Space Filling Curves and Geodesic Laminations

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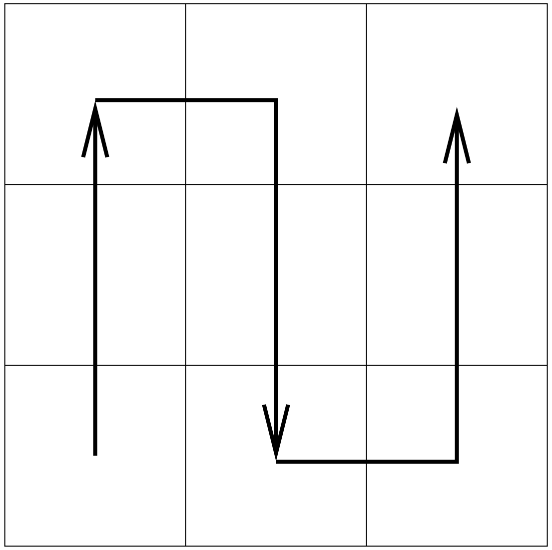
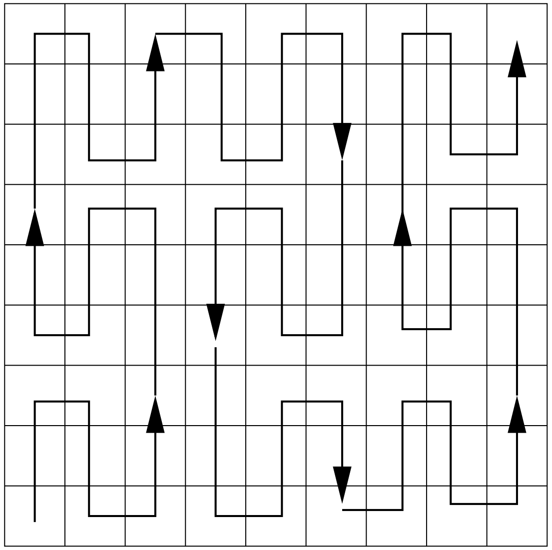
Classical Space Filling Curves

- Peano (1890, 1908)





Classical Space Filling Curves

- Peano (1890, 1908)
- Hilbert (1891)
- Lebesgue (1904)
- Sierpiński (1912)
- Schenberg (1938)

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Idea: Define $F_1, \ldots, F_k$, contractions on the plane, such that there exits a non-empty compact set $K \subset \mathbb{R}^2$ such that:

$$K = \bigcup_{j=1}^{k} F_j(K) \quad \text{int}F_i(K) \cap \text{int}F_j(K) = \emptyset.$$

The digits of the representation of numbers base k, in the interval, are used to know the order in which apply the maps F_j's.

More recently

- Pattern recognition algorithms.
- Data structures.
- Integral equations.
- Rauzy Fractals (Arnoux, S.).

Iterated Function System (IFS)

Let (X, d) be a complete metric space.

$f_j : X \to X$ contractions, $1 \leq j \leq k$.

Let $K(X)$ be the set of all non-empty compact subsets of X. Its topology is given by the Hausdorff metric.

Let $\mathcal{F} : K(X) \to K(X)$ defined as $\mathcal{F}(A) = \cup_{j=1}^{k} f_j(A)$.

$\mathcal{F}$ is a contraction.

Its fixed point is called the attractor of the IFS: $\{f_1, \ldots, f_k\}$.

Hilbert's classical space filling curve

Let $\{H_0, H_1, H_2, H_3\}$ be an IFS on $\mathcal{R} = \{x + iy \in \mathbb{C} \,|\, 0 \leq x, y \leq 1\}$, where

$$H_0(z) = \frac{\overline{z}i}{2}, \qquad H_1(z) = \frac{z}{2} + \frac{i}{2},$$

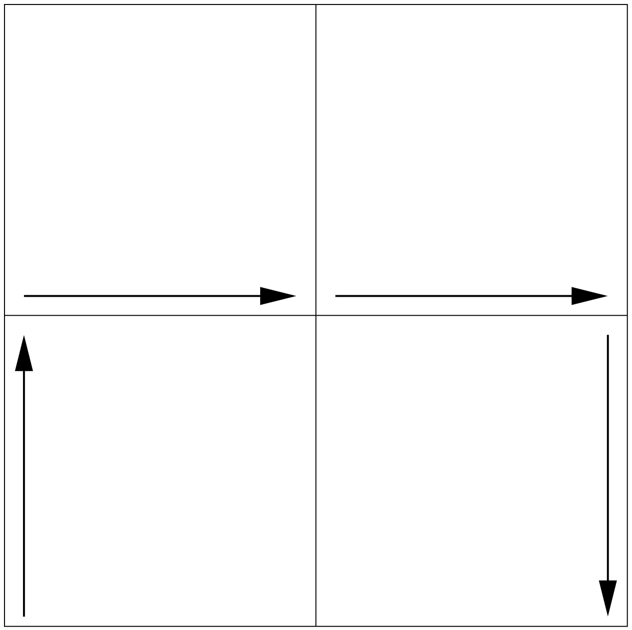
$$H_2(z) = \frac{z}{2} + \frac{1+i}{2}, \quad H_3(z) = -\frac{\overline{z}i}{2} + \frac{i}{2} + 1.$$

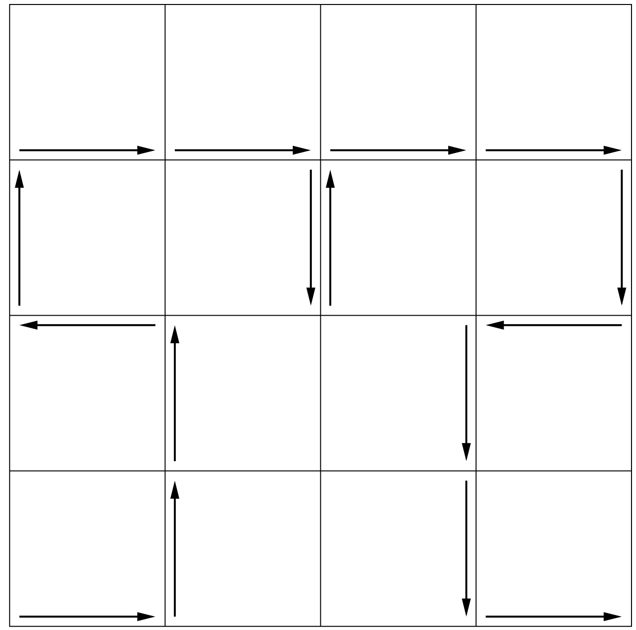
Let $\{h_0, h_1, h_2, h_3\}$ be an IFS on $I = [0, 1]$, where $h_k(t) = t/4 + k/4$, for $0 \leq k \leq 3$ This IFS on the interval defines the numeration system base 4, i.e. $t = \sum_{n=1}^{\infty} a_n/4^n$, with $0 \leq a_n \leq 3$ if and only if $t = \lim_{n\to\infty} h_{a_1} \cdots h_{a_n}(I)$.
The space filling curve $\xi_H : I \to \mathcal{R}$ is defined as

$$\{\xi_H(t)\} = \bigcap_{n=1}^{\infty} H_{a_1} H_{a_2} \cdots H_{a_n}(\mathcal{R}),$$

where $t = \sum_{n=1}^{\infty} a_n/4^n$. This map is continuous and surjective.

