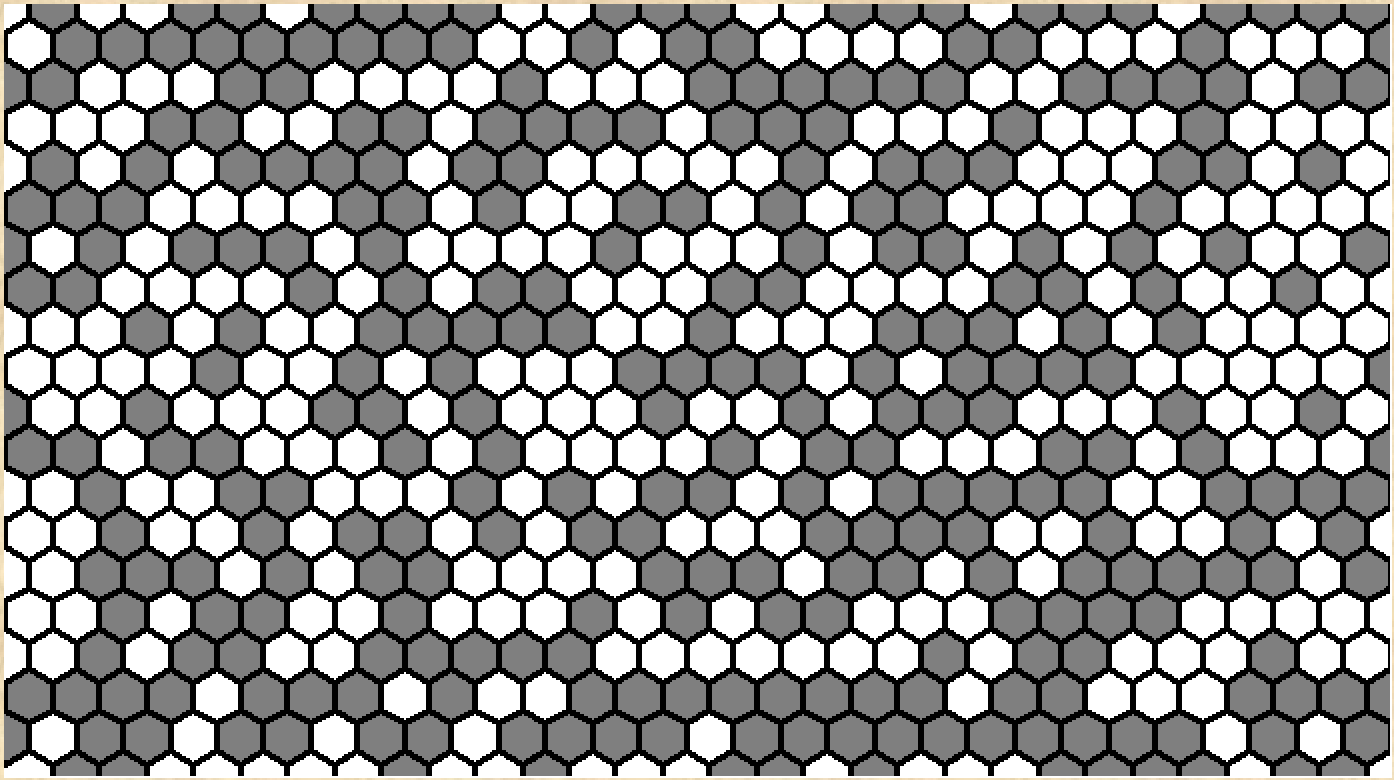
Dynamic percolation, exceptional times, and harmonic analysis of boolean functions

