Some random systems converging to SLE

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Plan

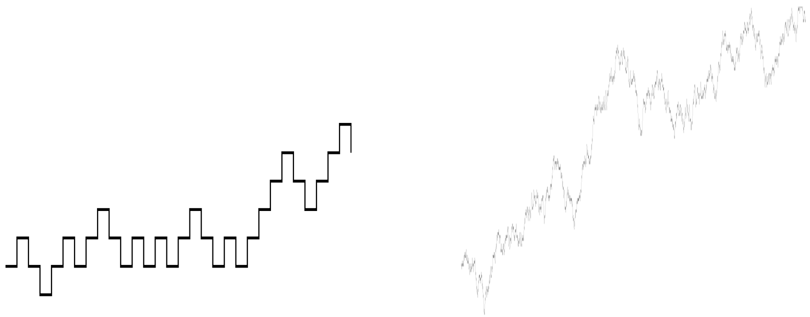
- Brownian Motion
- Percolation
- Motivation for SLE
- SLE Definition
- Other Processes converging to SLE
- Properties
- Open problems

Talk Series

1. Today: Overview of SLE and processes converging to SLE
2. Tomorrow (Wed): Hands on SLE
   (a) Cardy's formula via SLE
   (b) Proving that the harmonic explorer converges to SLE
3. Thursday: Dynamical percolation

One dimensional Brownian motion

A simple way to describe Brownian motion is as a scaling limit of simple random walk.



Let $S(0) = 0$ and given $S(1), \ldots, S(n)$ let $S(n+1) := S(n) \pm 1$ with probability $1/2$ each. Then $S(n)$ is simple random walk in $\mathbb{Z}$. For $\delta > 0$ let

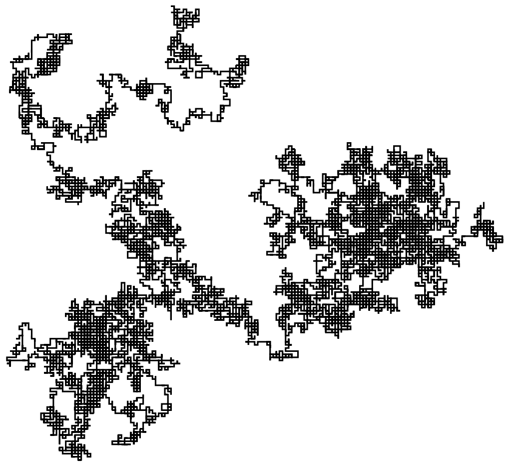
$$S_\delta(t) := \delta\, S\big(\lfloor t\, \delta^{-2} \rfloor\big)\,, \qquad t \geqslant 0\,.$$

As $\delta \downarrow 0$, S_δ converges to Brownian motion $B(t)$.

One-dimensional Brownian motion is a random continuous path $B : [0, \infty) \to \mathbb{R}$. It has the Markov property: given $B(t)$, the past, $B|_{[0,t]}$ and the future, $B|_{[t,\infty)}$, are independent.

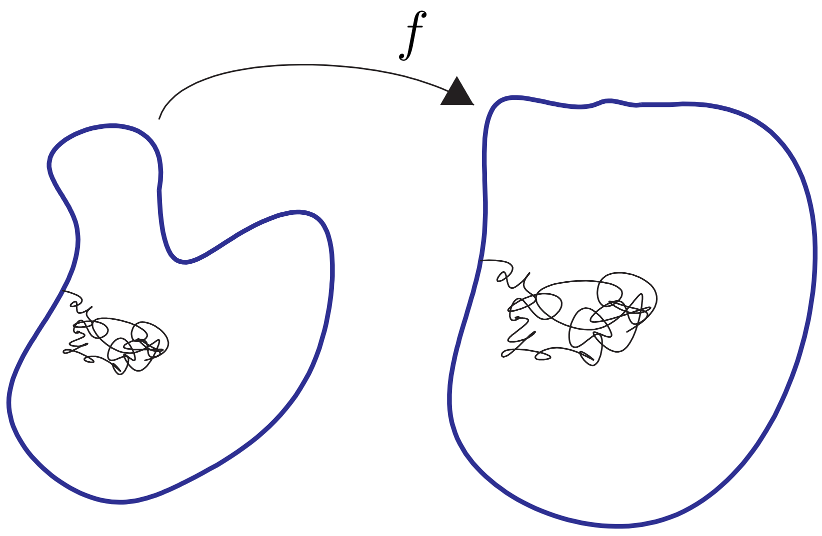
Two dimensional Brownian motion as a scaling limit

Two dimensional Brownian motion can be obtained as the scaling limit of the simple random walk on the $\mathbb{Z}^2$ grid (as well as other grids). The right scaling to take is $S_\delta(t) := \delta\, S\big(t\,\delta^{-2}\big)$.



Lévy's theorem: conformal invariance of Brownian motion

Let B_t be planar Brownian motion. If $D \subset \mathbb{C}$ is a domain containing 0 and $f : D \to \mathbb{C}$ is analytic, then $f(B_t)$ is time-changed Brownian motion.