Hilbert's classical space filling curve





An interesting example

Let $\{G_1, G_2, G_3, G_4\}$ be the IFS on $\mathcal{R}$:

$$G_1(x,y) = (-\alpha x + \alpha, -\alpha y + \alpha), \quad G_2(x,y) = (-\alpha y + \alpha, \beta x + \alpha),$$
$$G_3(x,y) = (\beta x + \alpha, \alpha y + \alpha), \qquad G_4(x,y) = (\beta y + \alpha, -\alpha x + \alpha)$$

where $\alpha, \beta > 0$, $\alpha + \beta = 1$ and $\alpha > \beta$.
Properties:

- int$(\mathcal{R}_i) \cap$ int$(\mathcal{R}_j) = \emptyset$ for $i \neq j$.
- $\cap_{i=1}^4 \mathcal{R}_i = \{(\alpha, \alpha)\}$.
- Each $p \in \mathcal{R}$ is of the form $\cap_{n=1}^{\infty} G_{a_1} \cdots G_{a_n}(\mathcal{R})$ for some $a_1 a_2 \cdots \in \{1,2,3,4\}^{\mathbb{N}^+}$. We call the sequence *itinerary* of p.

Let $\{g_1, g_2, g_3, g_4\}$ be the IFS on I defined as:

$$g_1(t) = \begin{cases} \alpha^2 t + (\alpha^2 - \alpha^4/2) & \text{if} \quad 0 \leq t < \alpha^2/2 \\ \\ \alpha^2 t - \alpha^4/2 & \text{if} \quad \alpha^2 \leq t < 1 \end{cases}$$

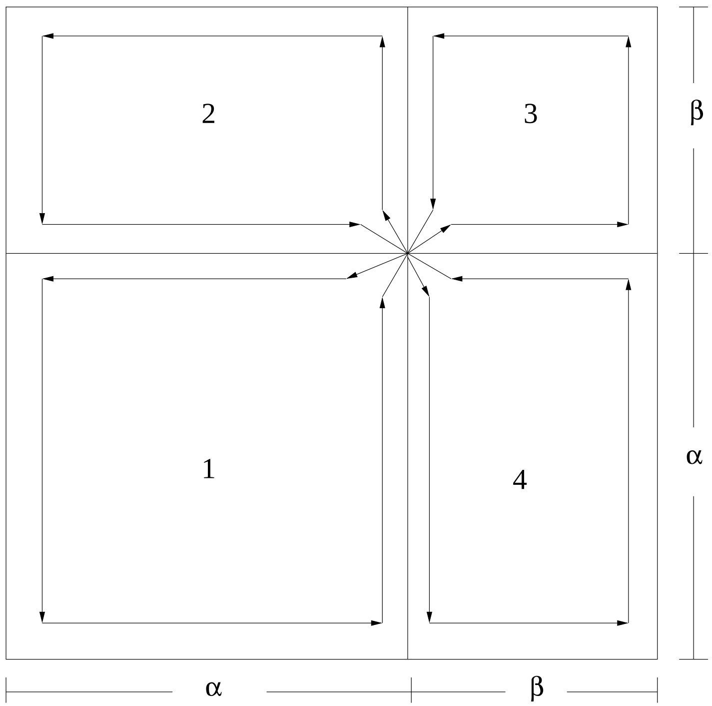
y

$$g_2(t) = (\beta/\alpha) g_1(t) + \alpha^2 \quad g_3(t) = (\beta/\alpha)^2 g_1(t) + \alpha^2 + \alpha\beta \quad g_4(t) = (\beta/\alpha) g_1(t) + 1 - \alpha\beta.$$

For each $t \in I$ there exists $a_1 a_2 \cdots \in \{1,2,3,4\}^{\mathbb{N}^+}$, such that $\{t\} = \cap_{n \geq 1} g_{a_1} \cdots g_{a_n}(I)$.
Let $\xi : I \to \mathcal{R}$, be the curve defined as: if $t \in I$ and $a_1 a_2 \cdots$ its itinerary,

$$\{\xi(t)\} = \bigcap_{n=1}^{\infty} G_{a_1} G_{a_2} \cdots G_{a_n}(\mathcal{R}).$$

ξ is continuous, surjective and measure preserving.



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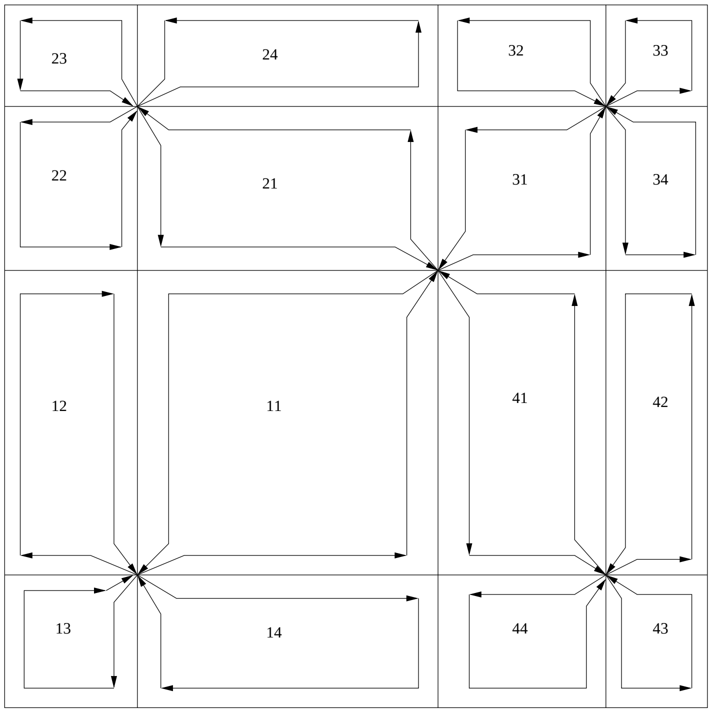

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Geodesic Lamination on the disk

Definition 1. *A* geodesic lamination *on* $\mathbb{D}^2$ *is a non-empty closed set of geodesics of the disk and that any two of these geodesics do not intersect except at their end points.*

Geodesic Lamination on the disk

Let $\mathbb{D}^2$ be the closed unit disk in the plane, and $\mathbb{S}^1$ its boundary.

We think of the IFS: $\{g_1, g_2, g_3, g_4\}$ as acting on $\mathbb{S}^1$.

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The construction of the geodesic lamination Λ is as follows:

We consider the extremities of the intervals defi ned by the IFS, i.e. t_k with $k = 1, 2, 3, 4$. And we join them by arcs of circles that meet the boundary of $\mathbb{D}^2$ perpendicularly.

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Let $a_1 \dots a_n$ be a word in the alphabet $\{1, 2, 3, 4\}$. We join by geodesics the points $g_{a_1} \cdots g_{a_n}(t_k)$ for $k = 1, 2, 3, 4$. We do this for all possible words in this alphabet and later we take the closure in the Hausdorff topology of $\mathbb{D}^2$.

