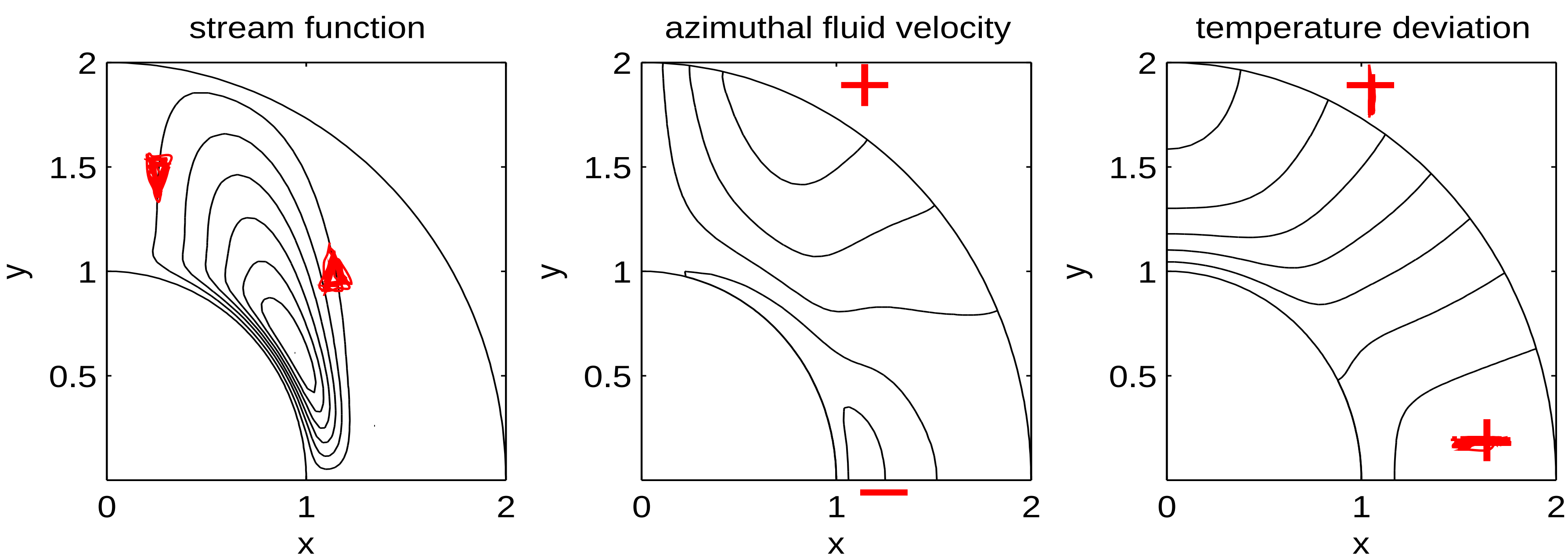
Steady Solution: $\eta = R/r_0 = 1/2$, $\Delta T = 0.026$