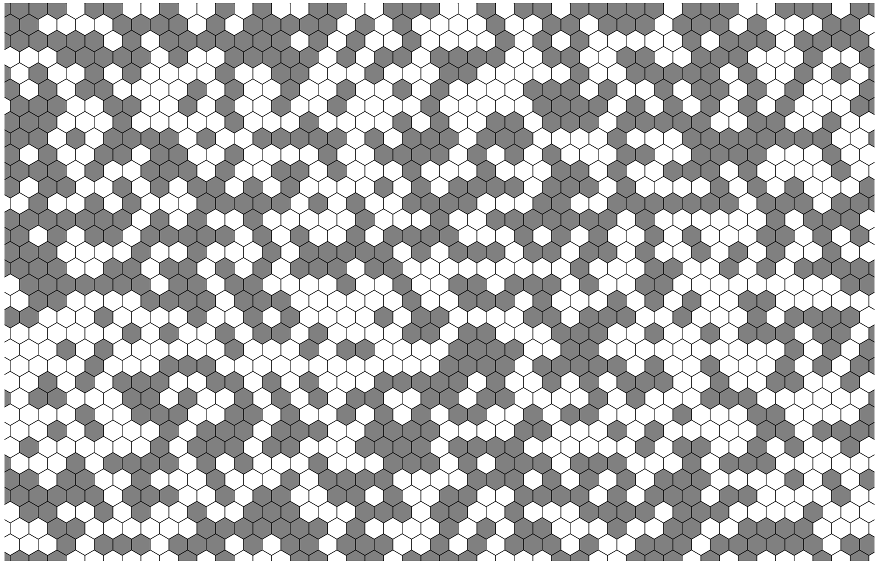


Reminder: conformal maps

A map $f : D \to D'$ is conformal if it is an angle-preserving and orientation preserving homeomorphism. Equivalently, f is analytic and f^{-1} is analytic. (Equivalently, f is analytic and injective.)

Riemann's mapping theorem tells us that for every simply connected $D \subset \mathbb{C}$, $D \neq \mathbb{C}$, there is a conformal map from D onto the unit disk, $f : D \to \mathbb{U}$. The collection of such maps is 3 dimensional.

Percolation



In Bernoulli(p) percolation, each hexagon is white (open) with probability p, independently. The connected components of the white regions are studied.

Various similar models include bond p-percolation on $\mathbb{Z}^d$.

Critical Percolation

There is some number $p_c \in (0,1)$ such that there is an infinite component with probability 1 if $p > p_c$ and with probability 0 if $p < p_c$.

The large-scale behaviour changes drastically when p increases past p_c. This is perhaps the simplest model for a phase transition.

Theorem (Harris 1960). At $p = 1/2$ there are no inifinite clusters a.s. Therefore, $p_c \geqslant 1/2$.

Theorem (Kesten 1980). $p_c = 1/2$.

A large critical cluster

At p_c, there are no infinite clusters. If we condition on the event that the cluster of the origin has more than 1000 vertices, then here's what it looks like.

Percolation exponents

Physicists have predicted some exponents describing asymptotics of critical percolation in 2D.

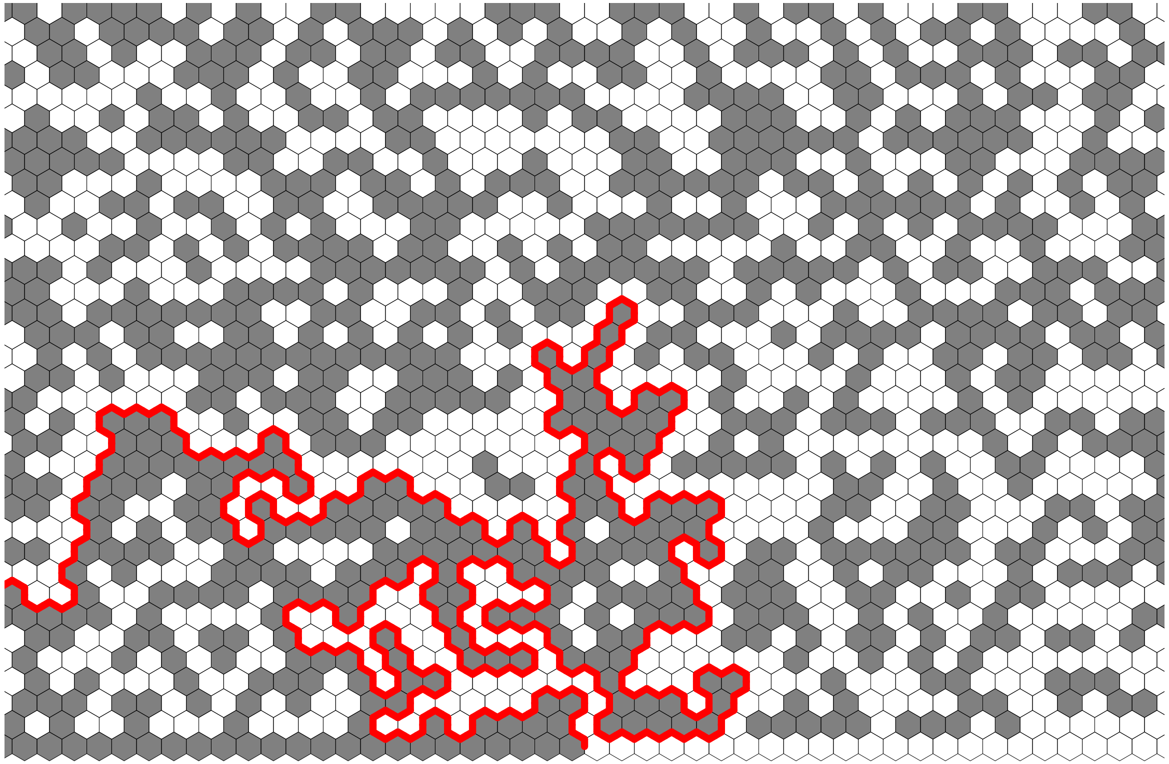
For example, Nienhuis conjectured that the probability that the origin is in a cluster of diameter $\geqslant R$ is

