Sample SLE argument and HE convergence

Oded Schramm
Microsoft Research
http://research.microsoft.com/~schramm

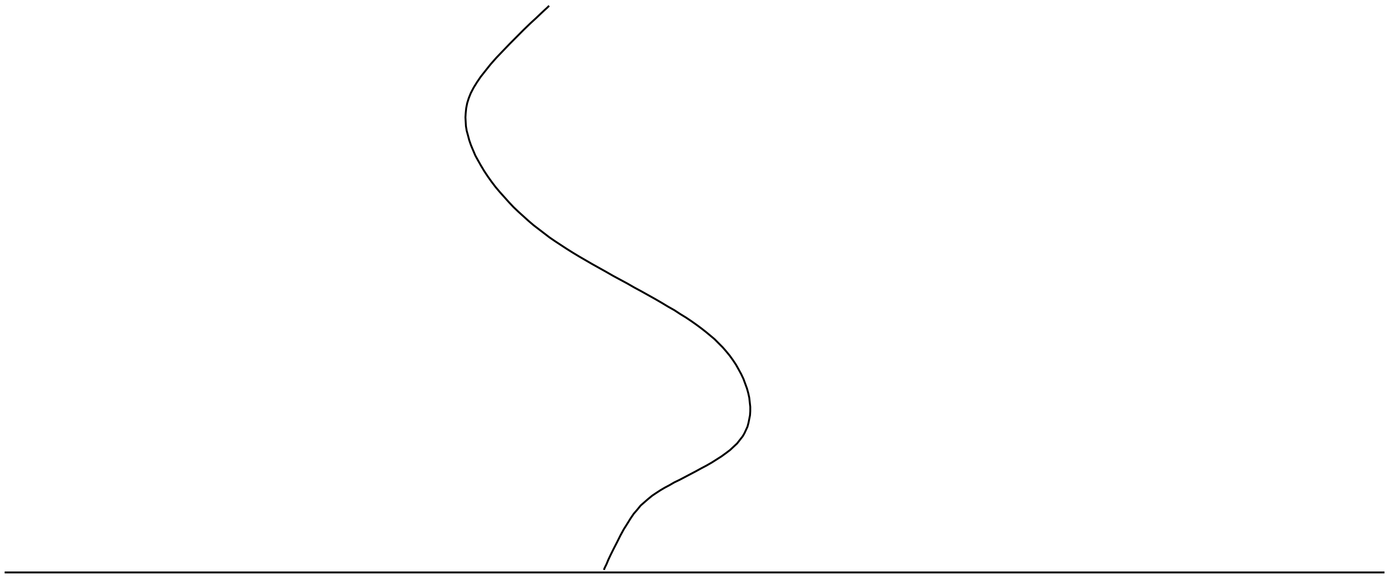
Plan

- Review (different perspective)
- An SLE calculation: Cardy's formula
- Harmonic explorer $\mapsto$ SLE(4)

Next talk tomorrow: Dynamical Percolation

Loewner's equation

Let $\gamma : [0, T] \to \overline{\mathbb{H}}$ be a simple path in the upper half plane, such that $\gamma(0) = 0$ and $\gamma(0, T] \subset \mathbb{H}$.



Set $K_t := \gamma[0, t]$.

By Riemann's mapping theorem, for $t \in [0, \infty)$ there is a unique conformal homeomorphism

$$g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$$

satisfying the hydrodynamic normalization

$$g_t(z) = z + a_1/z + a_2/z^2 + \cdots \qquad z \to \infty\,.$$

Capacity

The capacity is

$$A = A(g_t) = A(K_t) = a_1 = \lim_{z\to\infty} z(g_t(z) - z).$$

It is easy to check that if f and g are normalized in this way, then

$$A(g \circ f) = A(g) + A(f).$$

Looking at the boundary values, we find

$$\mathrm{Im}(g_t(z) - z) \leqslant 0.$$

This implies that $a_1 > 0$ and $A(g_t)$ is increasing.

Loewner's Thm

Assume γ parameterized by capacity: $A(g_t) = 2\,t$.

