Symbolic Dynamics and Geodesic Laminations

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Origins of Symbolic Dynamics

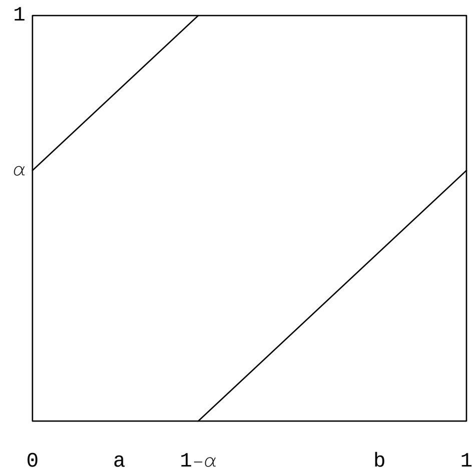
Let $f : X \to X$ a discrete dynamical system or $\phi_t : X \to X$ a continuous dynamical system.

We want to study the orbits of the dynamical system discretetizing the space X.

- Hadamard (1898), Morse (1921): Geodesic flows on constant negative curvature surfaces.
- Markov partition for Axiom A diffeomorphisms or Anosov Maps. (60's)

Irrational rotation on the circle

Let $R_\alpha : I \to I$, $R_\alpha(x) = x + \alpha \pmod 1$ with $\alpha \notin \mathbb{Q}$.



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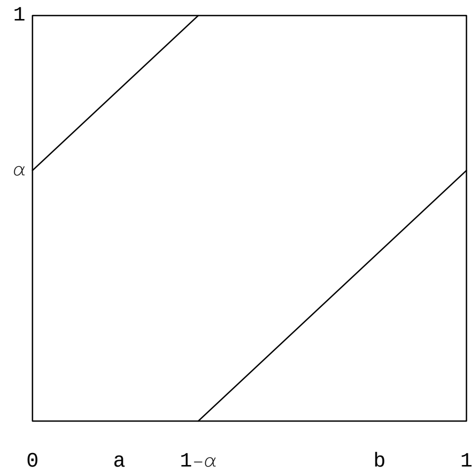

$I_a = [0, 1-\alpha)$, $I_b = [1-\alpha, 1)$.

$\nu : I \to \{a, b\}$, $\nu(x) = j$ if and only if $x \in I_j$.

$\mathbf{u} = \{\nu(R_\alpha^n(0))\}_{n\geq 0}$.

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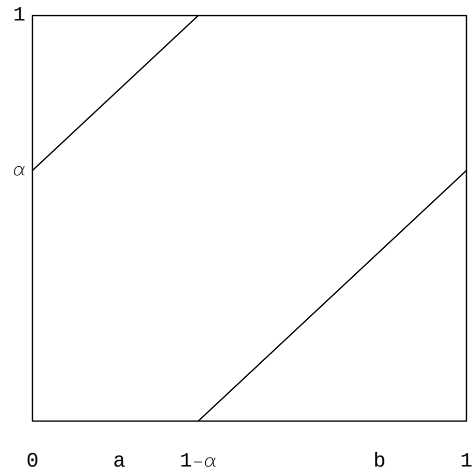

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If $\alpha = \frac{\sqrt{5}-1}{2}$ then $\mathbf{u} = abaababa\cdots$ The Fibonacci sequence.

It is the fixed points of the substitution $a \to ab$, $b \to a$.

Symbolic and geometrical systems

Let $\sigma(v_0 v_1 \ldots) = v_1 \ldots$ be the *shift*.

Let $\Omega = \overline{\{\sigma^n(\mathbf{u}) \mid n \geq 0\}}$, and the dynamical systems (Ω, σ).

There is a continuous and surjective map $\xi : \Omega \to I$ such that the diagram commutes:

$$\begin{array}{ccc} \Omega & \xrightarrow{\sigma} & \Omega \\ \xi \downarrow & & \downarrow \xi \\ I & \xrightarrow[R_\alpha]{} & I \end{array} \tag{1}$$

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$$\begin{array}{ccc} \Omega & \xrightarrow{\sigma} & \Omega \\ \xi \downarrow & & \downarrow \xi \\ I & \xrightarrow[R_\alpha]{} & I \end{array} \qquad (2)$$

We are interested in the "inverse problem". We start with a symbolic system and we would like to find a "geometrical representation".