$$R^{-5/48+o(1)}, \qquad R \to \infty$$

and Cardy conjectured that the probability that the origin is connected to distance R within the upper half plane is

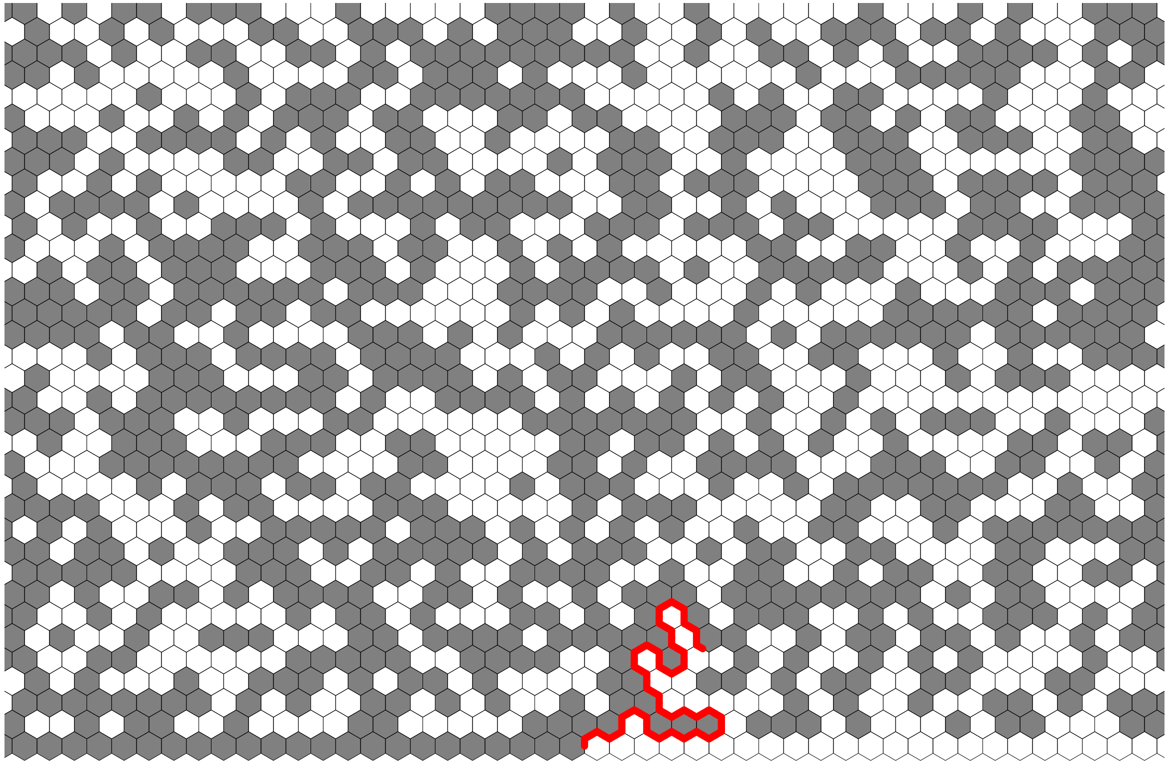
$$R^{-1/3+o(1)}, \qquad R \to \infty.$$

Critical percolation interface

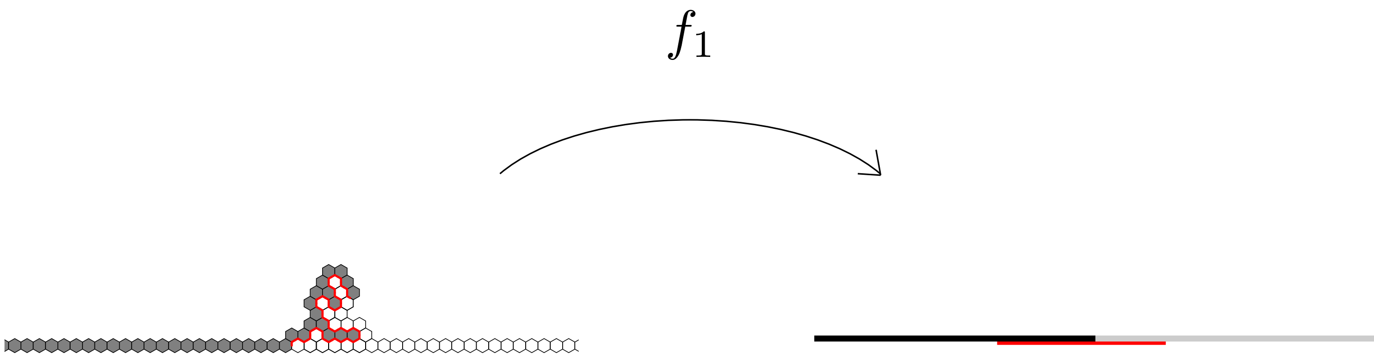


SLE motivation

Take fine scale, and stop when curve has size ϵ.



Apply a conformal map in the slitted half-plane to map back to the half-plane



$$f_1(z) = z + a_1\, z^{-1} + a_2 z^{-2} + \cdots$$

By conformal invariance, the image under f_1 of the continuation of the path on the left is approximately equal to the original distribution of the path, except that it is translated to the image of the tip under f_1.

Suppose that w_1 is the image of the tip. Then we may continue the path a bit further. In the next step we map by

$$G_2 = f_2 \circ f_1, \qquad\qquad f_2 \overset{\mathcal{L}}{=} T_{w_1} \circ f_1 \circ T_{-w_1}.$$

We may continue inductively, letting w_j be the image of the tip in the j-th stage. Then

$$G_n = f_n \circ f_{n-1} \circ \cdots \circ f_1,$$

where

$$T_{-w_j} \circ f_{j+1} \circ T_{w_j} \overset{\mathcal{L}}{=} f_1.$$

Each f_j is close to the identity map. So we may attempt to think of this as a flow, rather than discrete steps.

To understand the flow, let's look again at f_1

$$f_1(z) = z + a_1 z^{-1} + \cdots$$

We may choose $a_1 = 2\,\epsilon$. Then scaling implies

$$f_1(z) = z + 2\,\epsilon\,z^{-1} + O(\epsilon^{3/2}), \qquad f_{j+1}(z) = z + \frac{2\,\epsilon}{z - w_j} + O(\epsilon^{3/2}).$$