The elements of Λ are either geodesics or points in $\mathbb{S}^1$.

In the latter case the points are called *degenerate geodesics*.

Geodesic Lamination on the disk

Properties of Λ:

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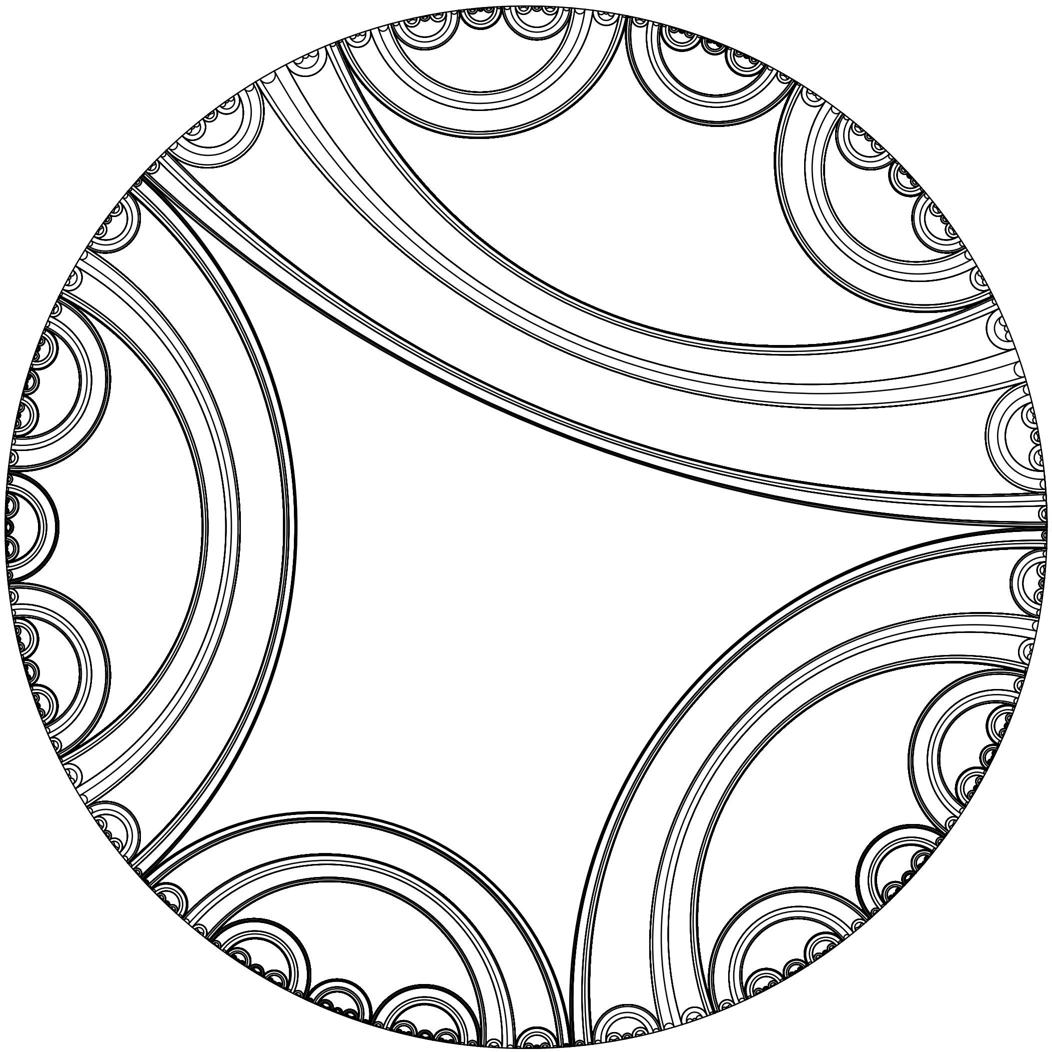
Geodesic Lamination on the disk

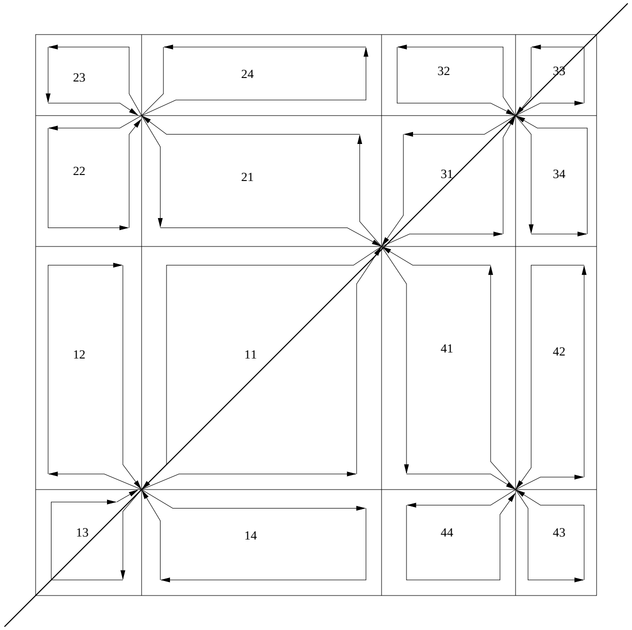
Properties of Λ:

- Λ is a geodesic lamination.
- Λ has an axis of symmetry.
- Let λ be an element of Λ with end points b and b'. Then $\xi(b) = \xi(b')$.
  These properties allow us to defi ne a map $\Xi : \Lambda \to \mathcal{R}$ as follows: Let λ be a geodesic of Λ with end points b, b'. So $\Xi(\lambda) := \xi(b)$.