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Hedlund-Morse (1938): $\mathbf{u}$ is sturmian if and only if it is obtained from coding the orbit of a point on the circle under a rotation by irrational number, using the partition given by the continuity intervals.

Rauzy fractal

Substitution Π: $1 \longrightarrow 12$, $2 \longrightarrow 13$, $3 \longrightarrow 1$. (Tribonacci substitution).

Fixed point $\mathbf{u} = \Pi^{\infty}(1) = 121312112\ldots$ Its complexity is $2n+1$.

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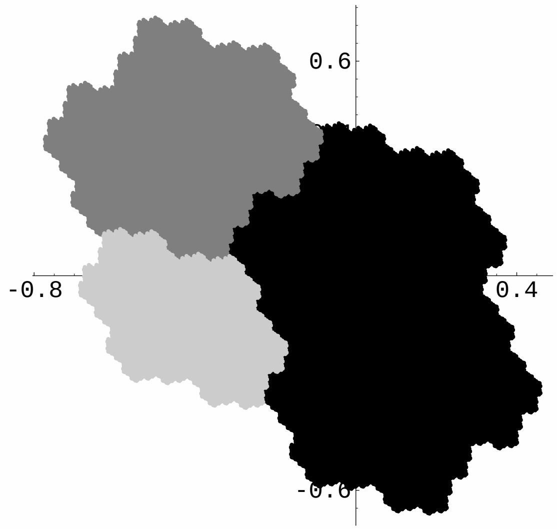
This fixed point gives the coding of the orbit of $T(x,y) = (x,y) + (\alpha, \alpha^2)$ according to the partition of the 2-dimensional torus given by the Rauzy fractal. Here $\alpha + \alpha^2 + \alpha^3 = 1$.



Arnoux-Rauzy sequences

Let $u = u_0 u_1 \ldots$ be a sequence in three symbols.

A word is *allowed or admissible* in $\mathbf{u}$ if it is a finite subword of the sequence $\mathbf{u}$.

We say that $\mathbf{u}$ is *Arnoux-Rauzy* (AR) if

- it has complexity $2n + 1$
- for all n there are allowed subwords of length n, V_n and W_n such that $V_n 1$, $V_n 2$, $V_n 3$ and $1 W_n$, $2 W_n$, $3 W_n$ are also allowed words.

How to construct these sequences?

Arnoux-Rauzy sequences

Let us consider the following substitutions:

$$\Pi_1 : \left\{ \begin{array}{ccc} 1 & \longrightarrow & 1 \\ 2 & \longrightarrow & 12, \\ 3 & \longrightarrow & 13 \end{array} \right. \quad \Pi_2 : \left\{ \begin{array}{ccc} 1 & \longrightarrow & 21 \\ 2 & \longrightarrow & 2, \\ 3 & \longrightarrow & 23 \end{array} \right. \quad \Pi_3 : \left\{ \begin{array}{ccc} 1 & \longrightarrow & 31 \\ 2 & \longrightarrow & 32 \\ 3 & \longrightarrow & 3 \end{array} \right.$$

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Theorem 1. *[Arnoux-Rauzy(1991)] Let $\mathbf{u}$ be a minimal sequence in the alphabet $\{1, 2, 3\}$. Then $\mathbf{u}$ is AR sequence, if and only if there exists a sequence $\{i_k\}_k$ with values in $\{1, 2, 3\}$ such that each symbol appears infinitely many times and*

$$\mathbf{u} = \lim_{k \to \infty} \Pi_{i_1} \cdots \Pi_{i_k}(\mathbf{u}).$$

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Theorem 2. *[Arnoux-Rauzy(1991)] Let $\mathbf{u}$ be a minimal sequence in the alphabet $\{1, 2, 3\}$. Then $\mathbf{u}$ is AR sequence, if and only if there exists a sequence $\{i_k\}_k$ with values in $\{1, 2, 3\}$ such that each symbol appears infinitely many times and*

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If the sequence $\{i_k\}_k$ is periodic then the sequence $\mathbf{u}$ is the fixed point of the substitution $\Pi_{i_1} \cdots \Pi_{i_l}$, where $\{i_k\}_k = \{i_1, \ldots, i_l, i_1, \ldots, i_l, \ldots\}$. This substitution is Pisot.
In the case Π: $1 \to 12$, $2 \to 13$, $3 \to 1$. We have $\Pi^3 = \Pi_1 \Pi_2 \Pi_3$.