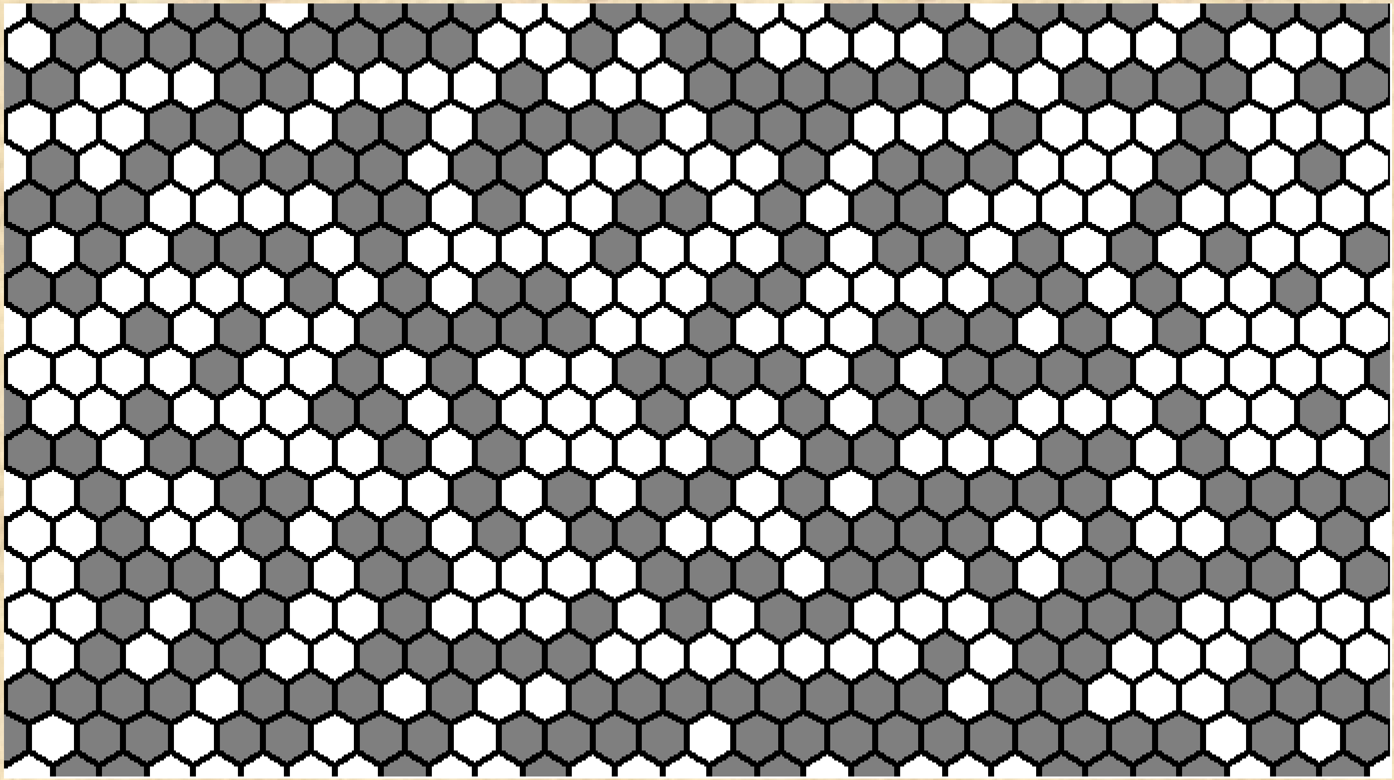
Oded Schramm
joint w/ Jeff Steif

Percolation



$\mathbf{P}[\text{white}] = 1/2$ independently

Dynamic Percolation



Infinite clusters?

For static percolation:

- Harris (1960): There is no infinite cluster
- Kesten (1980): There is if we increase *p*

For dynamic percolation:

- At most times there is no infinite cluster
- Can there be exceptional times?

- Any infinite graph G has a p_c:

$$p_c := \inf\{p \in [0,1] : \mathbf{P}_p[|C(0)| = \infty] > 0\}$$

Häggström, Peres, Steif (1997)

- Above p_c: $\mathbf{P}[\forall t\, \exists \text{ infinite cluster}] = 1$
- Below p_c: $\mathbf{P}[\exists t\, \exists \text{ infinite cluster}] = 0$
- The latter at p_c for Z^d, $d>18$.
- Some (non reg) trees with exceptional times.
- Much about dynamic percolation on trees…

Exceptional times exist

Theorem (SS): The triangular grid has exceptional times at p_c.

