

$$E_s = \{x \in \mathbb{R}^d : Q[x] \leq s\}$$

$\text{vol } B$ the Lebesgue measure of B

$\text{vol}_{\mathbb{Z}} B = |B \cap \mathbb{Z}^n|$ the number of integral points in B

$I(w) = I_r(w)$ the interval with endpoints r^2 and $r^2 + w$

By monotonicity for $0 < w < r^2$

$$V_{-w}(r) \leq \text{vol}_{\mathbb{Z}}(E_{r^2}) \leq V_w(r)$$

where

$$V_w(r) = \frac{1}{|w|} \int \text{vol}_{\mathbb{Z}}(E_v) dv$$

$$I(w)$$

$$R_{\pm w}(r) = \frac{1}{w} \int (\text{vol}_{\mathbb{Z}}(E_v) - \text{vol } E_v) dv$$

$$\mathcal{I}(\pm w)$$

We have, for $r \geq 1, 0 < w < r^2$,

$$\frac{1}{w} \int \text{vol}(E_v) dv = \text{vol } E_{r^2} + wr^{d-2} R$$

$$\mathcal{I}(\pm w)$$

where $|R| \ll_Q \text{vol } E_1$. Then we get

$$|\text{vol}_{\mathbb{Z}}(E_{r^2}) - \text{vol } E_{r^2}| \leq$$

$$\leq \max_{\pm} |R_{\pm w}(r)| + c_Q wr^{d-2}$$

$$\theta(z) = \sum_{m \in \mathbb{Z}^d} \exp \{-z Q[m]\} \quad \text{the theta series of } Q$$

$$\theta_0(z) = \int_{\mathbb{R}^d} \exp \{-z Q[x]\} dx = (\det Q)^{-1/2} \pi^{d/2} z^{-d/2}$$

the theta integral of Q .

$$\frac{1}{2\pi i} \int_{\gamma} \exp \{z T\} \frac{dz}{z^2} = \max \{T, 0\}, \quad \gamma: [b-i\infty, b+i\infty] \subset \mathbb{C}$$

$$\int_0^s \text{vol}_{\mathbb{Z}}(E_v) dv = \frac{1}{2\pi i} \int_{\gamma} e^{sz} \theta(z) \frac{dz}{z^2}$$

$$\int_0^s \text{vol}(E_v) dv = \frac{1}{2\pi i} \int_{\gamma} \exp \{sz\} \theta_0(z) \frac{dz}{z^2}$$

$$\begin{aligned}
 R_w(z) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} (\theta(z) - \theta_0(z)) \exp\{z^2 z\} \frac{\exp\{wz\}-1}{wz} \frac{dz}{z} = \\
 &= \int_{\gamma-i\infty}^{\gamma+i\infty} (\theta - \theta_0)(z) \varphi(z) dz , \quad 0 < w < z^2
 \end{aligned}$$

where $\varphi(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \exp\{z^2 z\} \frac{\exp\{wz\}-1}{wz^2}$

For $z^2 \geq w > 0, z \geq 1$, we have

$$|\varphi(z^{-2}+it)| \ll \frac{1}{w|z^{-2}+it|^2} \quad \text{and}$$

$$|\varphi(z^{-2}+it)| \ll |z^{-2}+it|^{-1}$$

We choose $\beta = \tau^{-2}$ and introduce $K_\tau = \pi \tau^{-\frac{1}{2}}$.

Consider the line segments $\gamma_0 = [\beta - i K_\tau, \beta + i K_\tau]$ and $\gamma_1 = (\beta + i R) \setminus \gamma_0$. Split

$$R_w(z) = \int_{\gamma_0} (\theta - \theta_0)(z) \varphi(z) dz - \int_{\gamma_1} \theta_0(z) \varphi(z) dz + \\ + \int_{\gamma_1} \theta(z) \varphi(z) dz = I_0 + I_1 + I_2.$$

The bound of I_2 involves the crucial dimension dependent part of the argument.

