Lattice Dynamical Systems

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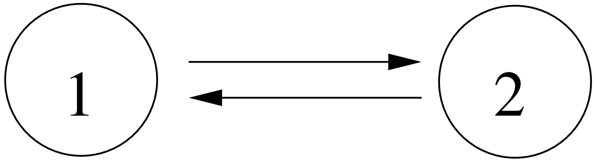
Two Identical Cells



$$
\begin{aligned}
\dot{x}_1 &= f(x_1, x_2) \\
\dot{x}_2 &= f(x_2, x_1)
\end{aligned}
$$

- $\sigma(x_1, x_2) = (x_2, x_1)$ is a symmetry

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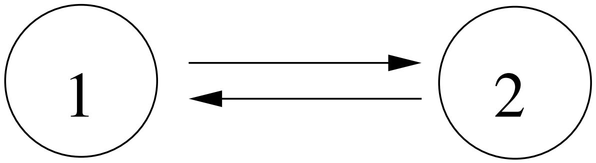

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- $\sigma(x_1, x_2) = (x_2, x_1)$ is a symmetry
- $\mathrm{Fix}(\sigma) = \{x_1 = x_2\}$ is flow invariant

  Synchrony is robust

Symmetry Overview

- A symmetry of a DiffEq $\dot{x} = f(x)$ is a linear map γ where

$$\gamma(\text{sol'n}) = \text{sol'n} \quad \Longleftrightarrow \quad f(\gamma x) = \gamma f(x)$$

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- Symmetry group Γ is a modeling assumption

  Network architecture is also a modeling assumption

Synchrony Subspaces

- A polydiagonal is a subspace

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- A **synchrony subspace** is a flow-invariant polydiagonal

  Synchrony subspaces are

  coupled cell analogs of fixed-point subspaces

Synchrony Subspaces (2)

- Let Δ be a polydiagonal

Stewart, G., and Pivato (2003); G., Stewart, and Török (2005)

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  Coloring is **balanced** if every pair of cells with same color receives equal numbers of inputs from cells of a given color
- **Theorem**: **synchrony subspace** $\Longleftrightarrow$ **balanced**

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Stable Equilibria Exist

Given balanced k-coloring with polydiagonal Δ. Let $X_0 \in \Delta$ be a generic point. Then X_0 is an asymptotically stable equilibrium for some admissible system

- Can assume balanced coloring is associated to homogeneous network with one-dimensional dynamics
- $X_0 \in \Delta$ has at most k distinct components $x_0^1, \ldots, x_0^k$. There exists polynomial $g : \mathbf{R} \to \mathbf{R}$ such that

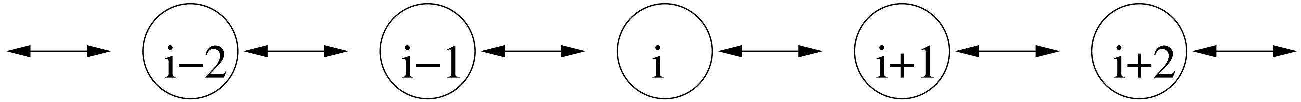
$$g(x_0^i) = 0 \quad \text{and} \quad g'(x_0^i) = -1$$

- System $\dot{x}_i = g(x_i)$ has equilibrium X_0 with Jacobian $-I$
- So X_0 is an asymptotically stable equilibrium.

G., Nicol, and Wang

1D-Lattice Dynamical Systems

- Assume nearest neighbor coupling



$$\dot{x}_i = f(x_i, x_{i-1}, x_{i+1}) \quad \text{where} \quad f(x,y,z) = f(x,z,y)$$

Antoneli, Dias, G. and Wang (2006)

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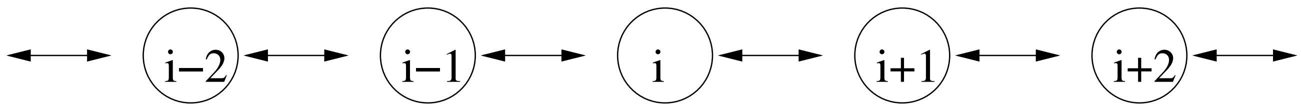