This is the only transitive graph for which it is known that there are exceptional times at p_c.

Proof idea 0: get to distance R

- Set

$$I_R := \{t \in [0,1] : 0 \leftrightarrow_t R\}$$

- We show

$$\mathbf{P}[\forall R\ I_R \neq \emptyset] = \inf_R \mathbf{P}[I_R \neq \emptyset] > 0$$

Namely, with positive probability the cluster of the origin is unbounded for t in [0,1].

2^{nd} moment argument

We show that

$$\sup_R \frac{\mathbf{E}[\mu(I_R)^2]}{\mathbf{E}[\mu(I_R)]^2} < \infty.$$

Then use Cauchy-Schwarz:

$$\mathbf{P}[\mu(I_R) > 0]\mathbf{E}[\mu(I_R)^2] \geq \mathbf{E}[\mu(I_R)]^2$$

2nd moment spelled out

$$\mu(I_R) = \int_0^1 \mathbf{1}_{t \in I_R}\, dt$$

$$\mathbf{E}[\mu(I_R)^2] = \int_0^1 \int_0^1 \mathbf{P}[t, s \in I_R]\, dt\, ds\,.$$

Consequently, enough to show

$$\int_0^1 \mathbf{P}[0, t \in I_R]\, dt \le O(1)\, \mathbf{P}[0 \in I_R]^2\,.$$

Interested in expressions of the form

$$\mathbf{E}[f(\omega_0)f(\omega_t)]$$

Where ω_t is the configuration at time *t*, and *f* is a function of a static configuration.

Rewrite

$$\mathbf{E}[f(\omega_0)\mathbf{E}[f(\omega_t)|\omega_0]] = \mathbf{E}[f\,T_t f]$$

where

$$T_t f(\omega) = \mathbf{E}[f(\omega_t)|\omega_0 = \omega]$$

Understanding T_t

Set

$$u_v(\omega) := \begin{cases} 1 & v \text{ is open} \\ -1 & v \text{ is closed} \end{cases}$$

Then

$$T_t u_v = e^{-t} u_v$$

Set $$u_S := \prod_{v \in S} u_v \qquad S \subset V$$

Then $$T_t\, u_S = e^{-t|S|} u_S$$

Write $$f(\omega) = \sum_{S \subset V} \widehat{f}(S)\, u_S(\omega)$$

$$\widehat{f}(S) := \mathbf{E}\Big[u_S\, f\Big]$$

$$\mathbf{E}\Big[f\,T_t\,f\Big] = \sum_S e^{-t|S|}\,\widehat{f}(S)^2$$

$$= \sum_k e^{-tk} \sum_{|S|=k} \widehat{f}(S)^2$$

Noise sensitivity

Theorem (BKS): When f_n is the indicator function for crossing an *n x n* square in percolation, for all positive *t*

$$\lim_{n\to\infty} \mathbf{E}\Big[f_n\, T_t\, f_n\Big] - \mathbf{E}\Big[f_n\Big]^2 = 0.$$

Equivalently, for all $t>0$

$$\lim_{n\to\infty} \mathrm{var}(T_t f_n) = 0.$$

Equivalently, for all k fixed

$$\lim_{n\to\infty} \sum_{|S|=k} \hat{f}_n(S)^2 = 0.$$

Need more quantitative

Conjectured (BKS):

$$\lim_{n\to\infty} \mathbf{E}\Big[f_n\, T_{t_n}\, f_n\Big] - \mathbf{E}\Big[f_n\Big]^2 = 0,$$

with $t_n = n^{-\epsilon}$.

We (SS) prove this (for Z^2 and for the triangular grid).