Geodesic Lamination on the disk



Symmetry



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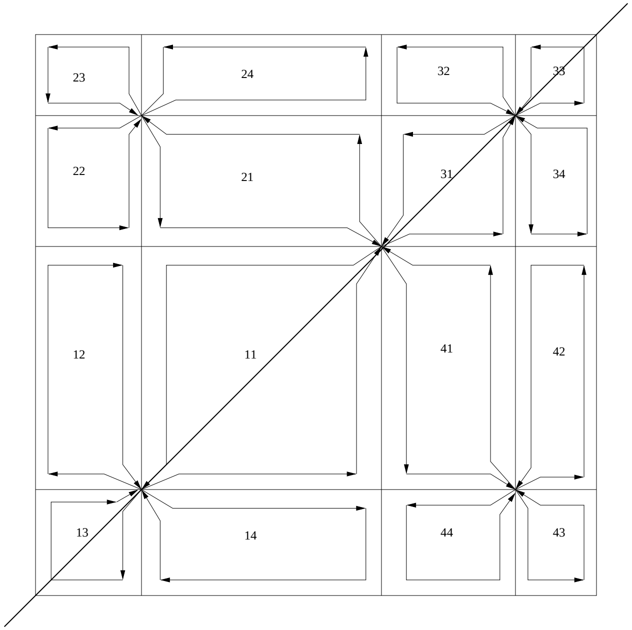
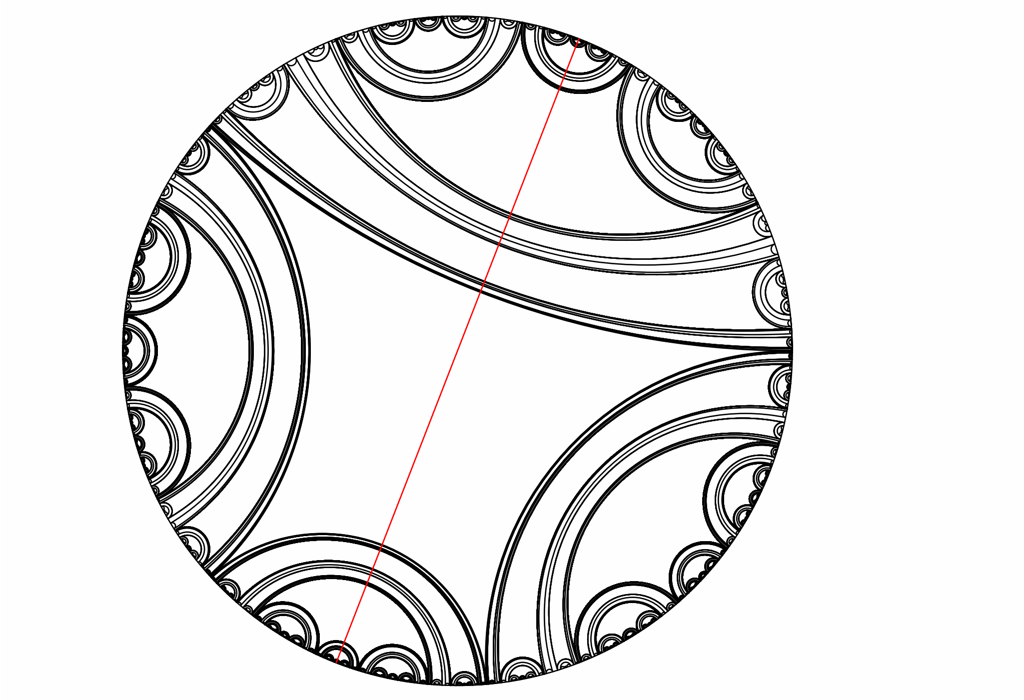
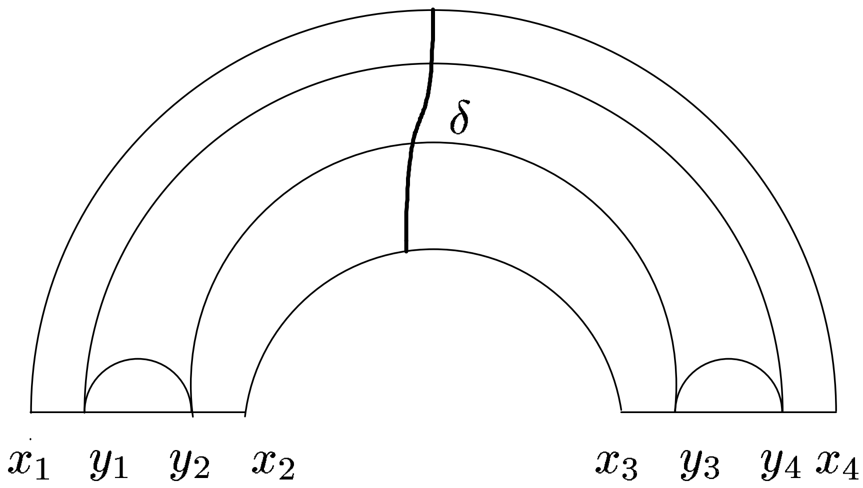




Transverse Measure to the Lamination

Let δ be any arc in $\mathbb{D}^2$ joining two distinct geodesics of the lamination. It can be slid along the geodesics towards the boundary of the disk according the two possible directions in which the geodesics can be oriented. This procedure gives rise to a Cantor set in the boundary of the disk, say C_δ.



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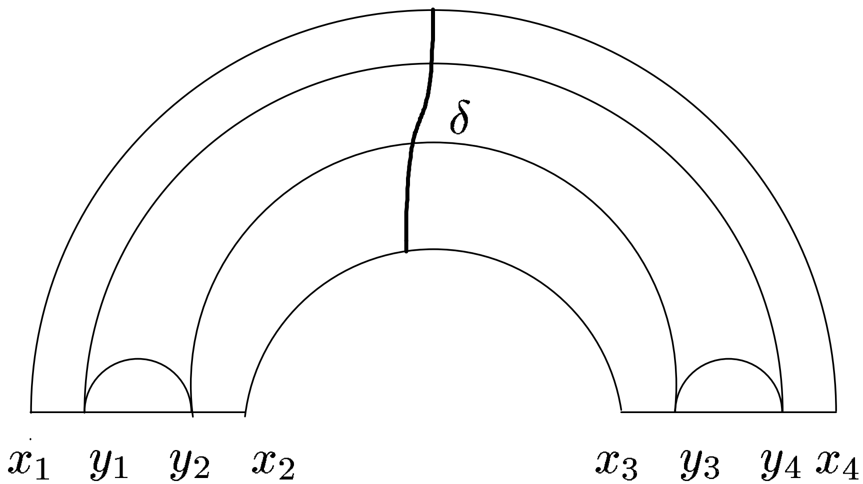
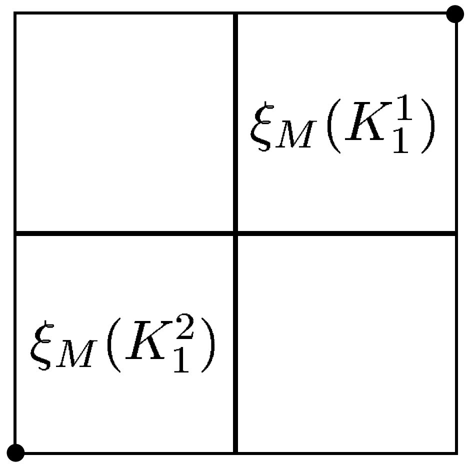


We define $\mu(\delta) = \mathcal{M}_{s_0}(C_\delta)$ where $\mathcal{M}_{s_0}$ is the s_0-Hausdorff measure and s_0 is the Hausdorff dimension of C_δ.

The Hausdorff dimension of C_δ is s_0 the solution of $\alpha^{2s} + \beta^{2s} = 1$.

Transverse Measure to the Lamination

Proposition 1. *Let* δ *be any arc transversal to* Λ *whose end points are in the geodesics* λ_1 *and* λ_2*. The image of the set* C_δ *under* ξ *is the line segment that joins* $\Xi(\lambda_1)$ *and* $\Xi(\lambda_2)$*.*



A dynamical system defined on Λ

Let γ be a geodesic with end points a and b.

We define $F(\gamma)$ as the geodesic that joins $f(a)$ with $f(b)$.

Where f is the expanding map defined by the inverses of the maps that defined the IFS:

$f(t) = g_i^{-1}(t)$, if $t \in I_i$.

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Where f is the expanding map defined by the inverses of the maps that defined the IFS:

$f(t) = g_i^{-1}(t)$, if $t \in I_i$.

- $F(\gamma) \in \Lambda$.
- $F : \Lambda \to \Lambda$ is continuous.
- $F_* \mu = (1 + 2(\alpha\beta)^{s_0})\mu$.