Thus, we arrive at Loewner's equation:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - w_t}\,.$$

In our case, w_t is a sum of independent stationary increments, and is symmetric and continuous. It follows that it is a multiple of Brownian motion.

SLE definition

Fix $\kappa > 0$. Let $w_t = B(\kappa\, t)$, where B is standard one dimensional Brownian motion. Define g_t in the upper half plane by solving the **ODE**

$$\partial_t g_t = \frac{2}{g_t(z) - w_t}\,, \qquad g_0(z) = z\,.$$

This is chordal SLE(κ).

The growing path is $\gamma(t) = g_t^{-1}(w_t)$.

The unbounded component in the complement of the path is $g_t^{-1}(\mathbb{H})$.

Radial SLE

Chordal SLE is a random path connecting two boundary points of a simply connected domain (0 and ∞ in $\mathbb{H}$).

Radial SLE connects a boundary point to an interior point (0) in the unit disk $\mathbb{U}$. It is obtained by solving

$$\partial_t g_t = -g_t(z)\,\frac{g_t(z) + w_t}{g_t(z) - w_t}\,, \qquad g_0(z) = z\,,$$

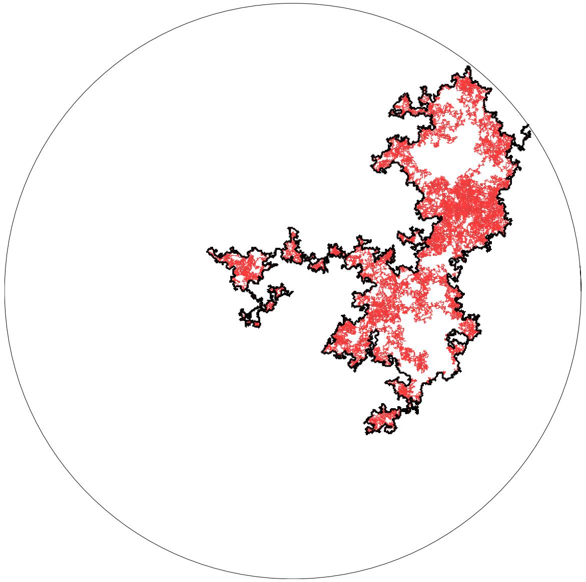
where $w_t = \exp\bigl(i\,B(\kappa\,t)\bigr)$ is Brownian motion on the unit circle.

Locality of SLE(6)

The value $\kappa = 6$ turns out to be special. It is the only value of κ for which the target point and the shape of the domain do not matter (up to a time change). The SLE(6) path does not feel the boundary before it hits it.

BM frontier is that of SLE(6)

Theorem (Lawler-Schramm-Werner). The outer boundary of 2D BM is the same as that of SLE(6) (when set up correctly.)