Set $W_t := g_t(\gamma(t))$. Then $W_t \in \mathbb{R}$.

Theorem (Loewner). (in the chordal setting)

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}.$$

$\gamma(t)$ can be recovered by $\gamma(t) = g_t^{-1}(W_t)$.

Inverse Loewner

Conversely, if we start with a continuous

$$W : [0, \infty) \to \mathbb{R}\,,$$

then we may solve Loewner's equation and get the maps

$$g_t : \mathbb{H} \setminus K_t \to \mathbb{H}\,,$$

where K_t is the set of points in $\overline{\mathbb{H}}$ hitting the singularity by time t under the Loewner flow.

However, K_t is not necessarily an arc of a path.

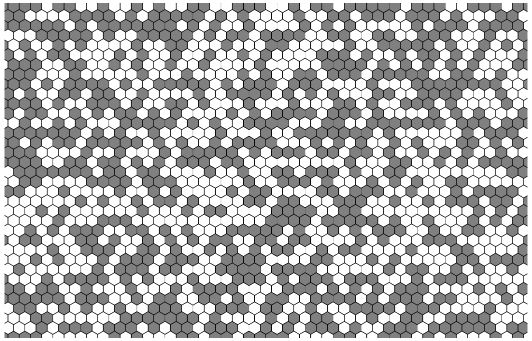
SLE (chordal)

SLE(κ) is obtained by choosing $W_t := B(\kappa t)$, where B is Brownian motion in $\mathbb{R}$ and $\kappa \geqslant 0$ is a parameter.

In this case, $\gamma(t)$ is well defined and continuous (Rohde-Schramm for $\kappa \neq 8$, Lawler-Schramm-Werner for $\kappa = 8$), but it is a simple path a.s. only in the range $\kappa \in [0, 4]$.

Cardy's formula

What is the probability of a white left-right crossing of a rectangle for critical percolation?



Cardy guessed that in the limit as the mesh goes to zero the answer is

$$\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{1}{3}\right)}\,\eta^{\frac{1}{3}}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{4}{3};\eta\right).$$

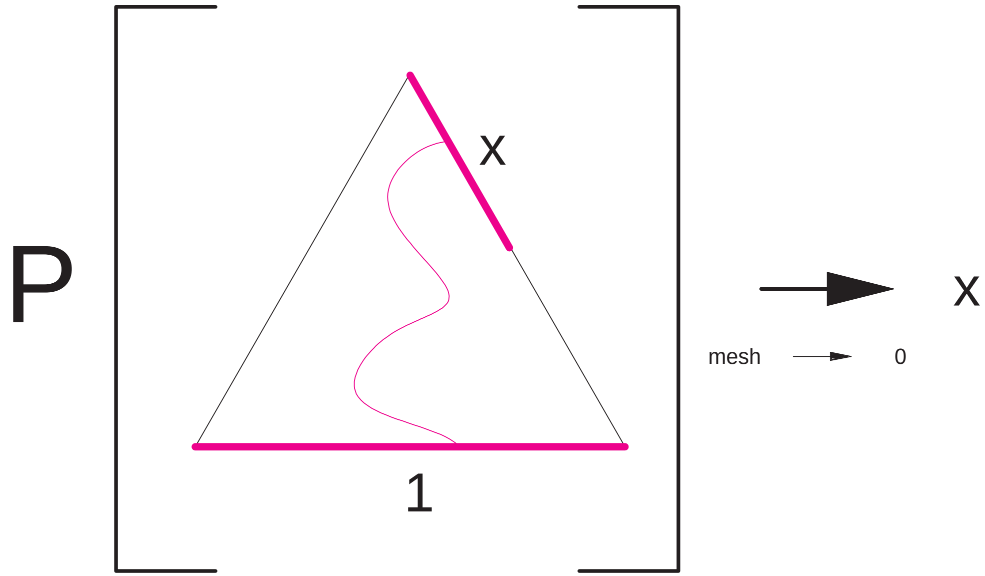
$$\text{Cardy's formula} = \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{1}{3}\right)}\,\eta^{\frac{1}{3}}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{4}{3};\eta\right).$$

Here, ${}_2F_1$ is the hypergeometric function and η is an explicit function of the aspect ratio of the rectangle.