Theorem 1. *There exists a geodesic lamination Λ on the disk, associated to the space filling curve ξ. This lamination has the following properties:*

1. *The end points of each element of Λ are mapped to the same point in the square by the space filling curve ξ_L.*
2. *Λ has an axis of symmetry.*
3. *For any transverse arc to Λ, there is a limit set on the boundary of the disk, whose Hausdorff dimension is s_1, the solution of $\alpha^{2s} + \alpha^s \beta^{2s} = 1$. And, the image of this limit set under ξ is a straight line between the points, which are the images under ξ of the end points of the geodesics, joined by the transverse arc.*
4. *The lamination has a transverse measure μ and there is a continuous map $F : \Lambda \to \Lambda$, so that $F_* \mu = (1 + 2(\alpha\beta)^{s_1})\mu$.*

Lebesgue version

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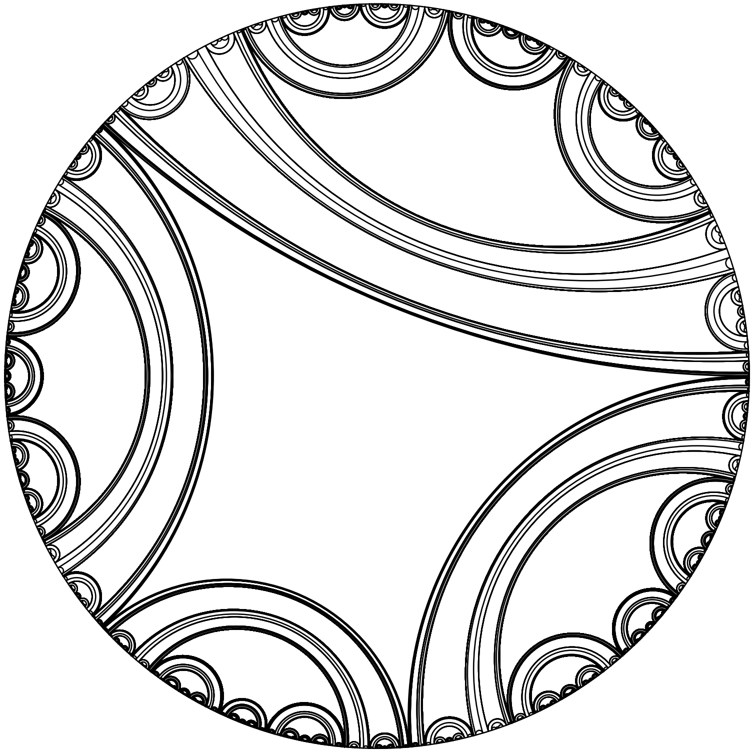


Lebesgue version

We consider the same IFS $\{G_i\}_{1=1}^4$ in $\mathcal{R}$ and on I the IFS $\{h_i\}_{i=1}^4$.

$$h_1(t) = \begin{cases} \alpha^3 t + (\alpha^2 - \alpha^5/2) & \text{if} \quad 0 \leq t < \alpha^2/2 \\ \alpha^3 t - \alpha^5 & \text{if} \quad \alpha^2 \leq t < (1+\alpha^2)/2 \\ \alpha^3 t - \alpha^5/2 + \alpha^2\beta & \text{if} \quad (1+\alpha)/2 \leq t < 1 \end{cases}$$

$$h_2(t) = (\beta/\alpha)h_1(t) + \alpha^2, \quad h_3(t) = (\beta/\alpha)^2 h_1(t) + \alpha^2 + \alpha\beta,$$

$$h_4(t) = (\beta/\alpha)h_1(t) + \alpha^2 + \alpha\beta + \beta^2.$$

The attractor of $\{h_1, \cdots, h_4\}$ is a Cantor set, C. $\dim_H(C) = s$ where s satisfies $\alpha^{3s} + 2\alpha^{2s}\beta^s + \alpha^s\beta^{2s} = 1$.

Let $t \in I$ and $a_1 a_2 \cdots$ be its itinerary.

We define

$$\begin{array}{rccl} \xi_L: & C & \longrightarrow & \mathcal{R} \\ & t & \longmapsto & \cap_{n\geq 1} G_{a_1} \cdots G_{a_n}(\mathcal{R}). \end{array}$$

This map is continuous, surjective and measure preserving.

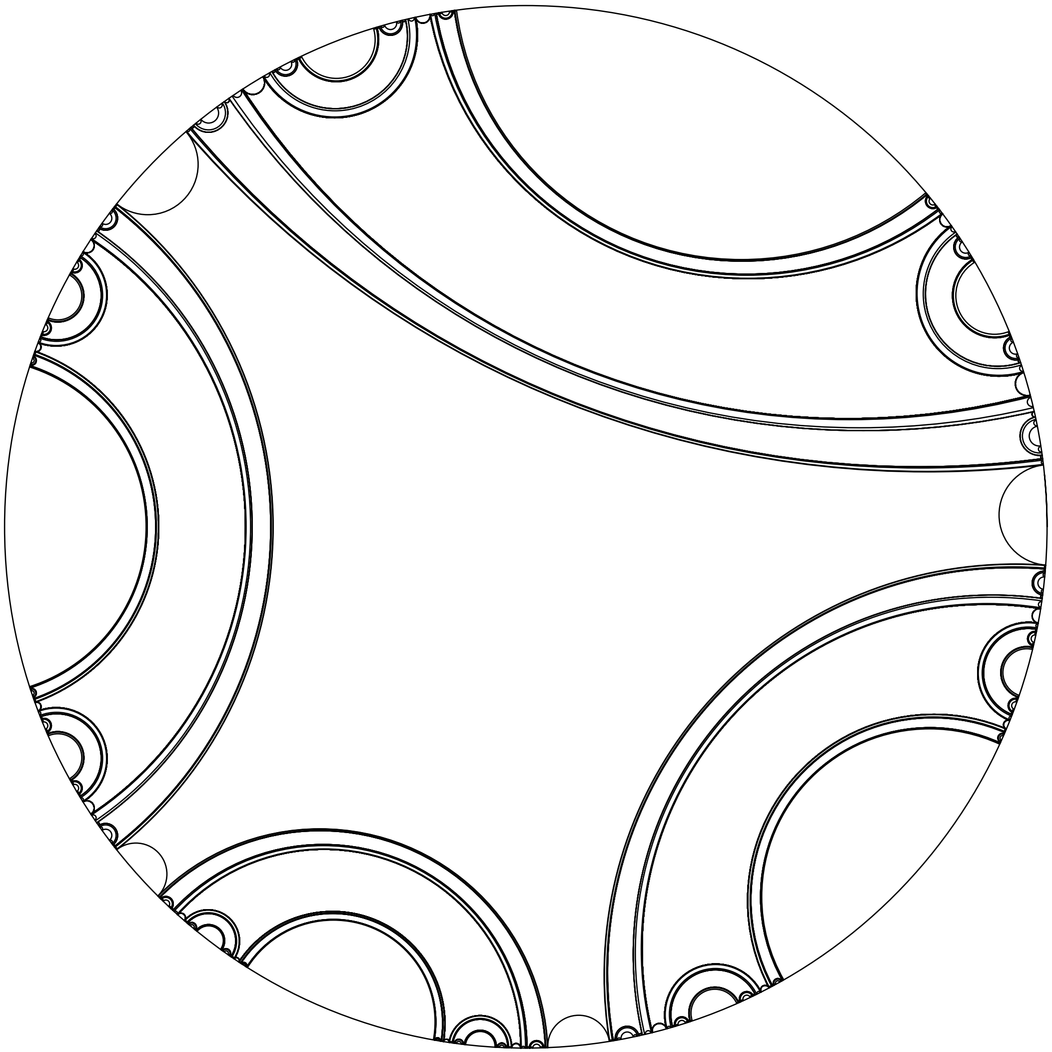
Now $\xi_L^{-1}(\alpha, \alpha)$ consists of 8 preimages instead of 4.