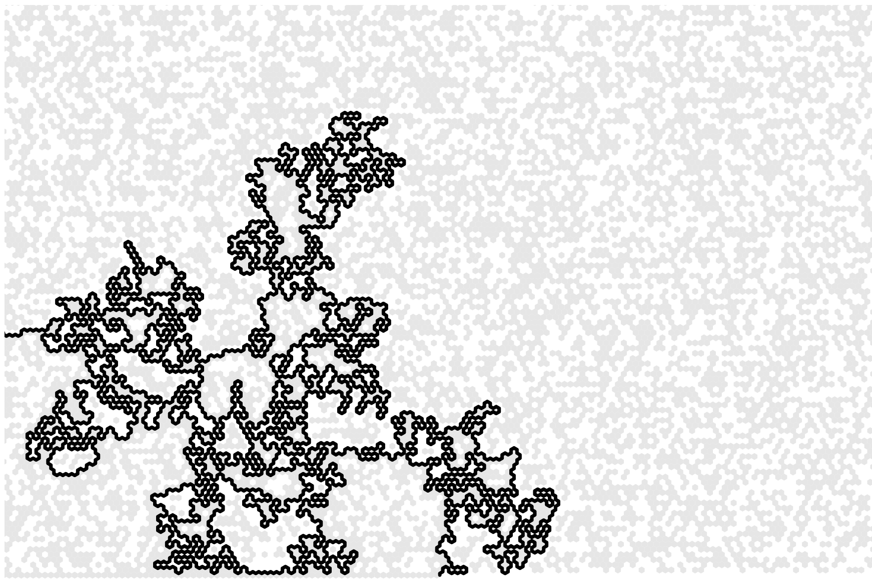


Corollaries about planar BM

Theorem (Lawler-Schramm-Werner). The outer boundry of SLE(6) (and therefore of planar BM) has Hausdorff dimension $4/3$ (as conjectured by Mandelbrot). The set of cut points has Hausdorff dimension $3/4$.

Percolation interface is SLE(6)

Smirnov's Theorem (2001). The above model of critical percolation is conformally invariant. The percolation interface scaling limit is SLE(6).