More precisely, η is the cross ratio of the images of the corners under any conformal map mapping the rectangle to the upper half plane $\mathbb{H}$.

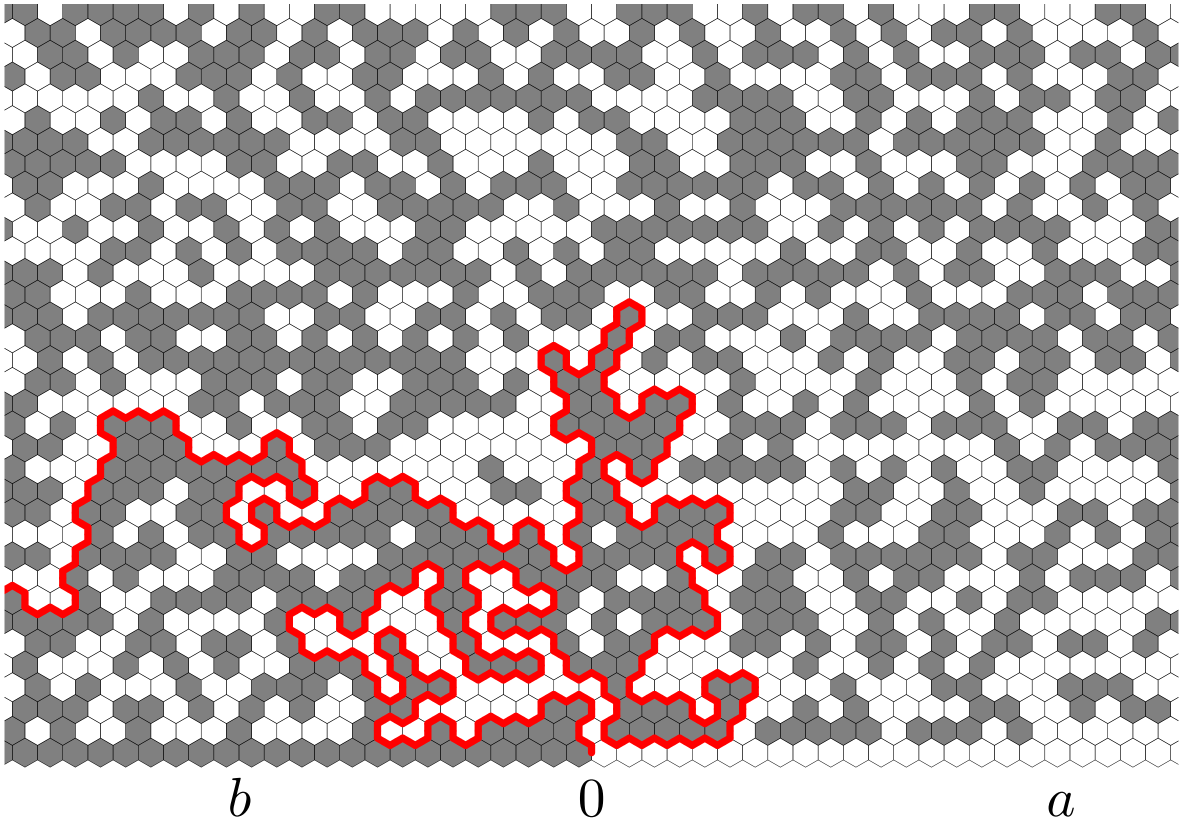
Carleson's version of Cardy's formula

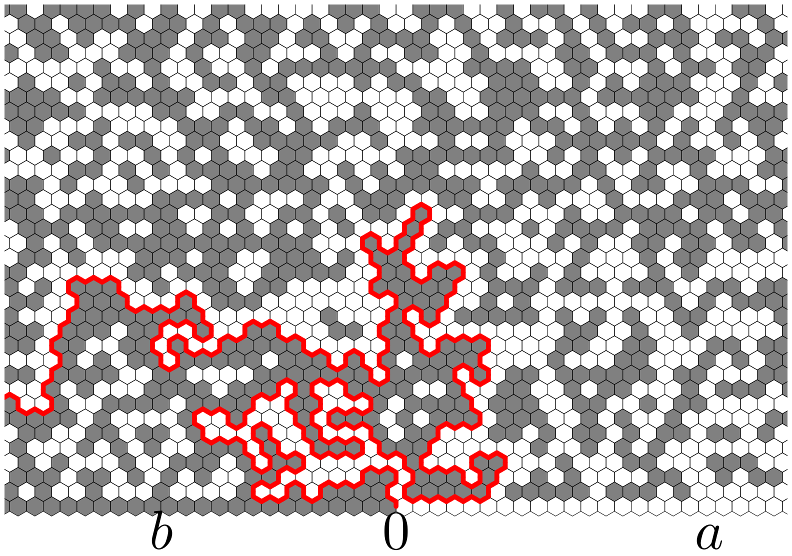
$\forall x \in [0,1]$,



The proof of this formula is central to Smirnov's work.

Cardy's formula for SLE(6)





Which point is swallowed first, a or b?

We basically ask, which tends to zero first, $g_t(a) - w(t)$, or $g_t(b) - w(t)$?

This is basically a one-dimensional problem, since it is enough to consider the restriction of the SLE to the real line.

Derivation of Cardy's formula for SLE(6) (LSW)