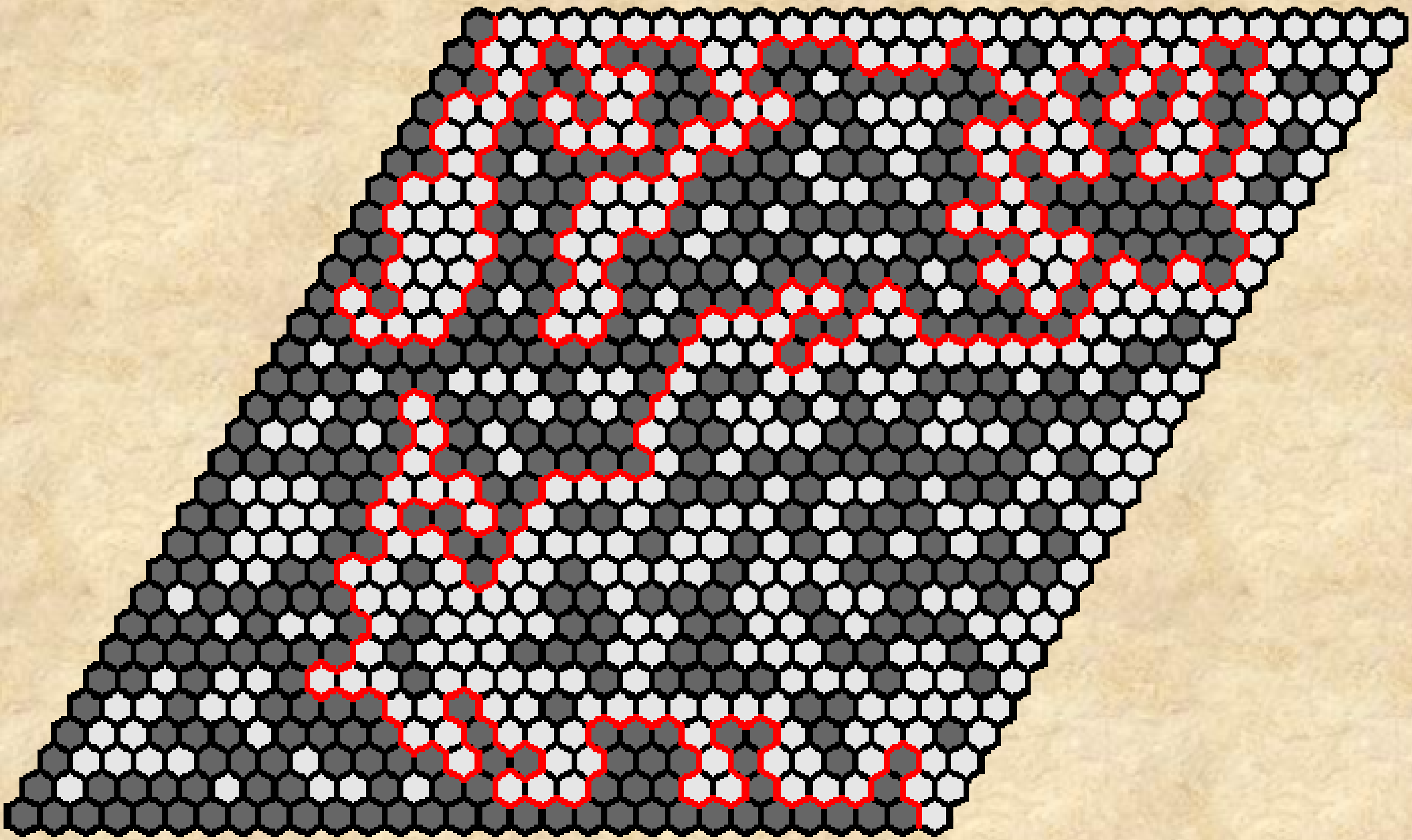
Estimating the Fourier weights

Theorem (SS): Suppose that there is a randomized algorithm for calculating *f* that examines each bit with probability at most δ. Then

$$\sum_{|S|=k} \hat{f}(S)^2 \leq \underset{?}{\cancel{k}}\,\delta\,\|f\|^2.$$

Probably not tight.