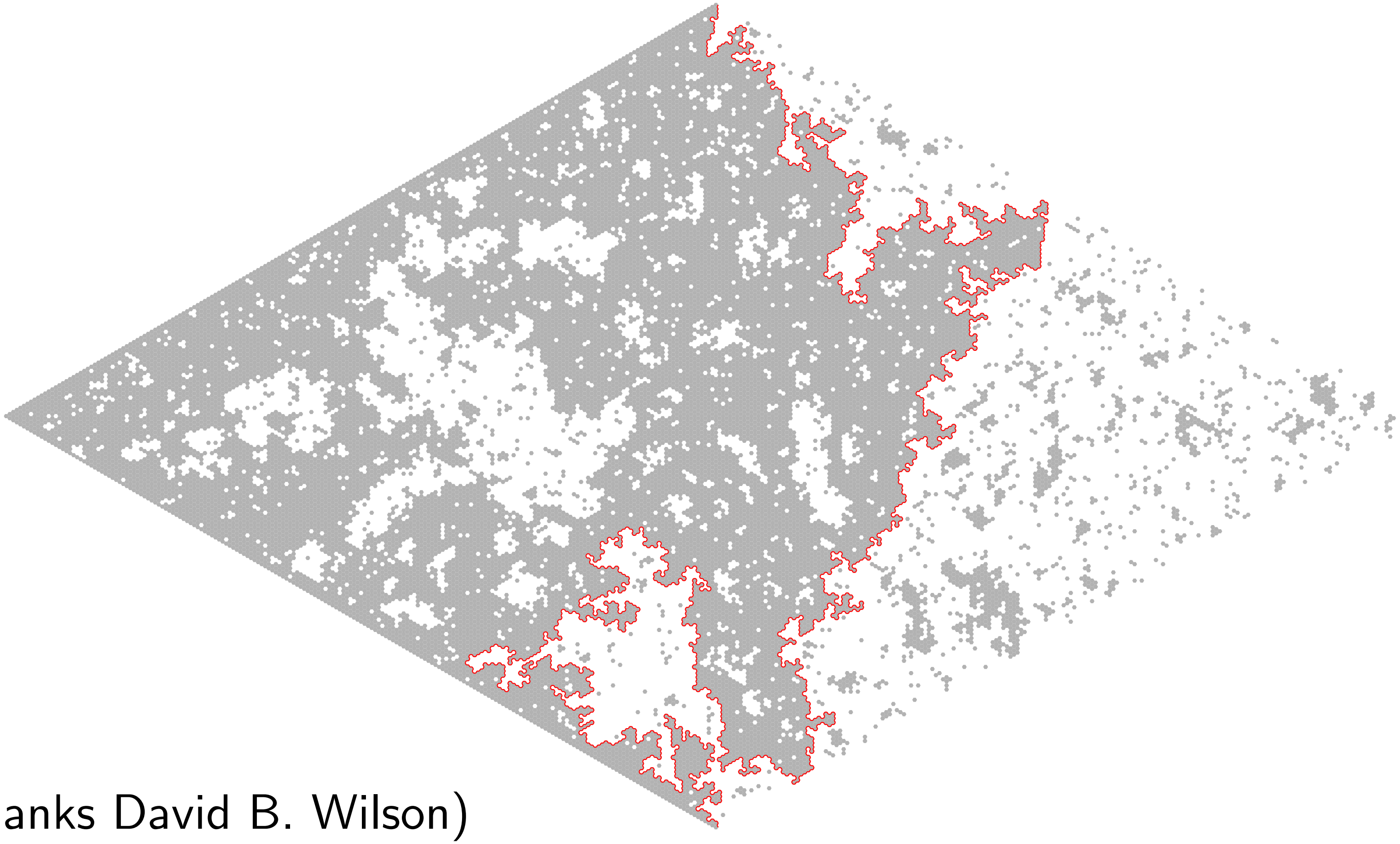


Percolation exponents

Theorem (Lawler-Schramm-Werner). The probability that the origin is connected to distance R is $R^{-5/48+o(1)}$ as $R \to \infty$.

Other exponents and properties too (Kesten, Smirnov-Werner).

Critical Ising model interface is SLE(3) (conj)

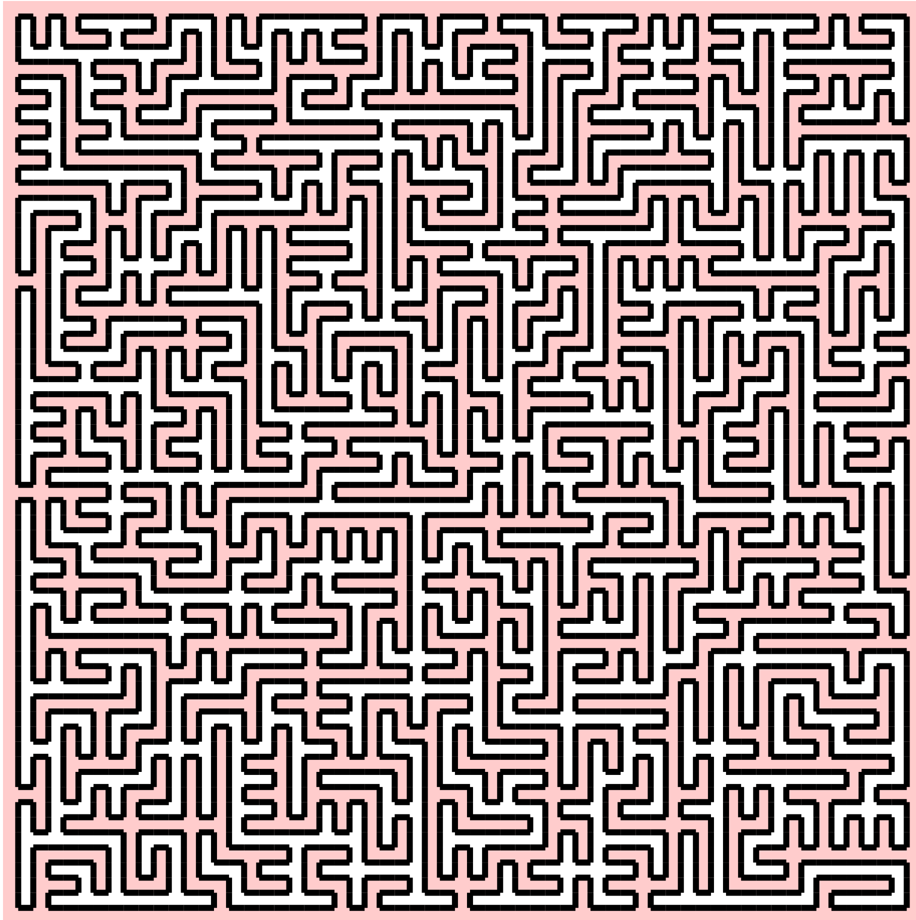


(Thanks David B. Wilson)

Uniform spanning tree

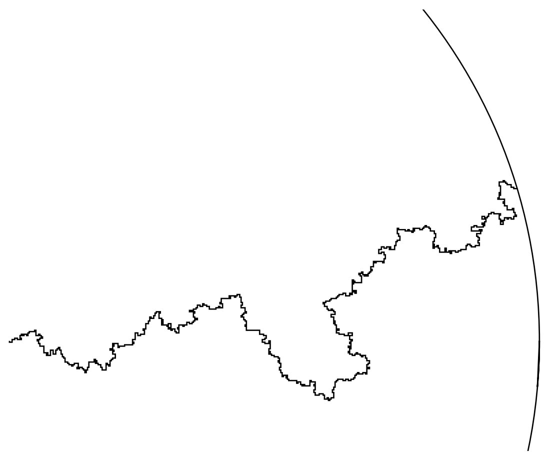
Loop-erased random walk

Peano path (Hamiltonian path on the Manhatten lattice)



Loop-erased walks

The loop-erasure of simple random walk (with some stopping time) is LERW.