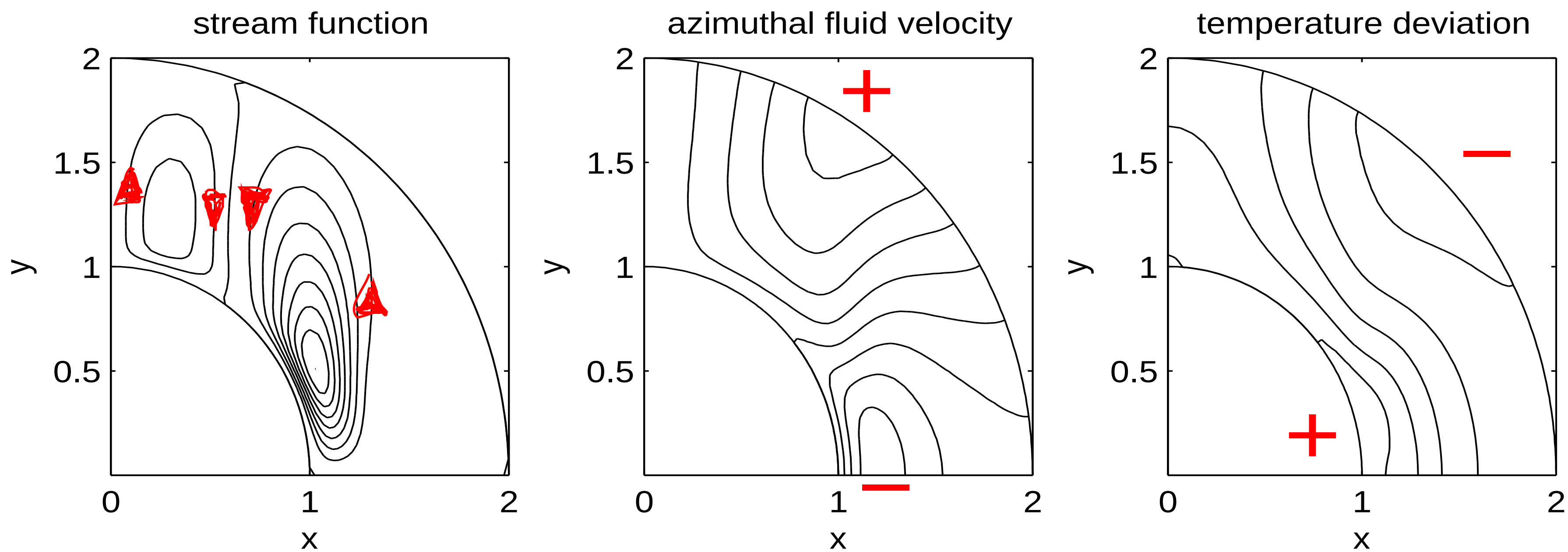
Steady Solution: $\eta = R/r_0 = 1/2$, $\Delta T = 0.0483$



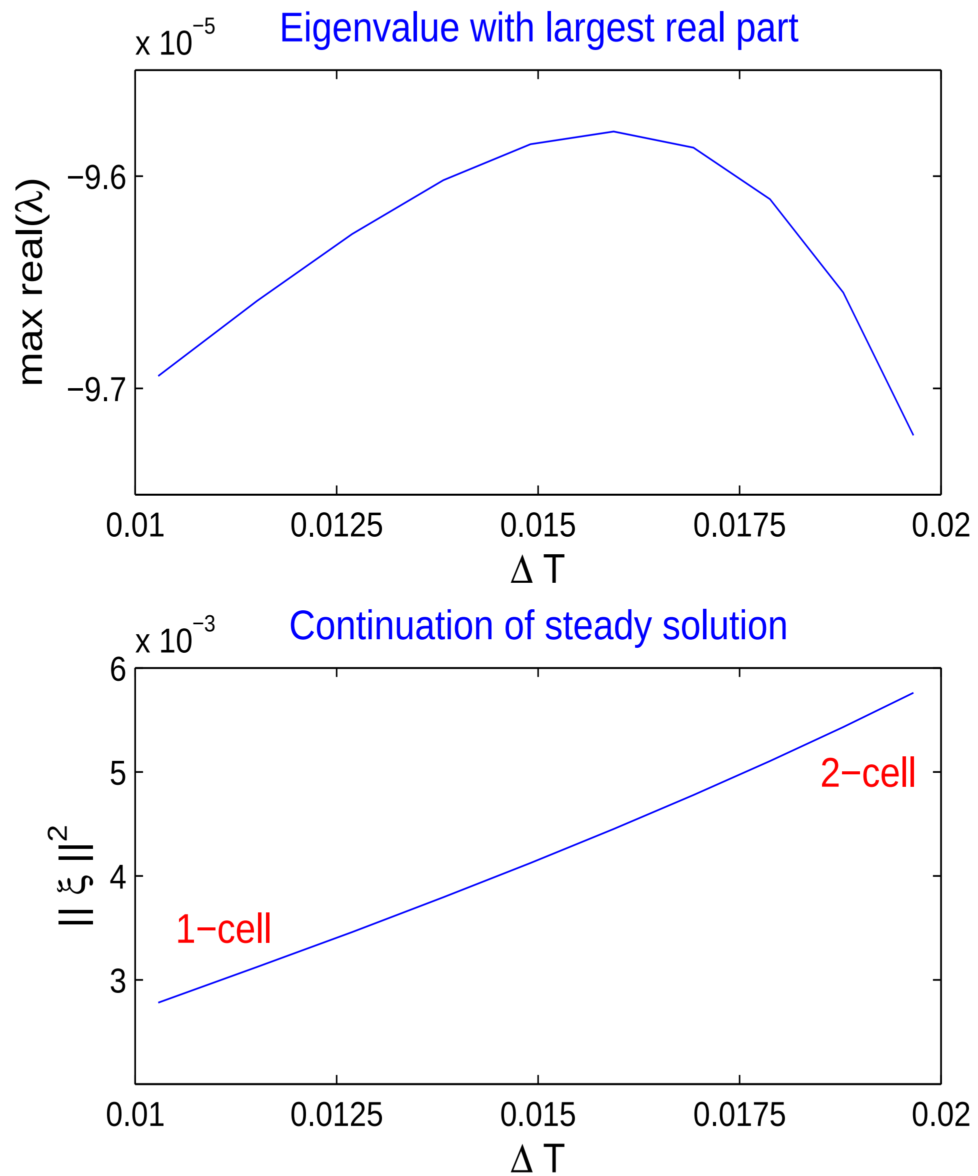
Steady Solution: $\eta = R/r_0 = 1, \Delta T = 0.002$