Poisson summation.

$$\begin{aligned} \sum_{m \in \mathbb{Z}^d} \exp \left\{ -z Q[m+a] + 2\pi i \langle m, b \rangle \right\} &= \\ = (\det(Q/\pi))^{-1/2} z^{-d/2} \exp \left\{ -2\pi i \langle a, b \rangle \right\} \\ \times \sum_{n \in \mathbb{Z}^d} \exp \left\{ -\frac{\pi^2}{z} Q^{-1}[n+b] - 2\pi i \langle a, n \rangle \right\} \end{aligned}$$

$\text{Re } z > 0, a, b \in \mathbb{R}^d$

Special case $a = 0$

$$\begin{aligned} \sum_{m \in \mathbb{Z}^d} \exp \left\{ -z Q[m] + 2\pi i \langle m, b \rangle \right\} &= \\ = (\det(Q/\pi))^{-1/2} z^{-d/2} \sum_{n \in \mathbb{Z}^d} \exp \left\{ -\frac{\pi^2}{z} Q^{-1}[n+b] \right\} \end{aligned}$$

Using Poisson summation and the inequality

$$|\varphi(r^{-2}+it)| \ll |r^{-2}+it|^{-1} \text{ we get}$$

$$\underline{I_0 \ll_Q r^{d/2}}$$

$$\underline{I_1 \ll_d \left| \int_{\gamma_1} \frac{\varphi(r^{-2}+it)}{(r^{-2}+it)^{d/2}} dt \right| \ll_d \int_{K_r}^{\infty} \frac{dt}{t^{1+d/2}} \ll_d r^{d/2}}$$

Collecting the bounds obtained so far, we get

$$\begin{aligned} |\text{vol}_{\mathbb{Z}}(E_{r^2}) - \text{vol } E_{r^2}| &\ll \\ &\ll_Q wr^{d-2} + r^{d/2} + I_2 \end{aligned}$$

Estimation of I_2 .

For $r \geq 1, t \in \mathbb{R}$, the following bound holds

$$|\theta(r^{-2} + it)| \ll_Q r^{d/2} \psi(r, t)^{1/2}, \text{ where}$$

$$\psi(r, t) = \sum_{m, n \in \mathbb{Z}^d} \exp\left\{-\frac{r^2}{2} Q^{-1} [\pi m + 2t Q n] - \frac{2}{r^2} Q[n]\right\}$$

$$\text{Let } g_r = \begin{pmatrix} r I_d & 0 \\ 0 & r^{-1} I_d \end{pmatrix} \text{ and } u_t = \begin{pmatrix} I_d & t Q \\ 0 & I_d \end{pmatrix}.$$

Then, for $r \geq 1$,

$$|\theta(r^{-2} + it\pi/2)| \ll_Q r^{d/2} \lambda_d^{2d} (g_r u_t)^{1/2}$$

Consider 2^d sublattices indexed by $\alpha = (\alpha_1, \dots, \alpha_d)$
 with $\alpha_j = 0, 1$, for $1 \leq j \leq d$:

$$\mathbb{Z}_\alpha^d \stackrel{\text{def}}{=} \{m \in \mathbb{Z}^d : m \equiv \alpha \pmod{2}\}.$$

$$\Theta_\alpha(z) \stackrel{\text{def}}{=} \sum_{m \in \mathbb{Z}_\alpha^d} \exp\{-z Q[m]\}.$$