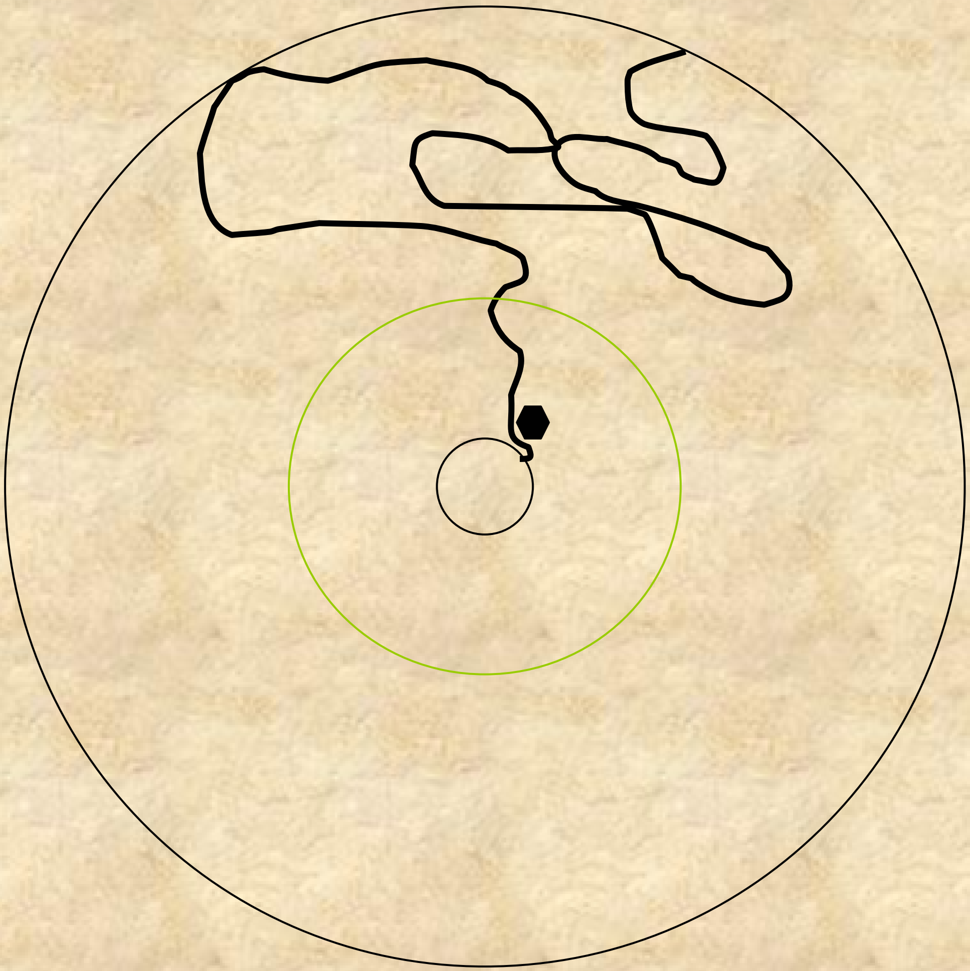
The δ of percolation



$\delta \approx \alpha_2(0,n) + \alpha^+(0,n)$ [LSW]

Not optimal, (simulations) [PSSW, Sim]

Annulus case δ



Annulus case δ

The δ for the algorithm calculating $\text{cross}(r \leftrightarrow R)(\omega)$ is approximately

$$\delta \approx \underbrace{\alpha(r, R)}_{\text{get to approx radius r}} \underbrace{\alpha_2(0, r)}_{\text{visit a particular hex}}$$

Putting it together

$$\mathbf{P}\Big[0,t \in I_R\Big] \le \alpha(0,r)\,\mathbf{P}\Big[r \overset{0,t}{\leftrightarrow} R\Big]$$

$$\mathbf{P}\Big[r \overset{0,t}{\leftrightarrow} R\Big] = \sum_{k\ge 0} e^{-kt} \sum_{|S|=k} \hat{f}_r^R(S)^2$$

$$\le \hat{f}_r^R(\emptyset)^2 + \sum_{k>0} e^{-kt} k\,\delta_r^R\,\|f_r^R\|^2$$

$$\approx \alpha(r,R)^2 + t^{-2}\alpha_2(0,r)\,\alpha(r,R)^2$$

Etc...