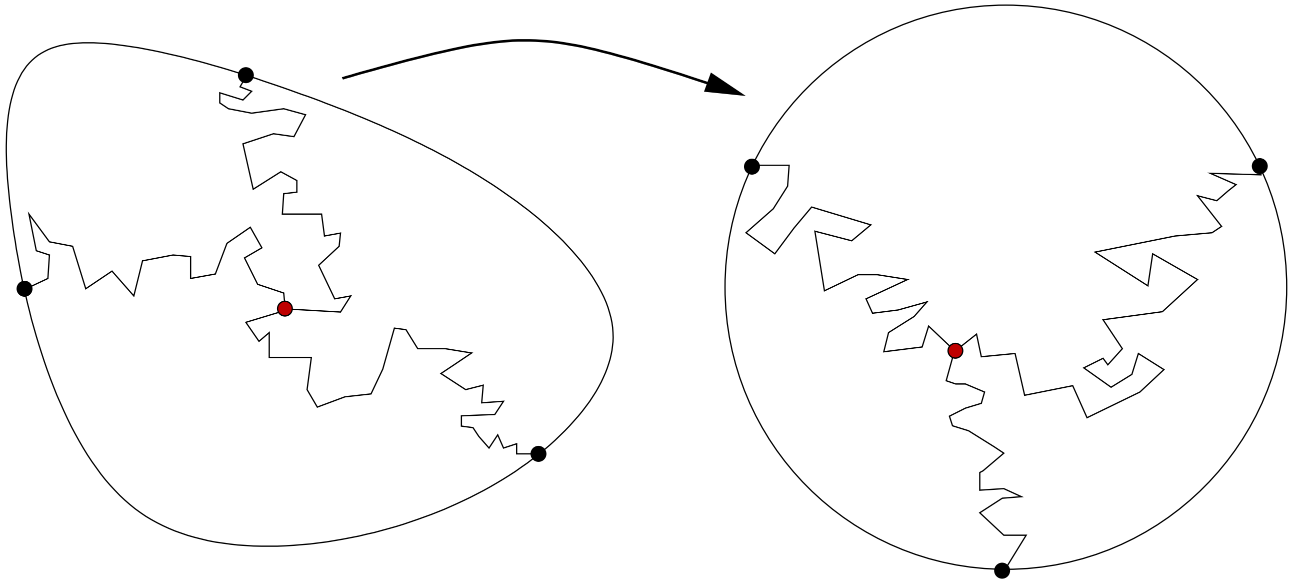


LERW and UST

Wilson's algorithm. The UST on any (finite, connected) graph can be built from a union of LERW paths: Order the vertices arbitrarily $v_0, v_1, \ldots, v_n$. Set $T_0 := \{v_0\}$. Inductively, let T_{j+1} be the union of T_j and the loop-erasure of simple random walk from v_{j+1} stopped when it first hits T_j. Then T_n is a UST on G.

Richard Kenyon proved that some properties of UST and LERW are conformally invariant in the scaling limit.

For example, he showed that the asymptotic distribution of the meeting point of three vertices adjacent to the boundary of a simply connected domain is conformally invariant.