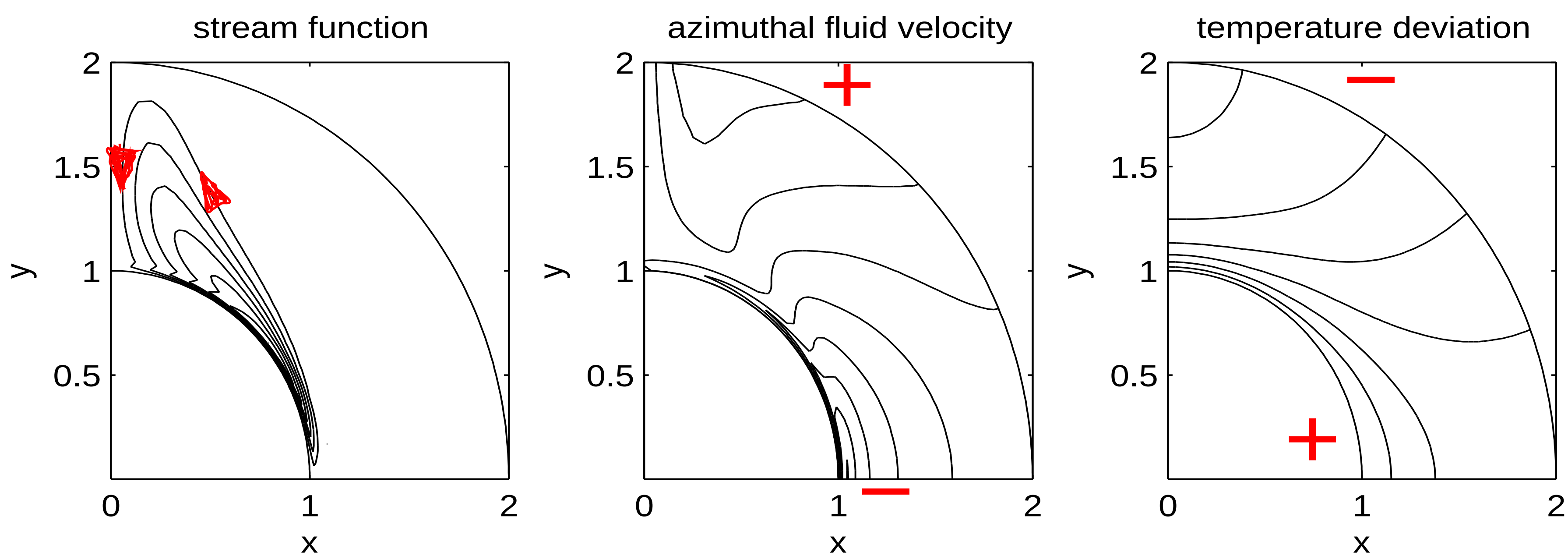
Steady Solution: $\eta = R/r_0 = 1$, $\Delta T = 0.029$