With some suitable modifications, we do similar constructions as before.

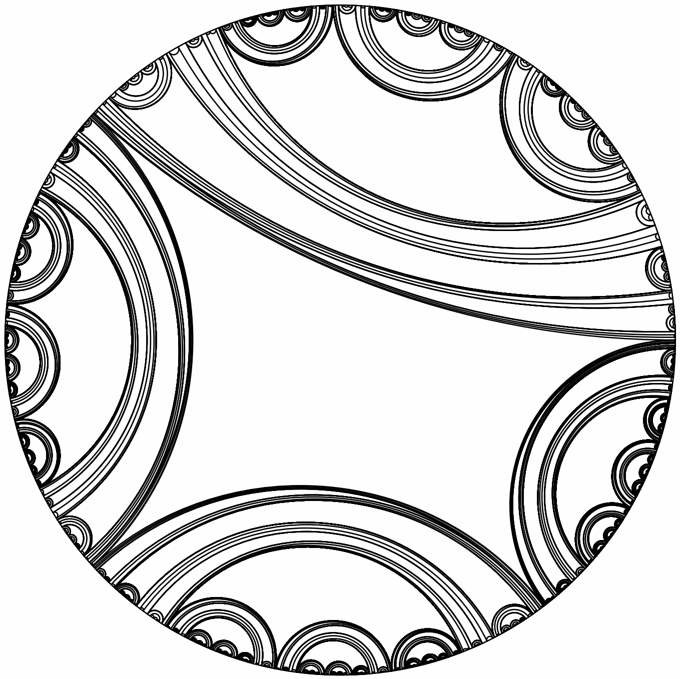
Theorem 2. *There exists a geodesic lamination Λ_L on the disk, associated to the modified Lebesgue space filling curve ξ_L. This lamination has the following properties:*

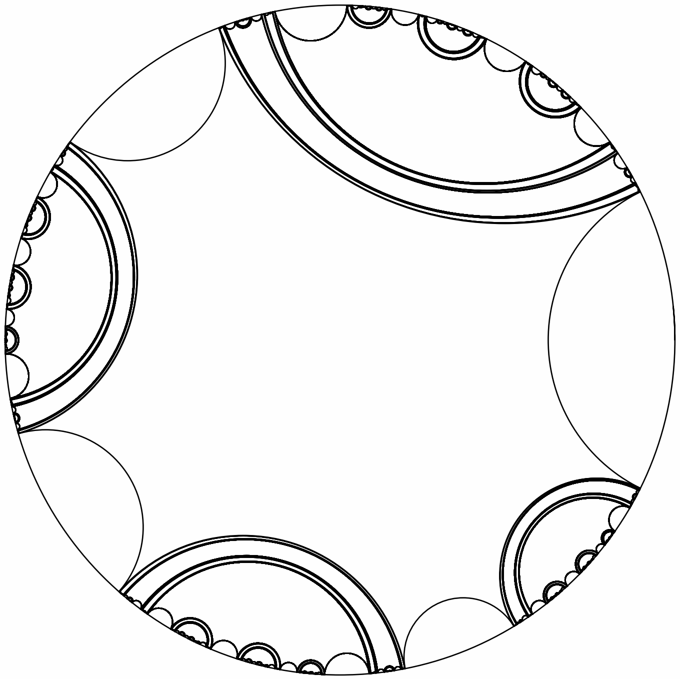
1. *The end points of each element of Λ_L are mapped to the same point in the square by the space filling curve ξ_L.*
2. *Λ_L has an axis of symmetry.*
3. *For any transverse arc to Λ_L, there is a limit set on the boundary of the disk, whose Hausdorff dimension is s_2, the solution of $\alpha^{3s} + \alpha^s \beta^{2s} = 1$. And, the image of this limit set under ξ_L is a straight line between the points, which are the images under ξ_L of the end points of the geodesics, joined by the transverse arc.*
4. *The lamination has a transverse measure μ and there is a continuous map $F : \Lambda_L \to \Lambda_L$, so that $F_* \mu = (1 + 2(\alpha\beta)^{s_2})\mu$.*

Lebesgue lamination



Other laminations





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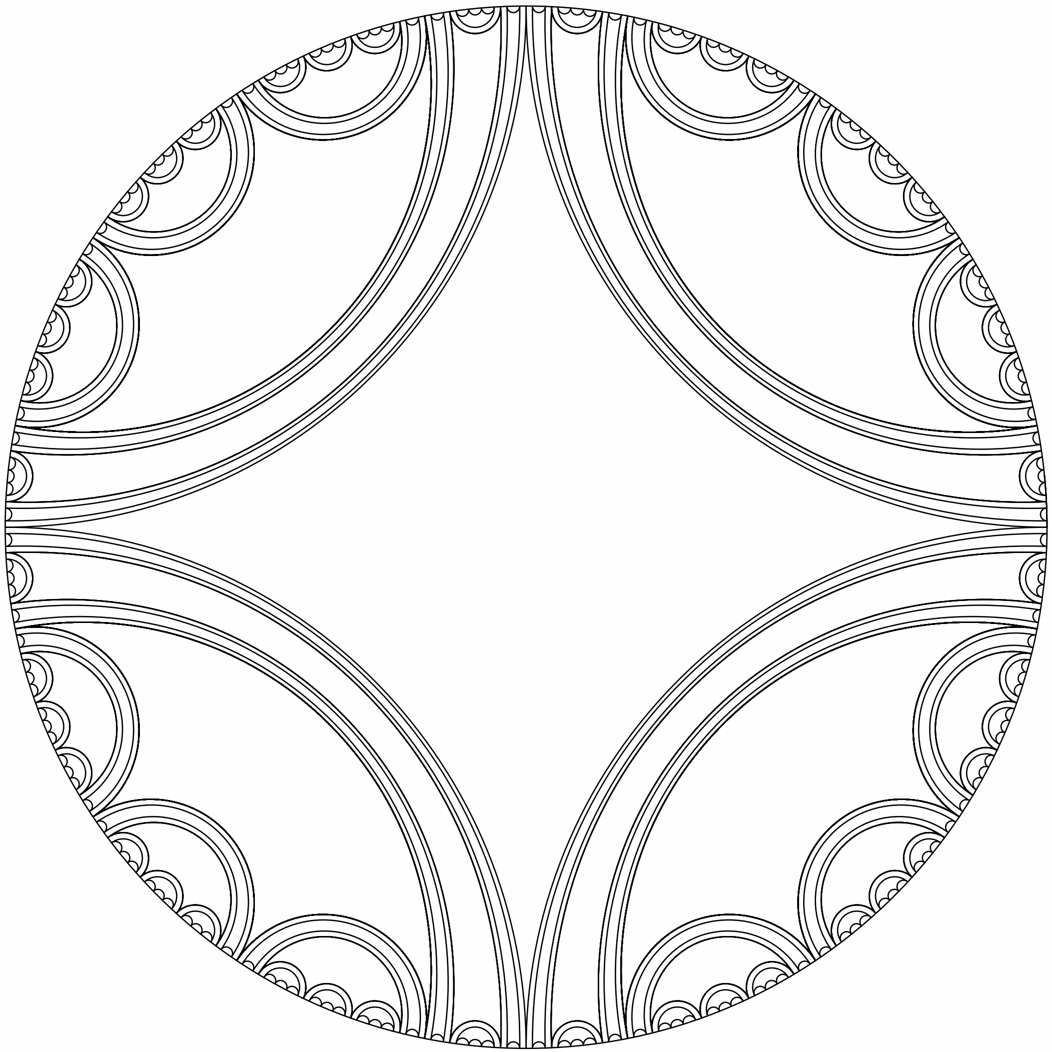