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- There are four balanced k colorings (when $k \geq 3$)

$$\cdots \mathbf{ABC}\ \mathbf{ABC} \cdots \qquad \cdots \mathbf{ABCB}\ \mathbf{ABCB} \cdots$$

$$\cdots \mathbf{ABCBA}\ \mathbf{ABCBA} \cdots \qquad \cdots \mathbf{ABCCBA}\ \mathbf{ABCCBA} \cdots$$

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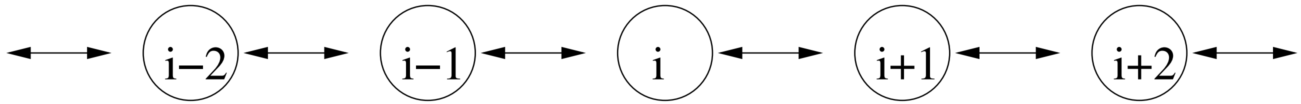

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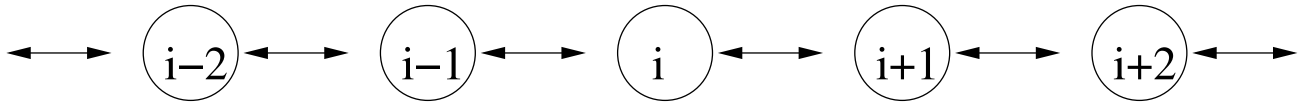

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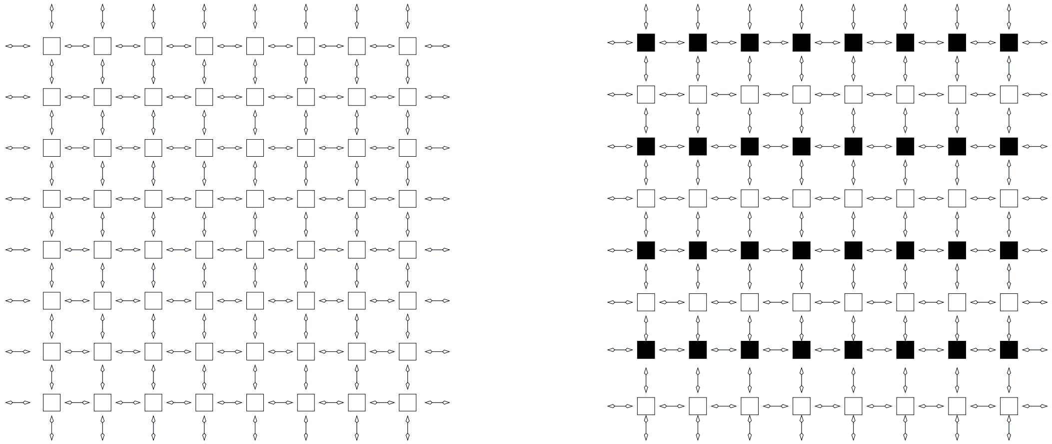
- Every synchrony subspace is a fixed-point subspace
- Every balanced coloring is periodic

Antoneli, Dias, G. and Wang (2006)

2D-Lattice Dynamical Systems

- Consider square lattice with nearest neighbor coupling
- Form a two-color balanced relation

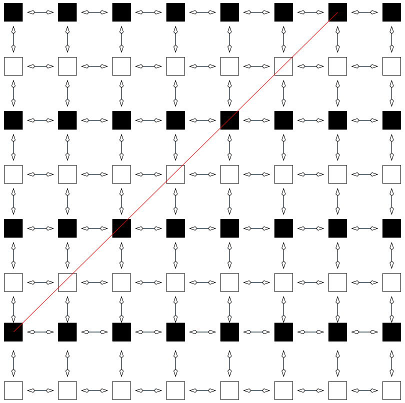
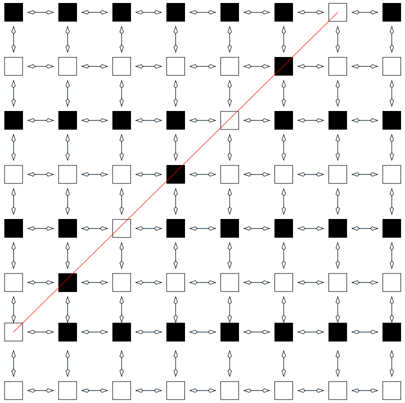