So: $\{i_k\}_k = \{1, 2, 3, 1, 2, 3, \ldots\}$

Arnoux-Rauzy sequences

Do all the AR sequences come from translations on the 2-torus?

Arnoux-Rauzy sequences

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No. Cassaigne, Ferenczi and Zamboni (2000).

Geometry of the dynamical systems (Ω, σ)?

where $\Omega = \overline{\{\sigma^n(\mathbf{u}) \,|\, n \geq 0\}}$, and $\mathbf{u}$ an AR sequence.

Interval exchange maps

Let $\mathbf{u}$ be an AR sequence and

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

be the matrices associated to the substitutions Π_1, Π_2, Π_3.

Let $\{i_k\}_k$ be the sequence associated to $\mathbf{u}$.

The image of the positive cone under the infinite product $M_{i_1} \cdots M_{i_k} \cdots$ is a straight line passing through the origin.

Let (α, β, γ) be the element of norm 1 in this line.

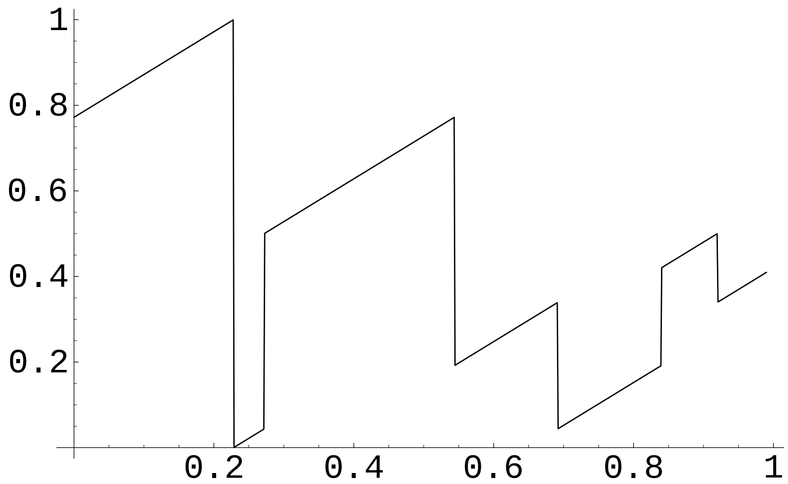
Interval exchange maps

Let $f = L_I \circ L_{I_1} \circ L_{I_2} \circ L_{I_3}$, where

$I = [0,1)$, $I_1 = [0,\alpha)$, $I_2 = [\alpha, \alpha+\beta)$, $I_3 = [\alpha+\beta, 1)$ and

L_J denotes the rotation of order 2 on the interval $J = [a,b)$, i.e.

$$L_J(x) = \begin{cases} x + \frac{b-a}{2} & \text{if } a \le x < \frac{a+b}{2} \\ x - \frac{b-a}{2} & \text{if } \frac{a+b}{2} \le x < b \\ x & \text{otherwise.} \end{cases}$$



Coding of IEMs

Let $\nu : I \to \{1, 2, 3\}$: $\nu(x) = i$ if and only if $x \in I_i$.

Let $\theta : I \to \Omega$, $\theta(x) = \{\nu(f^n(x))\}_{n \geq 0}$

ν is continuous to the right and $\theta(f(x)) = \sigma(\theta(x))$.

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Since f is invertible we can consider

$$\widetilde{\Omega} = \overline{\{\nu(f^n(x)) \,|\, x \in I,\ n \in \mathbb{Z}\}}$$

so we have the map $\tilde{\theta} : I \to \widetilde{\Omega}$ that send the point x to itinerary of its two sided infinite f-orbit.