What about Z^2 ?

- The argument almost applies to Z^2

1. Have $\alpha_2(r,R) \le \alpha(r,R)^2$

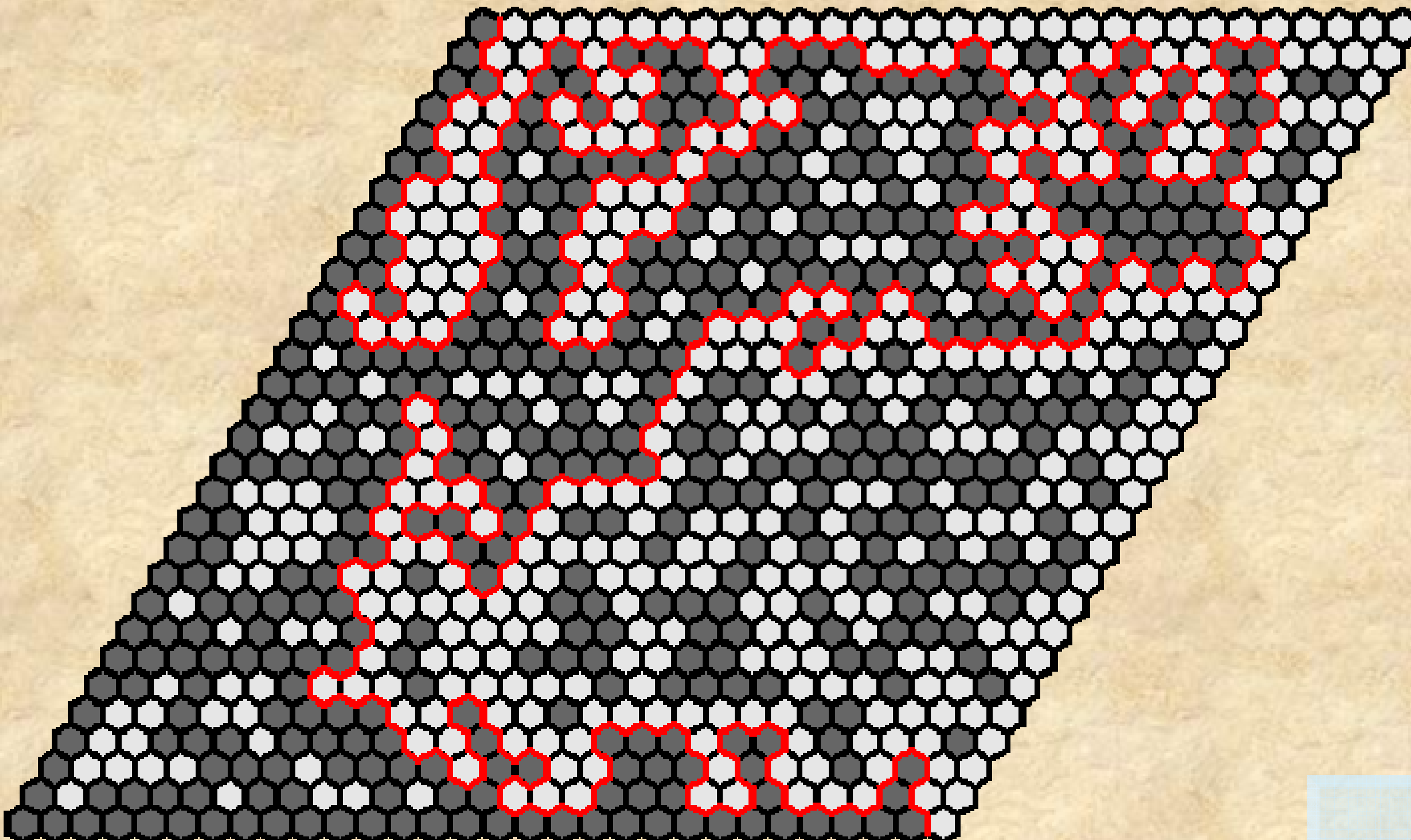
   Need $\alpha_2(r,R) \le \alpha(r,R)^2\,(r/R)^\epsilon$