- Each black cell connected to two black and two white
  Each white cell connected to two black and two white

Stewart, G. and Nicol (2004)

Lattice Dynamical Systems (2)

- On Black/White diagonal interchange black and white

Result is balanced

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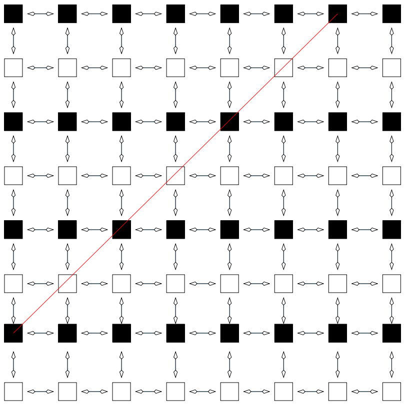
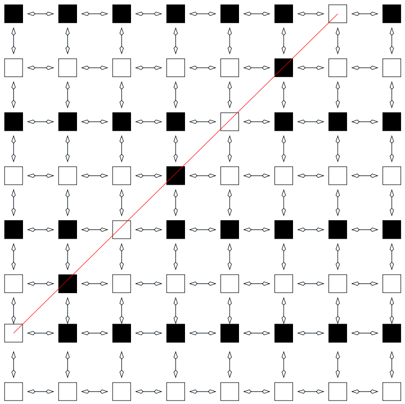
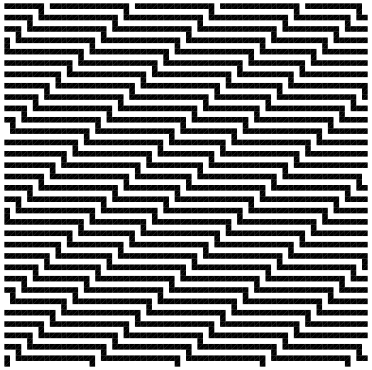
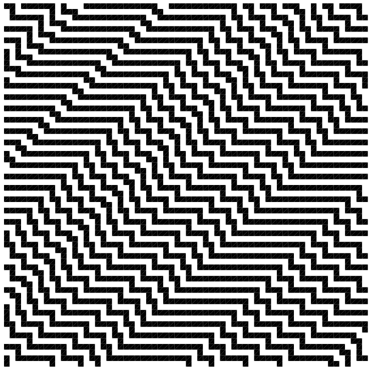




Result is balanced

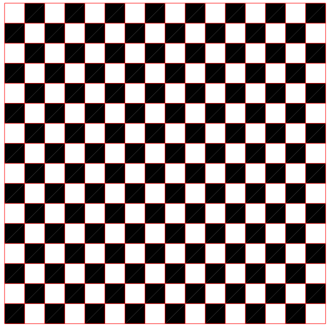
- Continuum of different synchrony subspaces



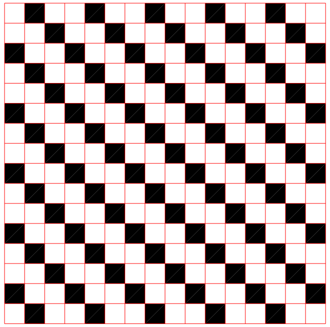


Lattice Dynamical Systems (3)

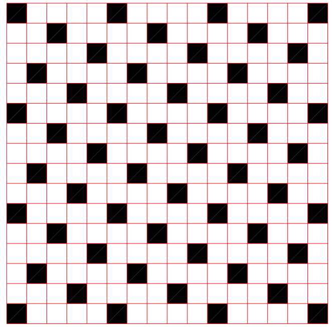
There are eight isolated balanced two-colorings on square lattice with nearest neighbor coupling