Given the restriction of w to $[0,t]$, this is just a function of $(g_t(a), g_t(b), w(t))$. Scale and translation invariance shows that it is, in fact, some function $f(\eta(t))$, where

$$\eta(t) = \frac{g_t(a) - w(t)}{g_t(b) - w(t)}.$$

Note that $f(\eta(t))$ is a martingale:

$$\mathbf{E}[f(\eta(t))] = f(\eta(0)).$$

$$\mathbf{E}[f(\eta(t))] = f(\eta(0)).$$

Therefore, at $t = 0$,

$$\partial_t \mathbf{E}[f(\eta(t))] = 0.$$

We know $\eta(t) = \frac{g_t(a)-w(t)}{g_t(b)-w(t)}$ and

$$\mathbf{E}[w(t)] = 0, \qquad \mathbf{E}[w(t)^2] = \kappa\, t, \qquad \partial_t g(x) = 2/(g(x) - w(t)).$$

Taylor series at $t = 0$ for $\mathbf{E}[f(\eta(t))]$ gives

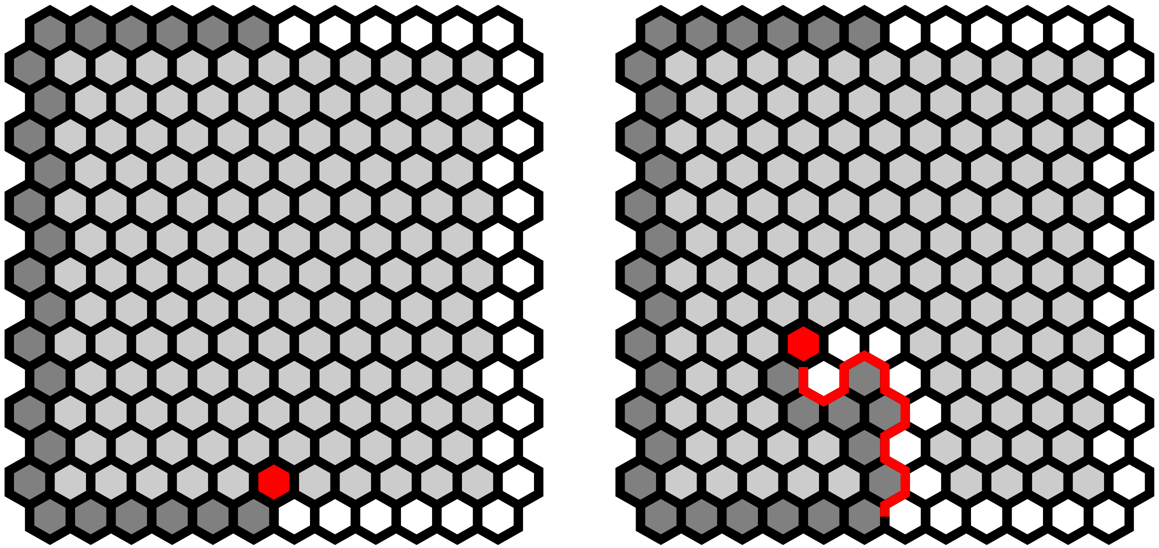
$$\kappa\,(\eta - 1)\,\eta\, f''(\eta) \,+ 2\,((\kappa - 2)\,\eta - 2)\, f'(\eta) = 0.$$