2. Improve δ (better algorithm)

3. Improve Fourier theorem

4. Calculate exponents for Z^2

Interface



How small can δ be

BSW:

- Example with $\delta \approx n^{-1/2}\sqrt{\log n}$
- Monotone example with $\delta \approx n^{-1/3}\log n$
- These are balanced
- Best possible, up to the log terms
- The monotone example shows that the Fourier inequality is sharp at k=1.

Next

- Proof of Fourier estimate
- Further results
- Open problems
- End

Theorem (SS): Suppose that there is a randomized algorithm for calculating *f* that examines each bit with probability at most **δ**. Then

$$\sum_{|S|=k} \hat{f}(S)^2 \le k\,\delta\,\|f\|^2.$$

Fourier coefficients change under algorithm

When algorithm examines bit i :

$$u_S \longrightarrow \begin{cases} \pm u_{S\setminus\{i\}} & i \in S, \\ u_S & i \notin S. \end{cases}$$

$$\text{new } \hat{f}(S) = \begin{cases} \hat{f}(S) \pm \hat{f}(S \cup \{i\}), & i \notin S, \\ 0, & i \in S. \end{cases}$$

Proof of Fourier Thm

Set $$g := \sum_{|S|=k} \hat{f}(S)\, u_S$$

$$\sum_{|S|=k} \hat{f}(S)^2 = \mathbf{E}\Big[g\, f\Big]$$

$$= \mathbf{E}\Big[\mathbf{E}[g\, f \mid A]\Big] = \mathbf{E}\Big[f\, \mathbf{E}[g \mid A]\Big]$$

$$\leq \|f\|\, \mathbf{E}\Big[\mathbf{E}[g \mid A]^2\Big]^{1/2} = \|f\|\, \mathbf{E}\Big[\widehat{g^A}(\emptyset)^2\Big]^{1/2}$$

$$\sum_{|S|=k} \hat{f}(S)^2 \le \|f\| \, \mathbf{E}\Big[\widehat{g^A}(\emptyset)^2\Big]^{1/2}$$

$$\mathbf{E}\Big[\widehat{g^A}(\emptyset)^2\Big] = \mathbf{E}\Big[\|g^A\|^2\Big] - \sum_{S \ne \emptyset} \mathbf{E}\Big[\widehat{g^A}(S)^2\Big]$$

$$\le \|g\|^2 - \sum_{|S|=k} \hat{g}(S)^2 \, \mathbf{P}\Big[S \cap A = \emptyset\Big]$$

$$= \sum_{|S|=k} \hat{g}(S)^2 \, \mathbf{P}\Big[S \cap A \ne \emptyset\Big] \le k\,\delta\,\|g\|^2$$