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Coding of IEMs

Let $x_1 = 0$, $x_2 = \alpha$, $x_3 = \alpha + \beta$ be the extremities of the canonical intervals and $y_1 = \alpha/2$, $y_2 = \alpha + \beta/2$, and $y_3 = \alpha + \beta + \gamma/2$ be the middle points of the canonical intervals. These points are the discontinuities of the iet f.

$$\theta(f(y_1)) = \theta(f(y_2)) = \theta(f(y_3)) = \mathbf{u}.$$

And the coding of the backward orbit of x_i is given by $\overline{\mathbf{u}} = \ldots u_2 u_1 u_0$

Geodesic Lamination on the disk

A *geodesic lamination* on $\mathbb{D}^2$ is a non-empty closed set of geodesics of the disk and that any two of these geodesics do not intersect except at their end points.

Construction of the geodesic lamination

Let $v_0 \dots v_k$ be an admissible word and

$$[v_0 \dots v_k] = I_{v_0} \cap f^{-1}(I_{v_1}) \cap \cdots \cap f^{-k}(I_{v_k})$$

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We join by geodesics consecutive extreme points that belong to different components of a given cylinder.

We do this for all cylinders and we take the closure.

As a result we get the space Λ.

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We are interested in points of $\widetilde{\Omega}$ with the same past and different futures and conversely in points with the same future and different pasts.

Properties of the geodesic lamination

Properties:

- Λ is a geodesic lamination.
- Λ is the closure of the geodesics γ such that the image under $\tilde{\theta}$ of the end points of γ have the same past and different futures.
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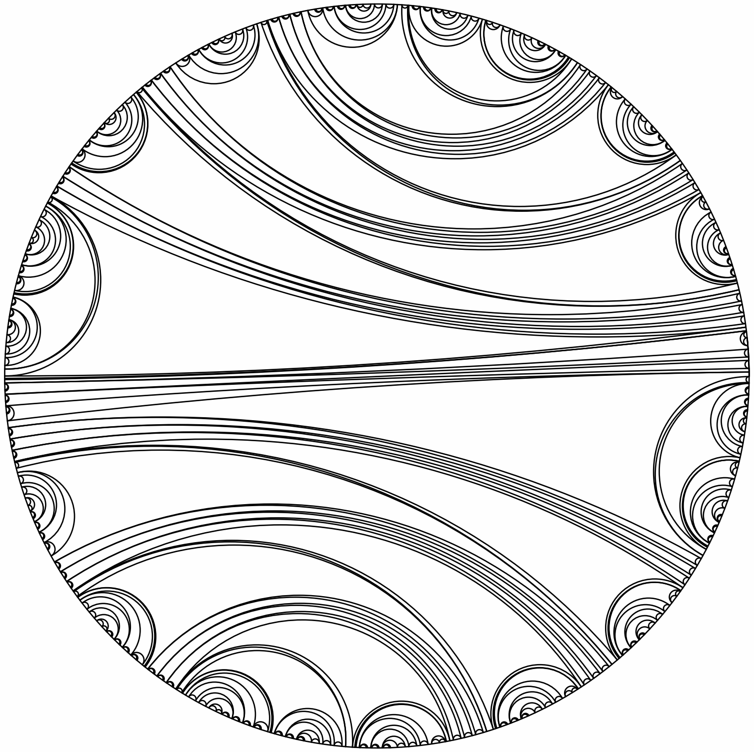
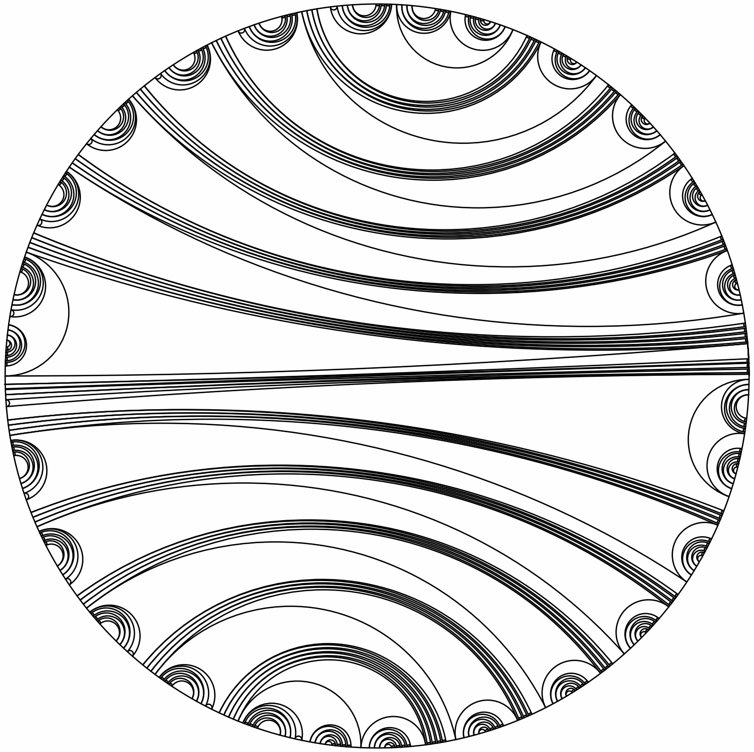
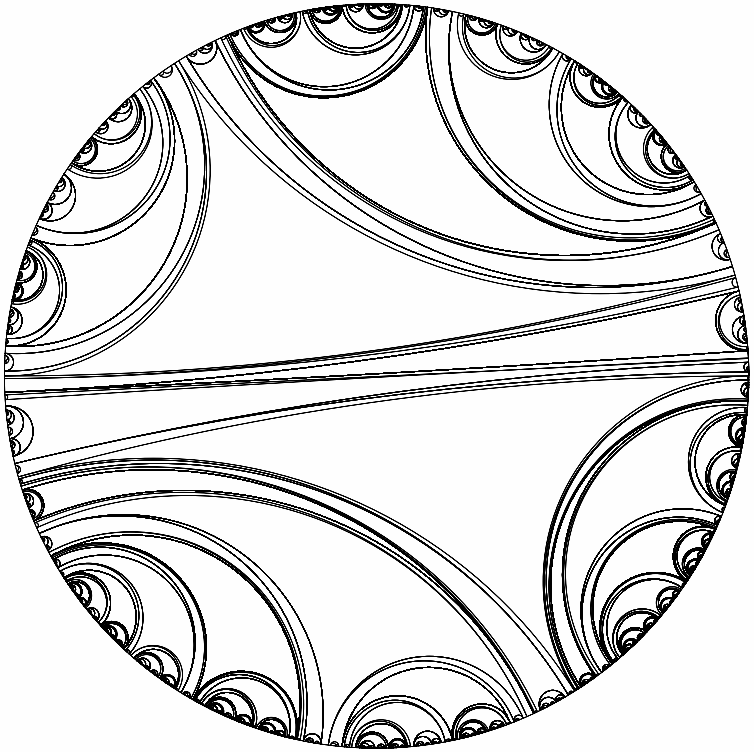
- Λ is invariant under the rotation by $1/2$.