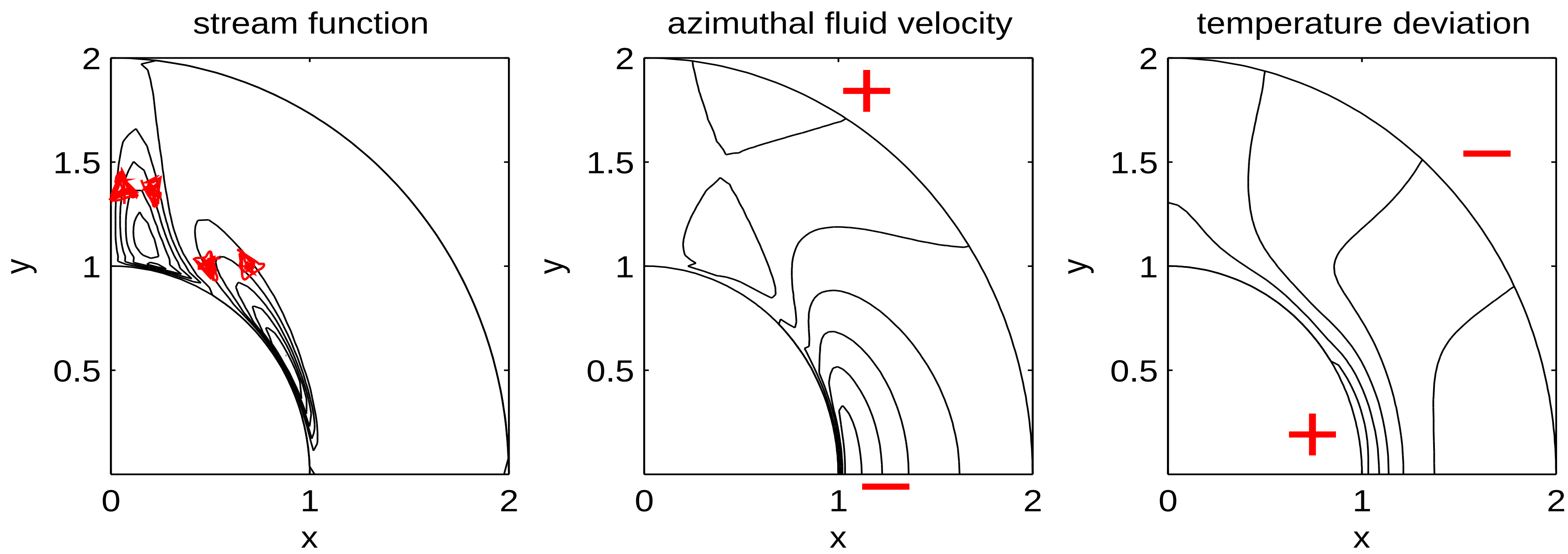
Bifurcation Diagram: $\eta = R/r_0 = 1$



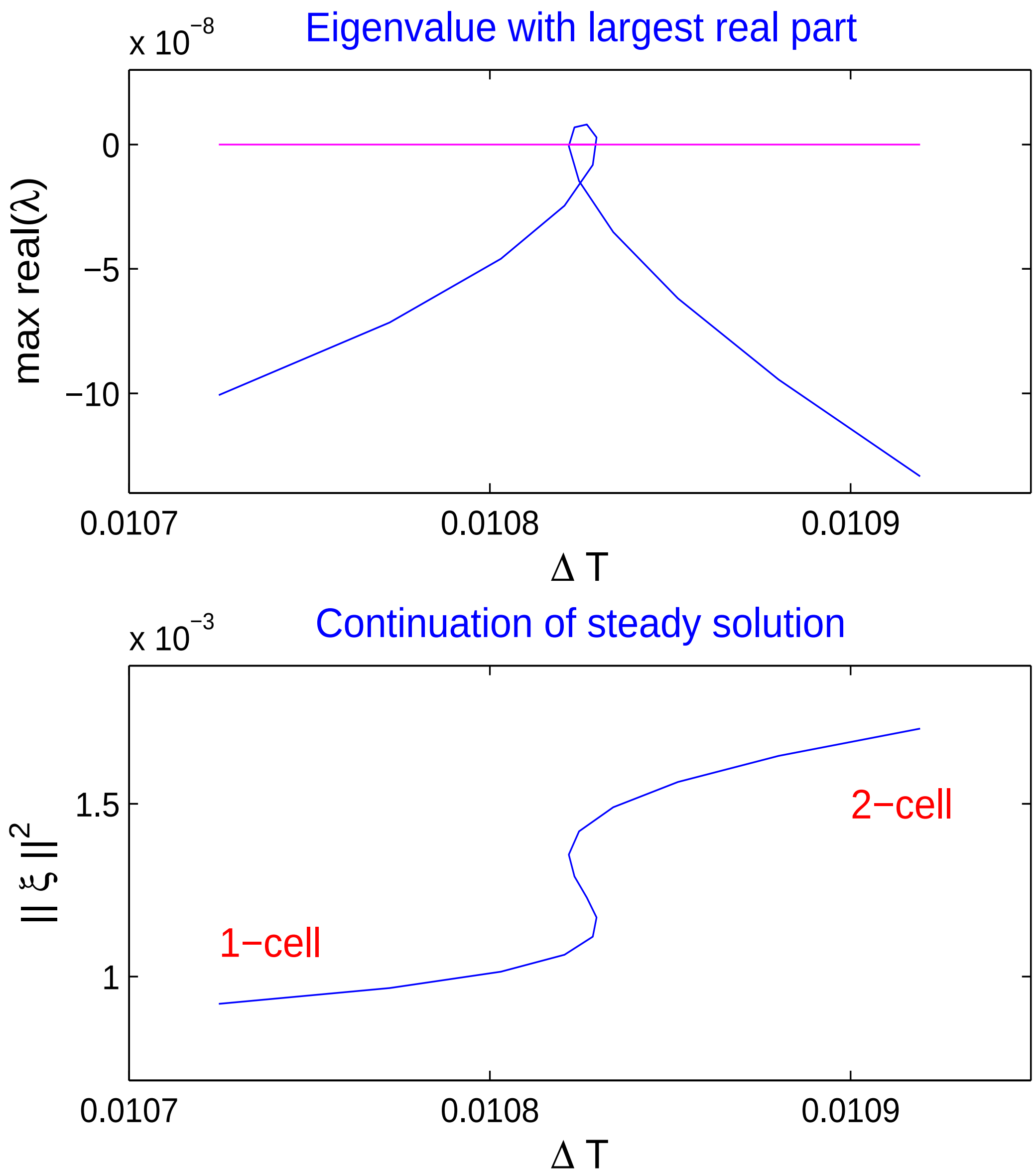
Steady Solution: $\eta = R/r_0 = 3.5$, $\Delta T = 0.001$