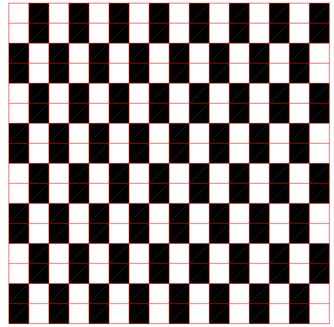
$4B \to W; 4W \to B$



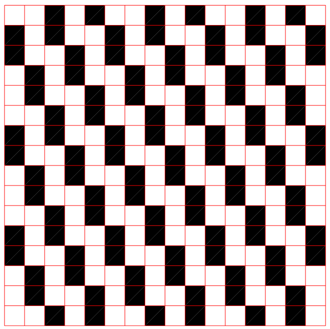
$2B \to W; 4W \to B$



$1B \to W; 4W \to B$



$3B \to W; 3W \to B$



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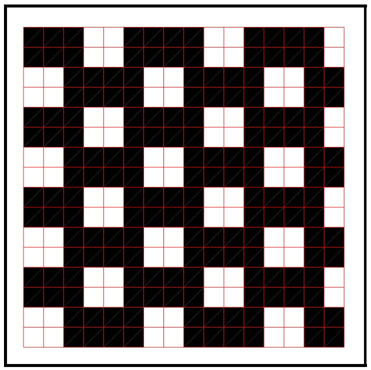

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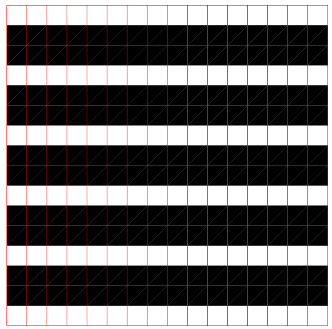

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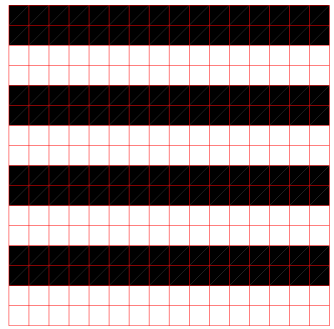


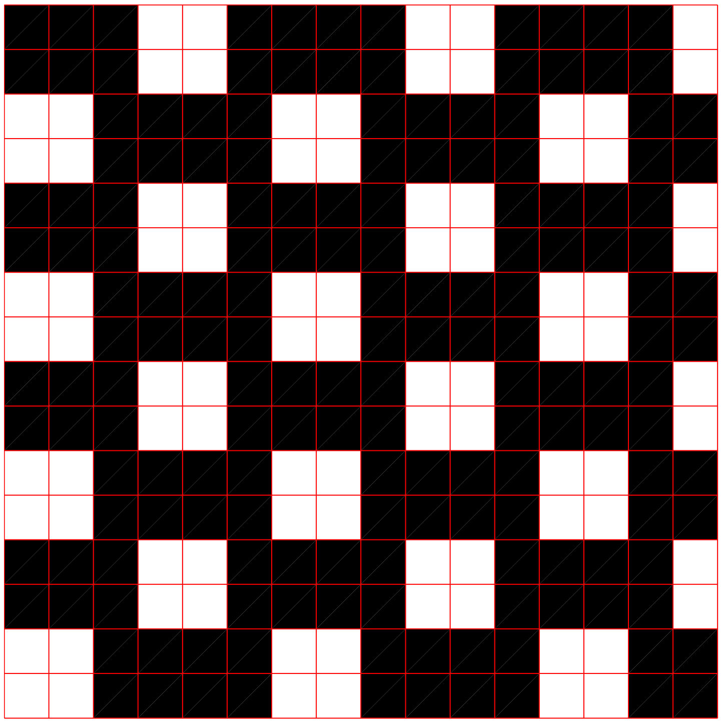
$1B \to W; 1W \to B$

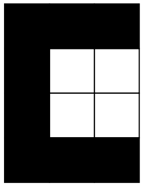
Wang and G. (2005)