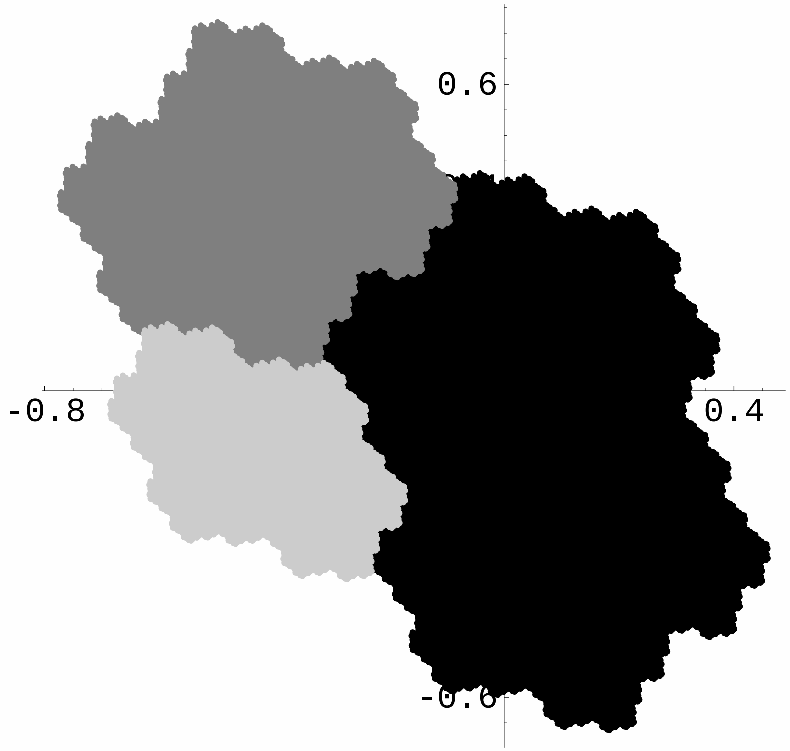
The geodesic lamination





Tribonacci lamination





Dynamical system on the lamination

Let $F : \Lambda \to \Lambda$ defined as let $\gamma \in \Lambda$ with end points a_γ and b_γ, $F(\gamma)$ is the geodesic that join $f(a_\gamma)$ with $f(b_\gamma^-)$.

Properties:

- F is well defined.
- F is continuous.
- (Λ, F) is semi-conjugate to (Ω, σ).
- (Λ, F) is semi-conjugate to $(\widetilde{\Omega}, \sigma)$.

Summary

Theorem 3. *Let* $\mathbf{u}$ *be an AR sequence and* (Ω, σ), $(\widetilde{\Omega}, \sigma)$ *their associated* $\mathbb{N}$ *and* $\mathbb{Z}$ *dynamical systems respectively. Then there exists* Λ *a geodesic lamination on* $\mathbb{D}^2$ *and a continuous dynamical system* (Λ, F) *such that:*

- (Λ, F) *is semi-conjugate to* (Ω, σ).
- (Λ, F) *is semi-conjugate to* $(\widetilde{\Omega}, \sigma)$.
- Λ *is invariant under the rotation by* $1/2$.

Tribonacci Case

The symbolic dynamical system (Ω, σ) is semi-conjugate to $(\mathbb{T}^2, T)$ an irrational translation.

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Let $\mathcal{R}$ be the Rauzy fractal.

It has a self-similar structure. It is the fixed point of the IFS $\{f_1, f_2, f_3\}$.

$$f_1(z) = \beta z, \quad f_2(z) = \beta^2 z + 1, \quad f_3(z) = \beta^3 z + \beta + 1.$$

The roots of $x^3 - x^2 - x - 1 = 0$ are
λ and β, $\overline{\beta}$, $\lambda > 1$, $|\beta| < 1$.

Tribonacci Case

The geodesic lamination Λ coincides with the lamination obtained using the self-similar structure (last week talk).

So Λ has a transverse measure μ and an expanding dynamical system $G : \Lambda \to \Lambda$.

$G_* \mu = \lambda^{s_0} \mu$, where $s_0 = \log \nu / \log \lambda$, $\nu^4 - 2\nu - 1 = 0$, $\nu > 1$.

Tribonacci Case

Let $G_i^{-1} : \Lambda \to \Lambda_i$ the inverse branches of G and

$$\Lambda_i = \{\gamma \in \Lambda \ : \ \text{the end points of } \gamma \in I_i\}$$

So the following diagram commutes:

$$\begin{tikzcd} \Lambda \arrow[r, "F"] \arrow[d, "G_i^{-1}"'] & \Lambda \arrow[d, "G_i^{-1}"] \\ \Lambda_i \arrow[r, "\widetilde{F}"'] & \Lambda_i \end{tikzcd}$$

where $\widetilde{F}$ is the induced map of F in Λ_i:

$$\widetilde{F}(\gamma) = F^{n_1}(\gamma) : \quad F^{n_1}(\gamma) \in \Lambda_i, \; F^n(\gamma) \notin \Lambda_i, \; \text{ if } 1 \leq n < n_1.$$