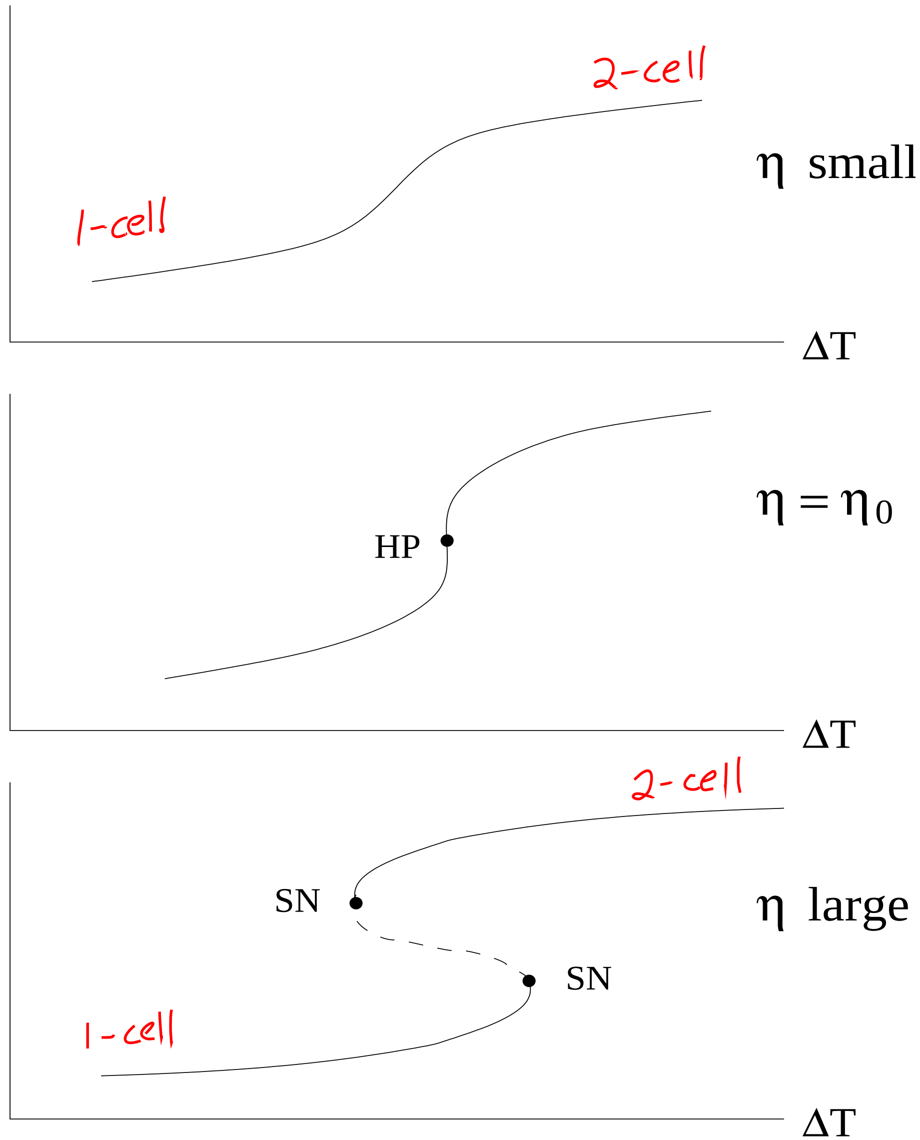
Steady Solution: $\eta = R/r_0 = 3.5$, $\Delta T = 0.019$