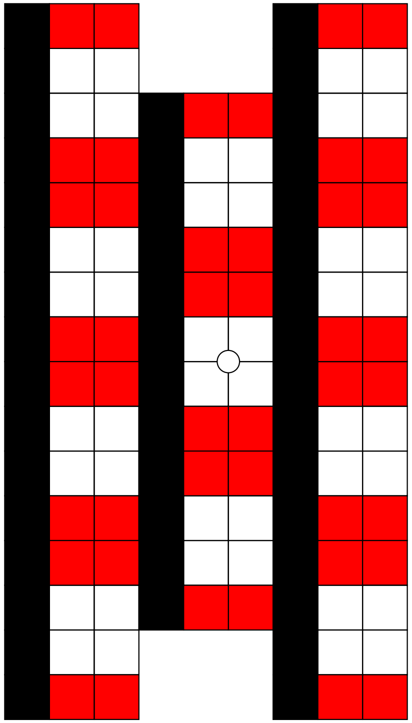
$\square$ indicates nonsymmetric solution

Symmetries

- $(i,j) \mapsto (i, j+4) \qquad (i,j) \mapsto (i+3, j+2)$
- $(i,j) \mapsto (-i, j) \qquad (i,j) \mapsto (i, -j)$

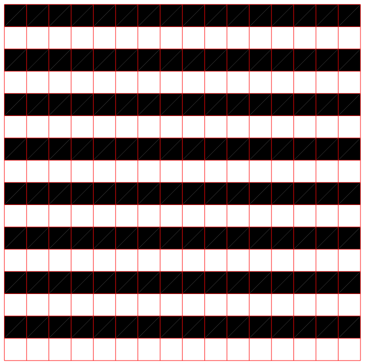






Lattice Dynamical Systems (4)

- There are two infinite families of balanced two-colorings



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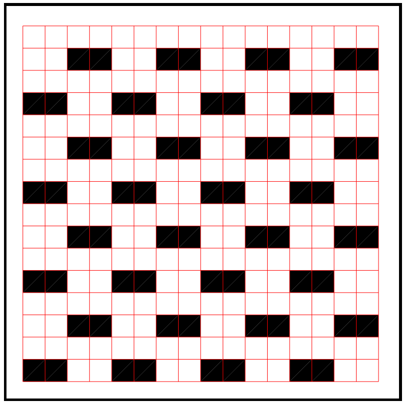

$1B \to W; 3W \to B$

- Up to symmetry these are all balanced two-colorings

Infinite Families

- There are many infinite families of balanced k-colorings

	B		D		B		C		A
A		C		A		D		B	
	D		B		C		A		D
C		A		D		B		C	
	B		C		A		D		B
A		D		B		C		A	
	C		A		D		B		C
D		B		C		A		D	
	A		D		B		C		A
B		C		A		D		B	

- We do not know how to classify balanced k-colorings

Lattice Dynamical Systems (5)

- Architecture is really important

Antoneli, Dias, G., and Wang (2005)

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- For square (and hexagonal) lattices with nearest and next nearest neighbor coupling
  - No infinite families

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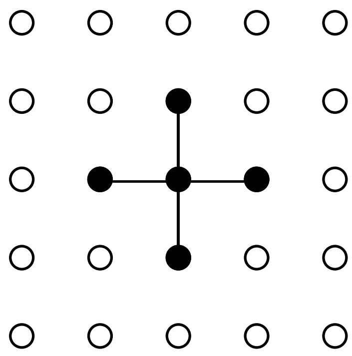
Lattice Dynamical Systems (5)

- Architecture is really important
- For square (and hexagonal) lattices with nearest and next nearest neighbor coupling
  - No infinite families
  - For each k a finite number of balanced k colorings
  - All balanced colorings are doubly-periodic

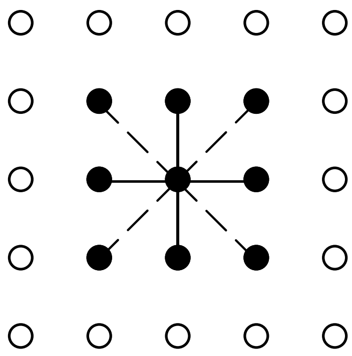
Antoneli, Dias, G., and Wang (2005)