The only solution for this satisfying $f(0) = 1, f(-\infty) = 0$ is given by Cardy's formula.

Hidden: Itô calculus

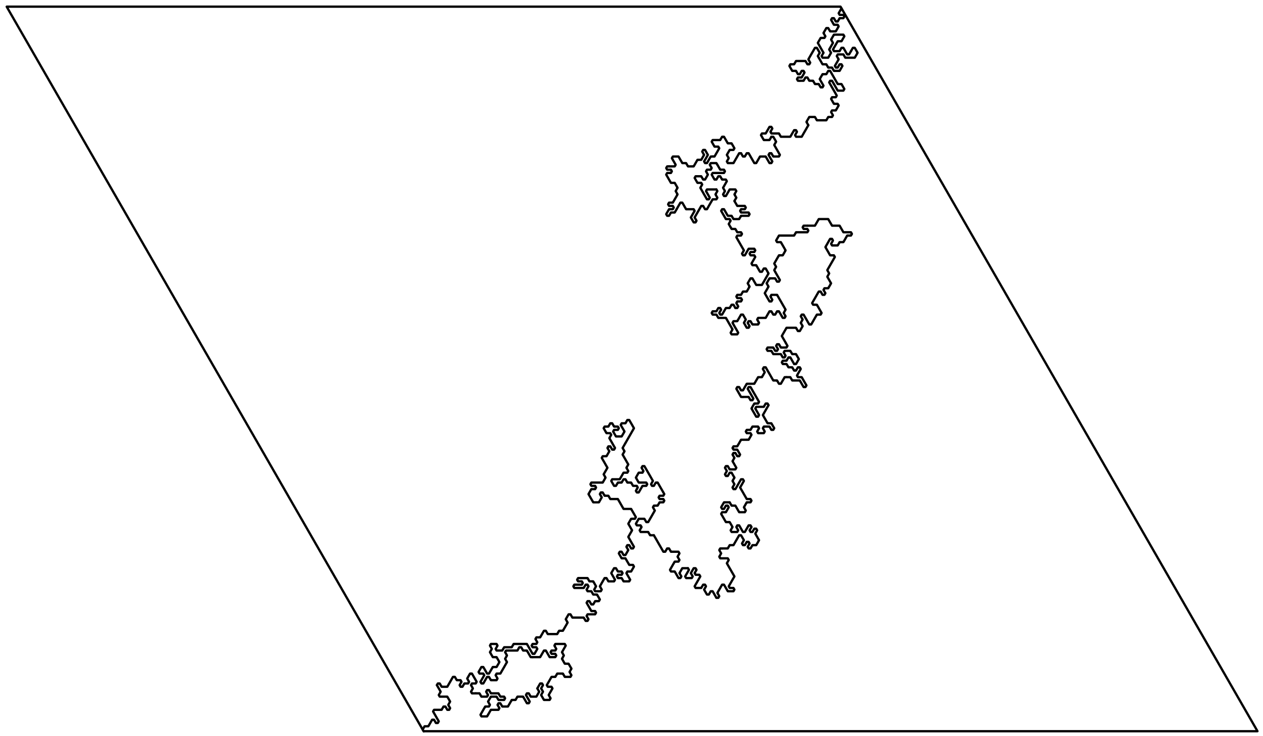
We have used Taylor's formula. But the pros use Itô calculus, which is a fancy version of Taylor's formula.