General set up

$\{f_1, \cdots, f_k\}$ IFS on $\mathbb{R}^d$, $d \geq 2$

- $f_i(x) = A_i x + \mathbf{v}_i$, $|\det A_i| < 1$.
- Open set condition.
- Let $\mathfrak{F}$ be the attractor of the IFS and s its Hausdorff dimension.
- Common point condition: $\exists y \in \mathfrak{F}$ such that $f_1(y) = \cdots = f_k(y)$.
- IFS is conformal: $\|f_i(x) - f_i(x')\| = c_i \|x - x'\|$.

General set up

IFS on $I = [0, 1]$:

Let $\alpha_i = c_i^s = (|\det A_i|)^{s/d}$, for $1 \leq i \leq k$ and $0 < \beta_1 < \alpha_1$.

$$h_i(t) = \begin{cases} \alpha_i t + \beta_i & \text{if} \quad 0 \leq t < t^* \\ \alpha_i t + \beta_i - \alpha_i & \text{if} \quad t^* \leq t < 1 \end{cases}$$

where $t^* = (\alpha_1 - \beta_1)/\alpha_1$ and $\beta_i = \sum_{j=1}^{i} \alpha_j - \alpha_i t^*$.

The IFS $\{h_1, \cdots, h_k\}$ satisfies the OSC.

The map $\xi : I \to \mathfrak{F}$ is defined as before.

Main Theorem

Let be an IFS on $\mathbb{R}^d$ such that

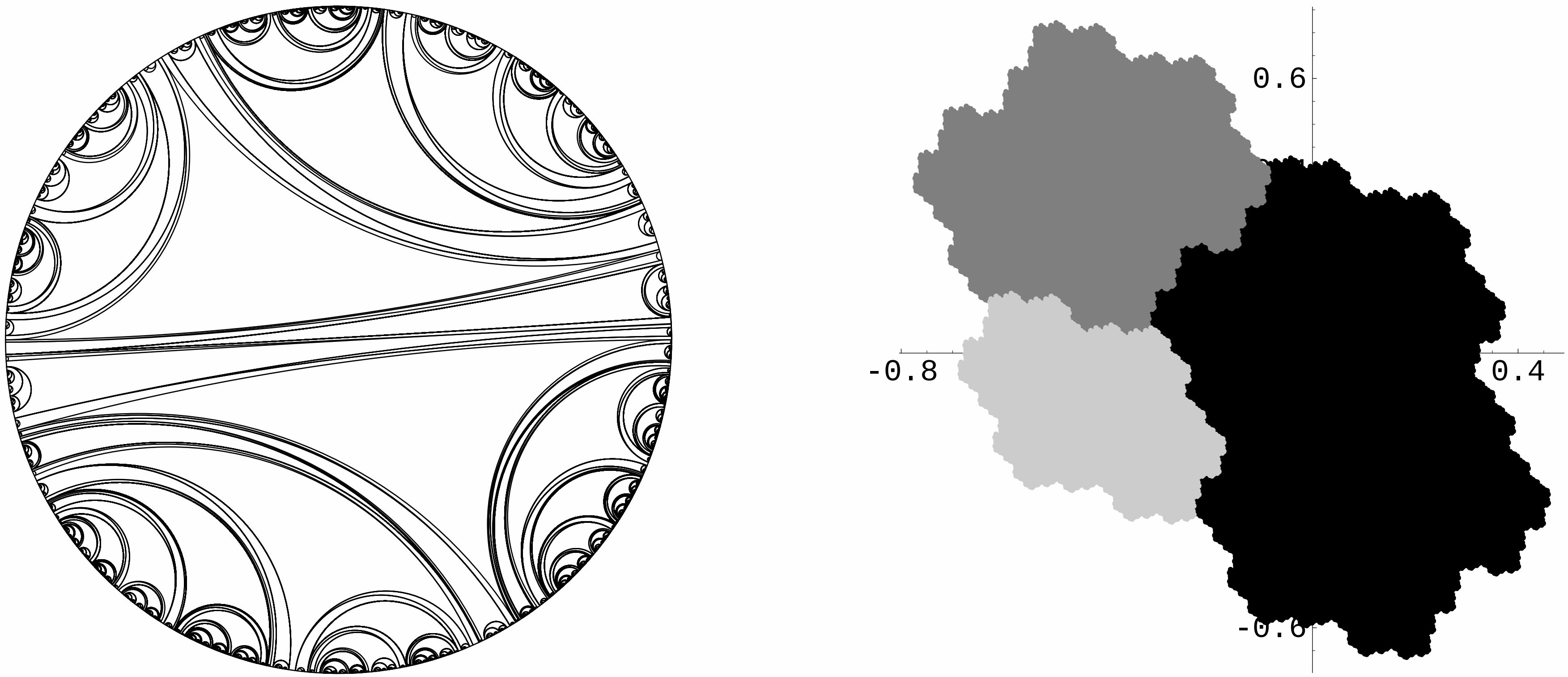
- It satisfies the open set condition.
- Let $\mathfrak{F}$ be its attractor and s its Hausdorff dimension.
- There exists a point $y \in \mathfrak{F}$ such that $f_1(y) = f_2(y) = \cdots = f_k(y)$.