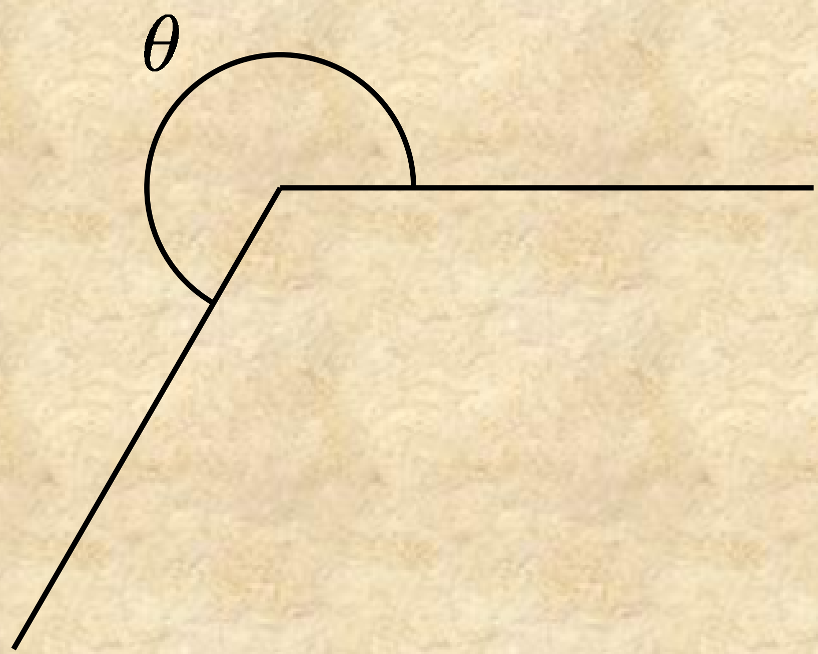
QED

Further results

$$\frac{1}{6} \leq \mathsf{dim}(\mathsf{exceptional\ times}) \leq \frac{31}{36}$$

Never 2 infinite clusters.

Wedges



Wedges

- Exceptional times with k different infinite clusters if

$$\theta > 8\,\pi\,\frac{k(2k-1)}{3}\,.$$

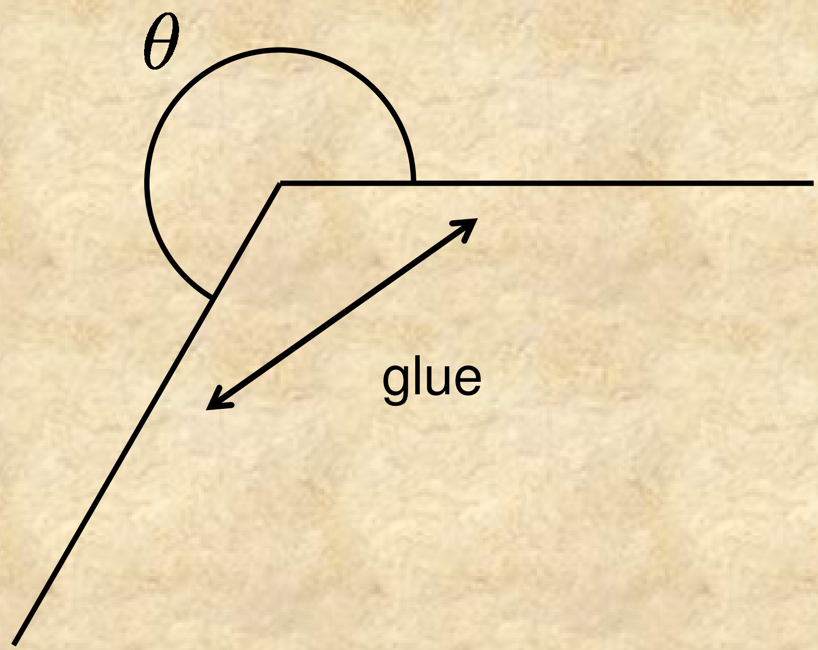
- None if

$$\theta < 4\,\pi\,\frac{k(2k-1)}{9}\,.$$

- HD:

$$1-8\frac{(2k-1)k\pi}{3\theta} \le \dim \le 1-\frac{4(2k-1)k\pi}{9\theta}\,.$$

Cones



Cones

- Exceptional times with k>1 different infinite clusters if

$$\theta > 4\,\pi\,\frac{4k^2-1}{3}\,.$$

- None if

$$\theta < 2\,\pi\,\frac{4k^2-1}{9}\,.$$

- HD:

$$1-4\,\pi\,\frac{4k^2-1}{3\theta} \le \mathrm{dim} \le 1-2\,\pi\,\frac{4k^2-1}{9\theta}\,.$$

Open problems

- Improve estimate for δ
- Improve Fourier Theorem
- Percolation Fourier coefficients
- Settle $\mathbb{Z}^2$
- Correct Hausdorff dimension?
- Space-time scaling limit

The End