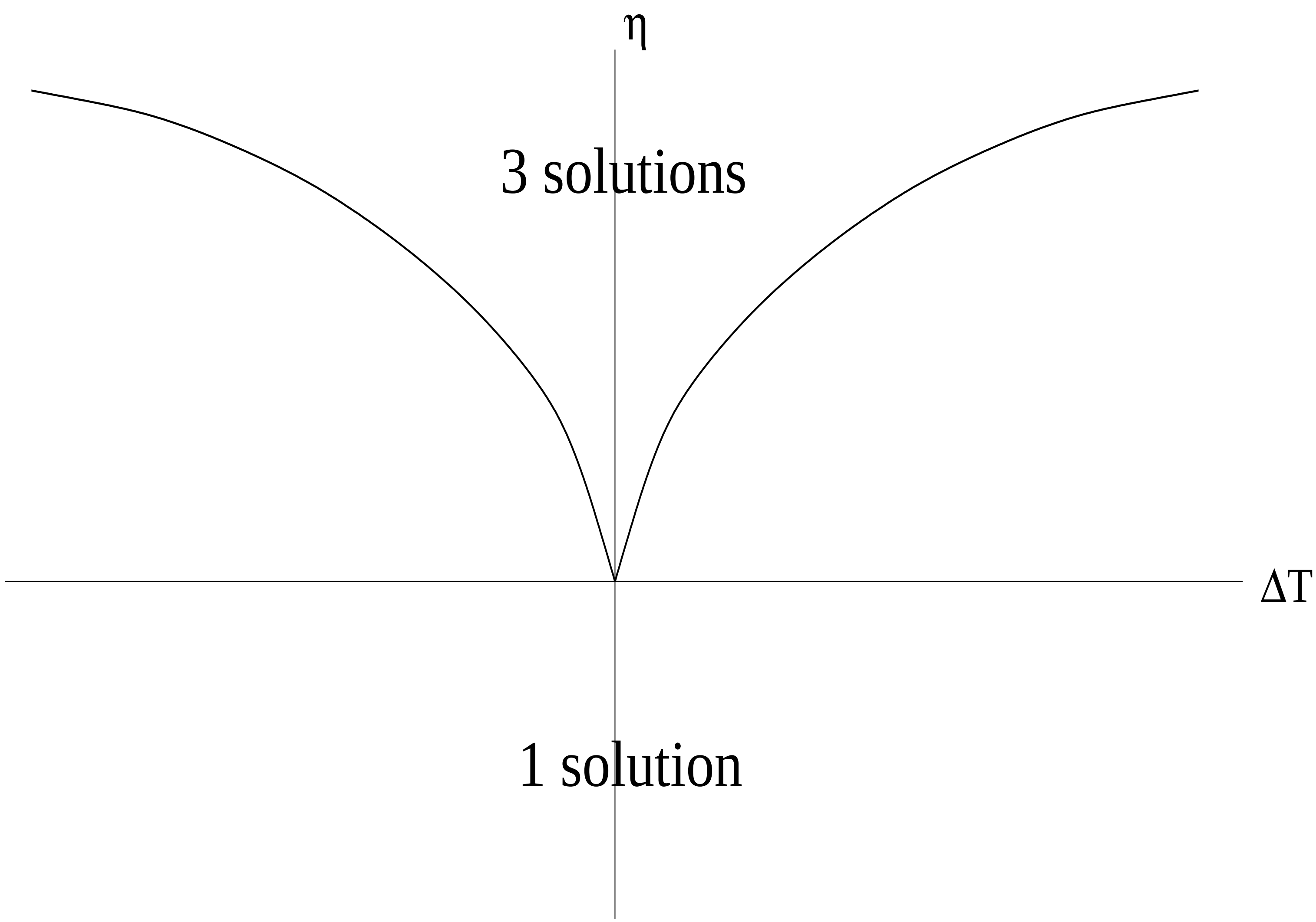
Bifurcation Diagram: $\eta = R/r_0 = 3.5$