Then there exist a space filling curve $\xi : I \to \mathfrak{F}$ and a geodesic lamination of Λ on the disk with the following properties:

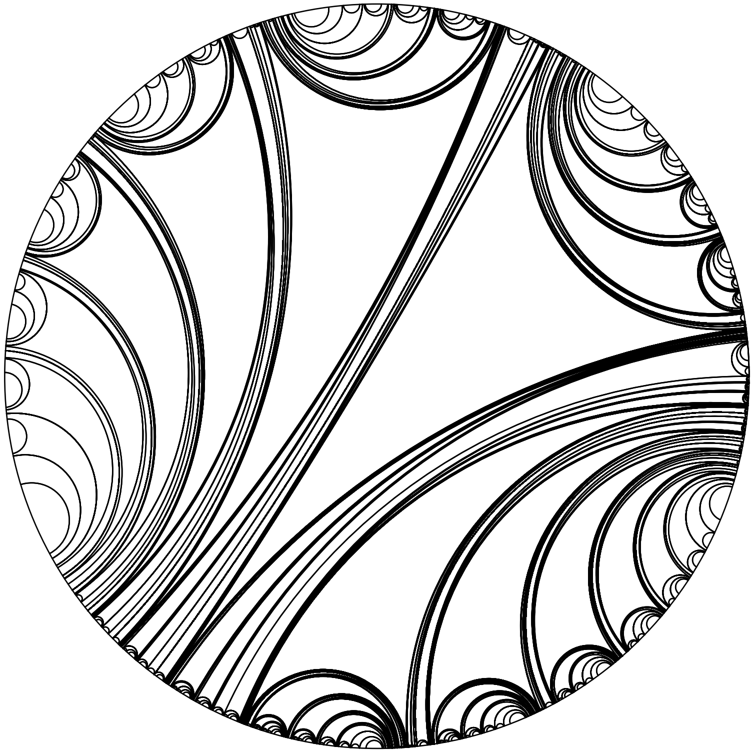
Main Theorem

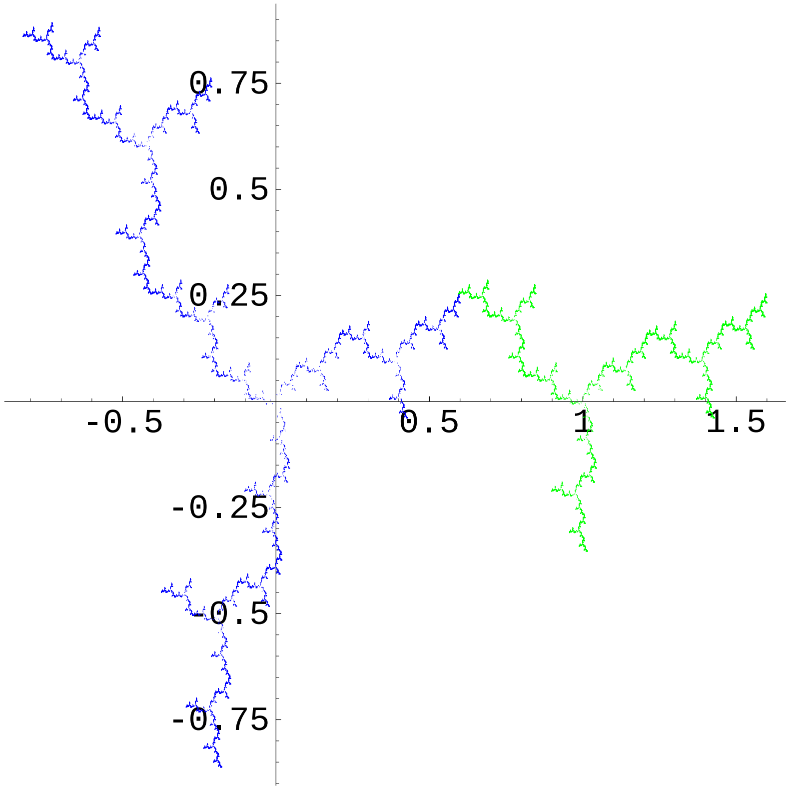
1. ξ is a measure preserving map between the Lebesgue measure on I and the s-dimensional Hausdorff measure of $\mathfrak{F}$.
2. If the IFS is conformal then ξ is Hölder continuous with exponent $1/s$.
3. The end points of each element of Λ are mapped to the same point in the square by the space filling curve ξ.
4. For any transverse arc to Λ, there is a limit set on the boundary of the disk.
5. The lamination has a transverse measure μ and there is a continuous map $F : \Lambda \to \Lambda$, so that $F_*\mu = (a_1^{s_0} + \cdots + a_k^{s_0})\mu$, where s_0 is the Hausdorff dimension of a limit on the boundary of the disk obtained by any transverse arc to Λ.

Lamination of The Rauzy Fractal

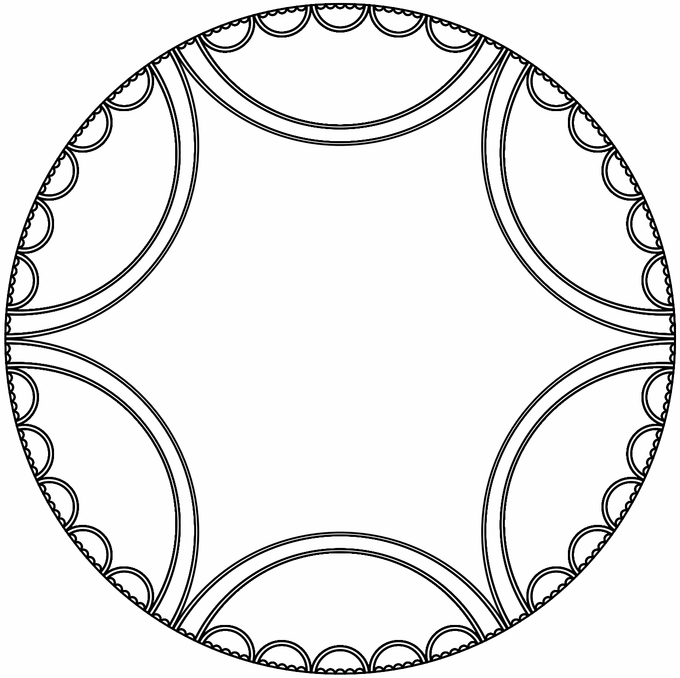
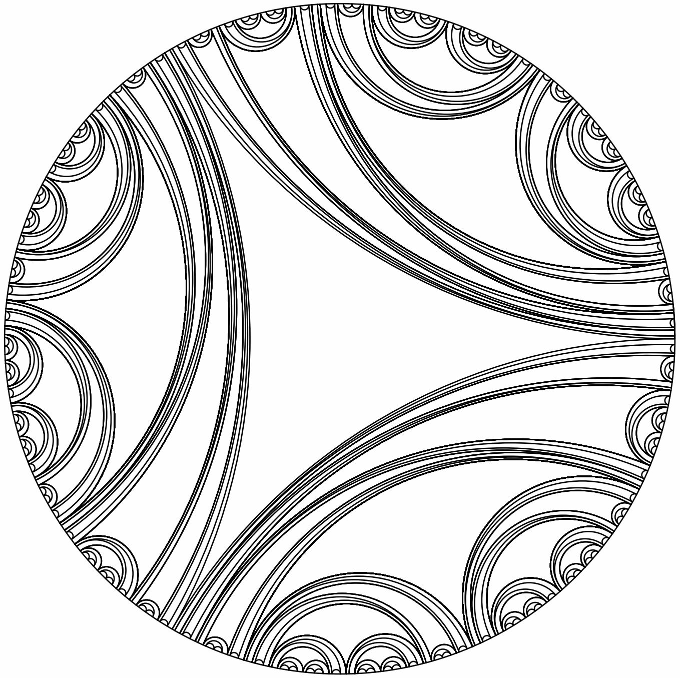


Other lamination





Other laminations

The open set condition is not satisfied in these examples.

Work in progress

Generalization of the previous results for Graph directed IFS-s.