Harmonic Explorer



Theorem (Schramm-Sheffield 2003). The harmonic explorer scaling limit is SLE(4).

Harmonic explorer simulation



Evolution of discrete harmonic measure

Fix a hexagon H. Let h_t be the discrete harmonic measure of white versus black from H.

Claim. h_t is a martingale: $h_t = \mathbf{E}\big[\, h_{t+1} \;\big|\; h_t \,\big]$ and $\mathbf{E}\big[\, h_t \,\big] = h_0$.

Proof: Let H' be the hexagon decided between times t and $t+1$. If a random walk starting at H does not visit H', then it behaves the same at time t as $t+1$. If it does visit H', its probability to end white is the same as the probability that H' is white.

Corollary. h_t is the probability that H is on right hand side.

Same for SLE(4)!

Now consider a very fine mesh. Let

$$\psi : D \to \mathbb{H},$$

normalized, and let g_t and W_t be the Loewner family and driving term for the path $\psi \circ \gamma$, now parameterized by capacity. Loewner's equation holds.

Since discrete harmonic measure converges to harmonic measure,

$$\tilde{h}_t := \arg(z_t - W_t)$$

is approximately a martingale, where $z_t = g_t(z_0)$. But at this point, we have no idea how bad W_t is.

First step

Fix $\delta > 0$ and mesh much smaller. Let

$$\tau := \min\{t \geqslant 0 : |W_t - W_0| = \delta \text{ or } t = \delta^2\}.$$

We know by Loewner

$$z_\tau = z_0 + \frac{2\,\tau}{z_0 - W_0} + O(\delta^3).$$

If the mesh is small, we have $\mathbf{E}[\tilde{h}_\tau] = \tilde{h}_0 + O(\delta^3)$. Thus,

$$\mathbf{E}\big[\arg(z_\tau - W_\tau)\big] - \arg(z_0 - W_0) = O(\delta^3)\,.$$

First step, continued

$$\mathbf{E}\big[\arg(z_\tau - W_\tau)\big] - \arg(z_0 - W_0) = O(\delta^3)\,.$$

Using Loewner's equation $\partial_t z_t = 2/(z_t - W_t)$, Taylor expansion gives

$$\mathrm{Re}(W_0 - z_0)\,\mathbf{E}\big[\,4\,\tau - \Delta W^2\,\big] + |z_0 - W_0|^2\,\mathbf{E}\big[\,\Delta W\,\big] = O(\delta^3),$$

where $\Delta W = W_\tau - W_0$. Since we have some freedom in choosing z_0, this gives

$$\mathbf{E}\big[\,4\,\tau - \Delta W^2\,\big] = O(\delta^3), \qquad \mathbf{E}\big[\,\Delta W\,\big] = O(\delta^3).$$

These expectations would be zero if $W_t/2$ is BM and τ any bounded stopping time.

One step at a time

It is not hard to see that having

$$\mathbf{E}\big[\,4\,\tau - \Delta W^2\,\big] = O(\delta^3), \qquad \mathbf{E}\big[\,\Delta W\,\big] = O(\delta^3)$$

and the analogous statements for later steps implies that $W_t/2$ converges to BM as $\delta \to 0$.

Note that we must have $\mathbf{E}[\tau]$ of order δ^2, by the choice of τ. Thus we have order δ^{-2} steps to get to time $t > 1$. Thus, the $O(\delta^3)$ terms may be ignored.

Are we done?

Knowing that $W_t/2$ converges to BM is enough to show that the HE path converges to SLE(4) with respect to the Hausdorff metric.

Improving this to pathwise convergence requires other methods.

Conclusion

To prove convergence of a random path to SLE, one does not need much.

It is enough to prove the asymptotic formula for a single observable.

In the above case, the observable is the probability to end up on the right hand side.

In the case of LERW this may be taken as the expected number of hits to v.

For the DGFF, it is the expected $h(v)$.

The catch is that this must be done for an essentially arbitrary simply connected domain.