Cusp bifurcation



Cusp bifurcation (schematic)



Computation of cusp point

- Codimension two bifurcation
  - Need two parameters: ΔT and η
- Write equations as:

$$\dot{U} = LU + N(U, U)$$

where U is dependent variable,
LU is linear part, $N(U, U)$ is nonlinear part,
and $\dot{U}$ is derivative with respect to time

Computation of cusp point

- Cusp point is characterized by:
  1. $LU_0 + N(U_0, U_0) = 0$
  2. zero eigenvalue of L_0 where
     $L_0 V = LV + N(V, U_0) + N(U_0, V)$
  3. vanishing of the coefficient of 2nd-order term of equation on centre manifold (or reduced equation)

Reduced equation

- Reduced equation

$$\dot{w} = \beta_1 + \beta_2 w + a w^2 + c w^3$$

where

$$a = 1/2 \langle \Phi^*, N(\Phi, \Phi) \rangle = 0$$

Φ is the eigenfunction corresponding to $\lambda = 0$,
Φ^* is the corresponding adjoint eigenfunction,
$\langle \cdot, \cdot \rangle$ is the inner product

Defining system

$$LU_0 + N(U_0, U_0) = 0, \quad g = 0, \quad g' = 0$$

where g and g' are scalars given by

$$L_0 V + gB = 0, \quad \langle C, V \rangle = 1$$

$$L_0 V' + g'B = -N(V, V), \quad \left\langle C, V' \right\rangle = 0$$

where B not in range of L_0,
and C not in range of the adjoint operator L_0^*.

- Solve to get $a = 0$ at $\eta = 3.46$, $\Delta T = 0.011$

Computation of cusp point

- Cusp point is characterized by:
  1. $LU_0 + N(U_0, U_0) = 0$
  2. zero eigenvalue of L_0 where
     $L_0 V = LV + N(V, U_0) + N(U_0, V)$
  3. vanishing of the coefficient of 2nd-order term of equation on centre manifold (or reduced equation)

Summary

- Looked for steady solutions with symmetry
- Flow transition without eigenvalue crossing imaginary axis
- Cusp point
- Future Work
  - Cusp for 2 to 3 cell transition
  - Symmetry breaking bifurcations (waves)
  - Hopf bifurcations
  - Other higher codimension bifurcations
    - steady-Hopf bifurcations