Windows 1

$$W_0 = \{0\} \quad \text{and} \quad W_{i+1} = I(W_i)$$



NEAREST NEIGHBOR



NEXT NEAREST NEIGHBOR

- Input set of $U = I(U) = \{c \in \mathcal{C} : c$ connects to cell in $U\}$
- Input set contains lattice generators: $\mathcal{L} = W_0 \cup W_1 \cup \cdots$
- W_{k-1} contains all k colors of a balanced k-coloring

Windows 2

- $\mathrm{bd}(U) = I(U) \setminus U$

  $c \in \mathrm{bd}(U)$ is 1-determined if color of c is determined by colors of cells in U and fact that coloring is balanced

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  $c \in \mathrm{bd}(U)$ is 1-determined if color of c is determined by colors of cells in U and fact that coloring is balanced

- Define p-determined inductively

- Boundary cells with NN coupling are not 1-determined

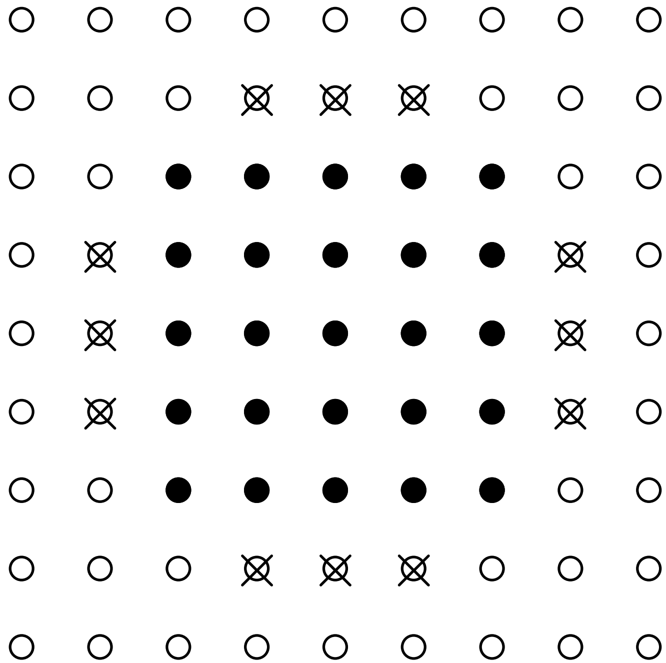
  Boundary cells with NNN coupling are 2-determined

Windows 3

Square lattice

Nearest and next nearest neighbor coupling

$\times$ indicates
1-determined cells of W_2



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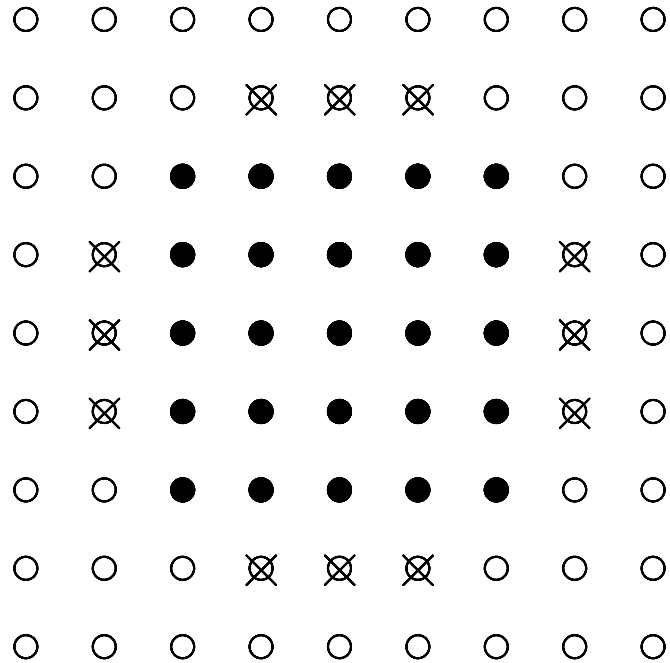