We have $\Theta(z) = \sum_\alpha \Theta_\alpha(z)$ and hence by

the Cauchy-Schwarz inequality

$$|\Theta(z)|^2 \leq 2^d \sum_\alpha |\Theta_\alpha(z)|^2.$$

Using the equalities

$$2(Q[x] + Q[y]) = Q[x+y] + Q[x-y] \text{ and}$$

$$\langle Q(x+y), x-y \rangle = Q[x] - Q[y]$$

we get

$$\begin{aligned} |\theta_\alpha(z^{-2}+it)|^2 &= \theta_\alpha(z^{-2}+it) \overline{\theta_\alpha(z^{-2}+it)} = \\ &= \sum_{m \in \mathbb{Z}_\alpha^d} \exp\{(z^{-2}+it)Q[m]\} \sum_{n \in \mathbb{Z}_\alpha^d} \exp\{(z^{-2}-it)Q[n]\} = \\ &= \sum_{m, n \in \mathbb{Z}_\alpha^d} \exp\{z^{-2}(Q[m]+Q[n])+it(Q[m]-Q[n])\} = \\ &= \sum_{m, n \in \mathbb{Z}_\alpha^d} \exp\left\{-\frac{1}{2z^2}(Q[m+n]+Q[m-n])+it\langle Q(m+n), m-n \rangle\right\} = \\ &= \sum_{\bar{m}, \bar{n} \in \mathbb{Z}^d} \exp\left\{-\frac{2}{z^2}(Q[\bar{m}]+Q[\bar{n}])+4it\langle Q(\bar{m}), \bar{n} \rangle\right\} \end{aligned}$$

$$\text{where } \bar{m} = \frac{m+n}{2}, \bar{n} = \frac{m-n}{2}, \bar{m}, \bar{n} \in \mathbb{Z}^d.$$

In this double sum fix \bar{n} and sum over $\bar{m} \in \mathbb{Z}^d$. Using Poisson summation, we get

$$\begin{aligned}\theta(z, \bar{n}) &\stackrel{\text{def}}{=} \sum_{\bar{m} \in \mathbb{Z}^d} \exp\left\{-\frac{2}{z^2} Q[\bar{m}] + 4it \langle Q(\bar{m}), \bar{n} \rangle\right\} = \\ &= \left(\det\left(\frac{2}{\pi z^2} Q\right)\right)^{-1/2} \sum_{\bar{m} \in \mathbb{Z}^d} \exp\left\{-\frac{z^2}{2} Q^{-1} [\pi \bar{m} + 2t Q \bar{n}]\right\},\end{aligned}$$

and therefore we have

$$\begin{aligned}|\theta(z)|^2 &\ll \\ &\ll_Q z^d \sum_{\bar{m}, \bar{n} \in \mathbb{Z}^d} \exp\left\{-\frac{z^2}{2} Q^{-1} [\pi \bar{m} + 2t Q \bar{n}] - \frac{2}{z^2} Q[\bar{n}]\right\},\end{aligned}$$

which proves the desired estimate for $|\theta(z)|$.

Theorem. Let β be a positive number such that $\beta d > 2$. Then for a lattice Δ in \mathbb{R}^d

$$\int_0^1 \alpha_d(g_r u_t \Delta)^\beta dt \ll_Q \varepsilon^{\frac{\beta d - 2}{2}} \alpha_d(\Delta)^\beta$$

$$\begin{aligned} \int_a^{a+a^2} \alpha_d(g_r u_t \Delta)^\beta dt &= \int_0^{a^2} \alpha_d(g_{ar} u_{s+a} \Delta)^\beta ds = \\ &= \int_0^{a^2} \alpha_d(g_{ar} (g_a^{-1} u_s g_a)(g_a^{-1} u_a \Delta))^\beta ds = \\ &= \int_0^{a^2} \alpha_d(g_{ar} u_{a^{-2}s} (g_{a^{-1}} u_a \Delta))^\beta ds = \\ &= a^2 \int_0^1 \alpha_d(g_{ar} u_{s'} (g_{a^{-1}} u_a \Delta))^\beta ds' \ll \\ &\ll_Q (ar)^{\frac{\beta d - 2}{2}} a^2 \alpha_d(g_{a^{-1}} u_a \Delta)^\beta \end{aligned}$$

$$\int_a^{2a} \alpha_d(g_r u_t \mathbb{Z}^{2d})^\beta dt \ll_Q (\alpha r)^{\beta d-2} a$$

$$\int_a^{2a} \alpha_d(g_r u_t \mathbb{Z}^{2d})^\beta \frac{dt}{t^1} \ll_Q (\alpha r)^{\beta d-2} = r^{\beta d-2} a^{\beta d-2}$$

In the case of irrational Q one has to use also the upper bounds for

$\alpha_d(g_r u_t \mathbb{Z}^{2d})$ which follow from

"the degree of irrationality" of Q .