- Three cells in corners of square are 2-determined

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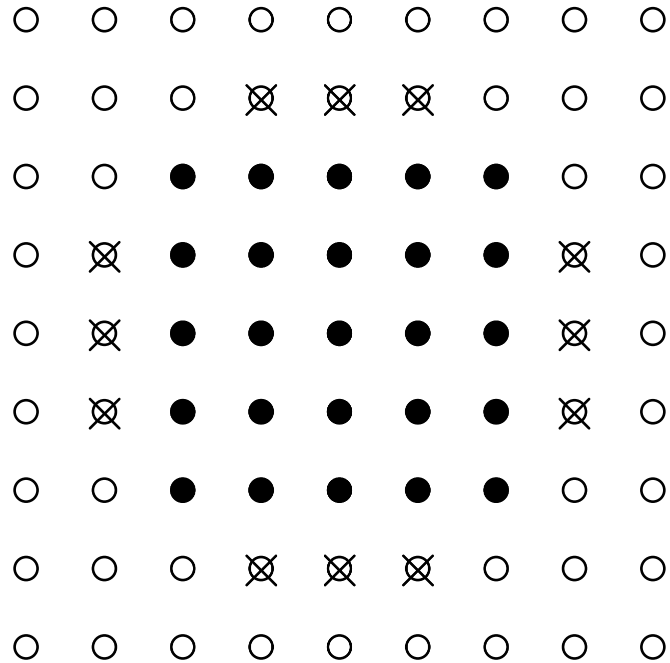

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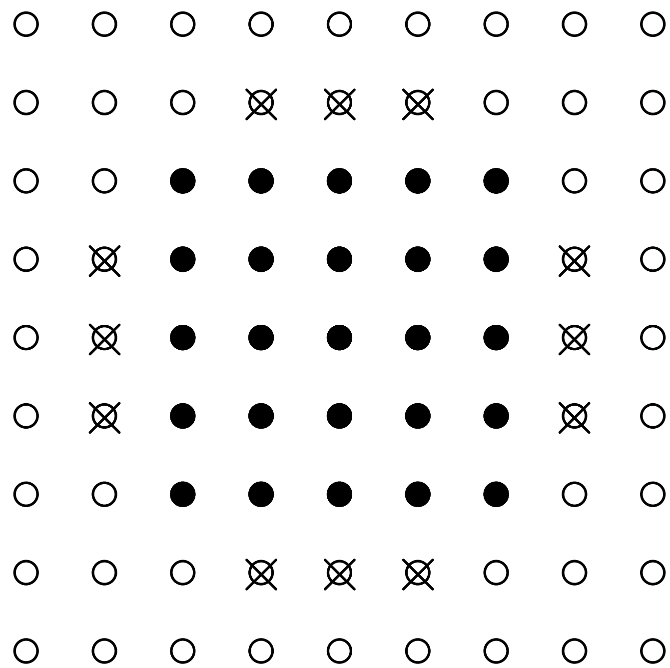
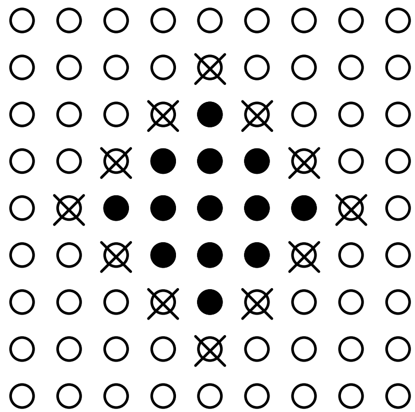


- Three cells in corners of square are 2-determined
- U determines its boundary if all $c \in \mathrm{bd}(U)$ are p-determined for some p
- W_i determines its boundary for all $i \geq 2$

Windows 4

Square lattice with Nearest neighbor coupling

W_2 is not 1-determined



Windows 5

- W_{i_0} is a window if W_i determines its boundary $\forall\ i \geqslant i_0$
- Suppose a balanced k-coloring restricted to $\mathrm{int}(W_i)$ for some $i \geqslant i_0$ contains all k colors. Then
  - k-coloring is uniquely determined on whole lattice by its restriction to W_i
- Thm: Suppose lattice network has window. Fix k. Then
  - Finite number of balanced k-colorings on $\mathcal{L}$
  - Each balanced k-coloring is multiply-periodic

Antoneli, Dias, G., and Wang (2004)