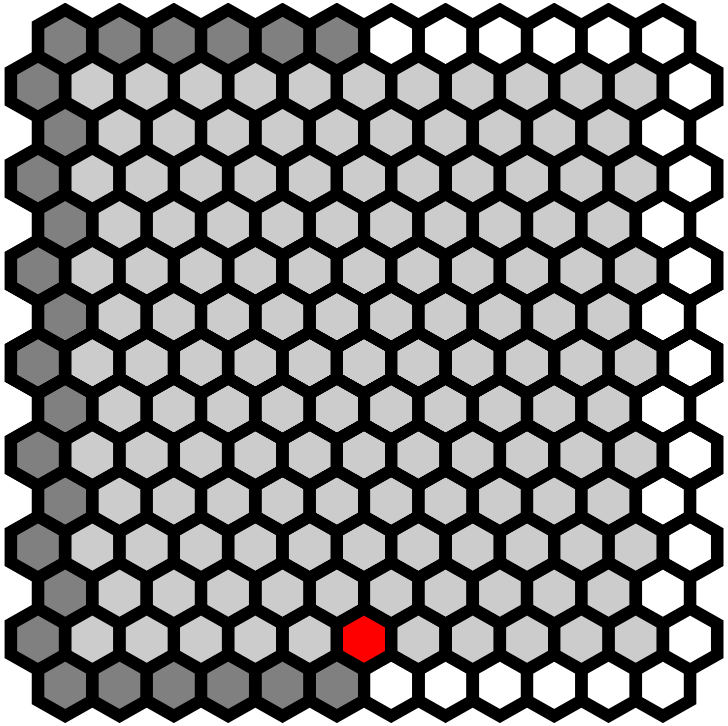
LERW, UST, Peano

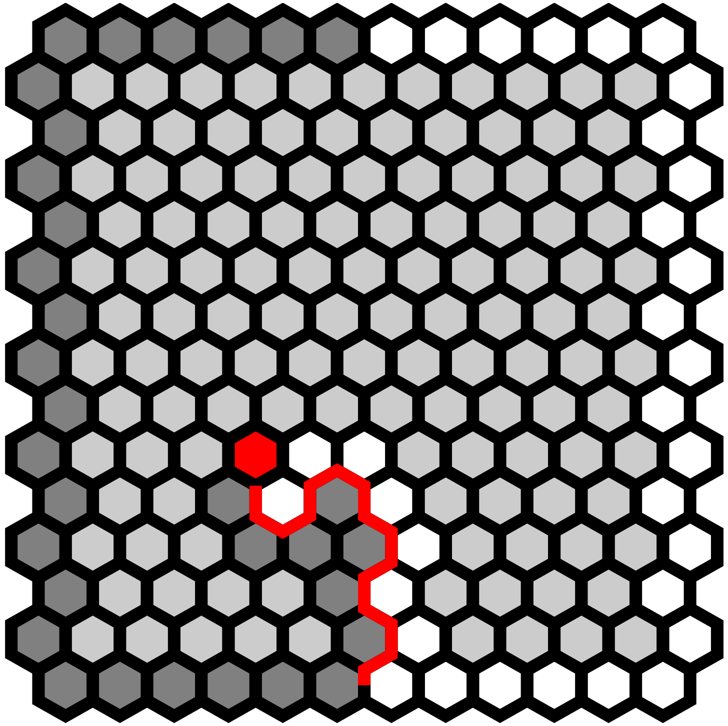
Theorem (Lawler-Schramm-Werner 2002). The LERW scaling limit is SLE(2). The UST Peano path scaling limit is SLE(8).

Corollary. The UST, LERW and UST Peano path are conformally invariant.

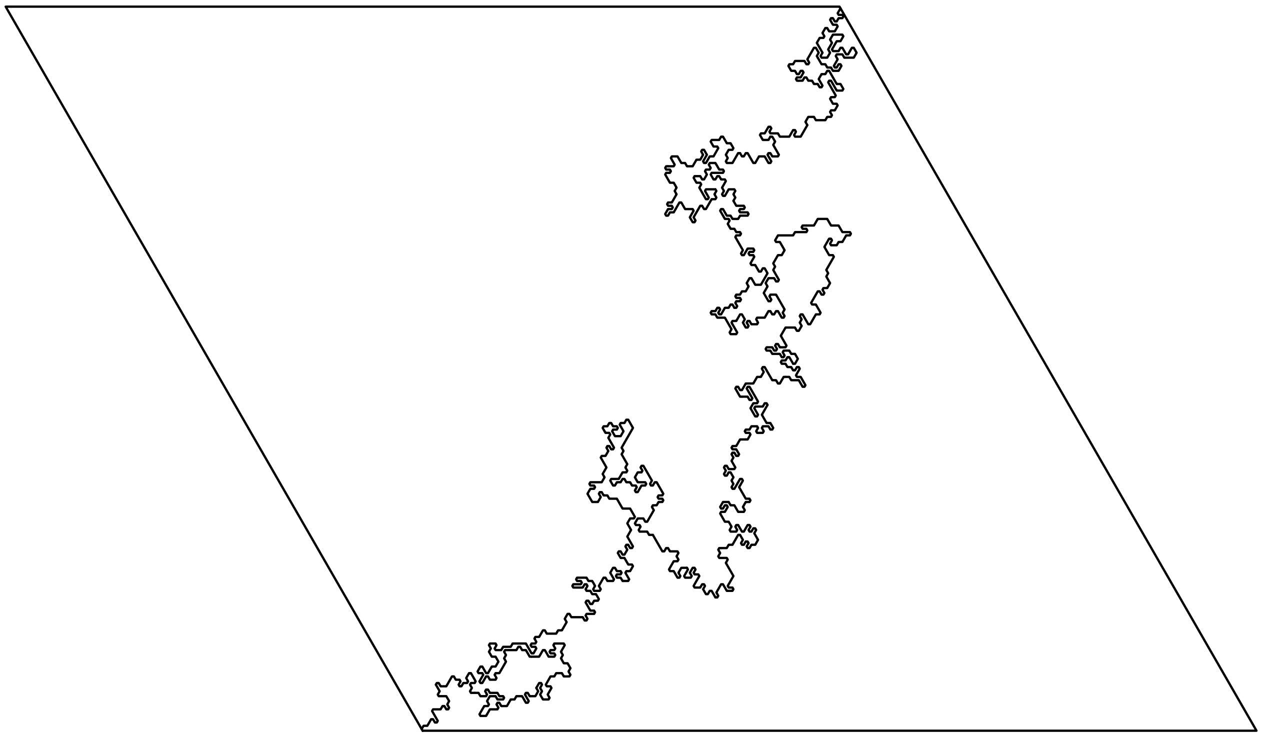
Harmonic explorer converges to SLE(4)

Theorem (Schramm-Sheffield 2003). The harmonic explorer scaling limit is SLE(4).





Harmonic explorer simulation



Discrete GFF

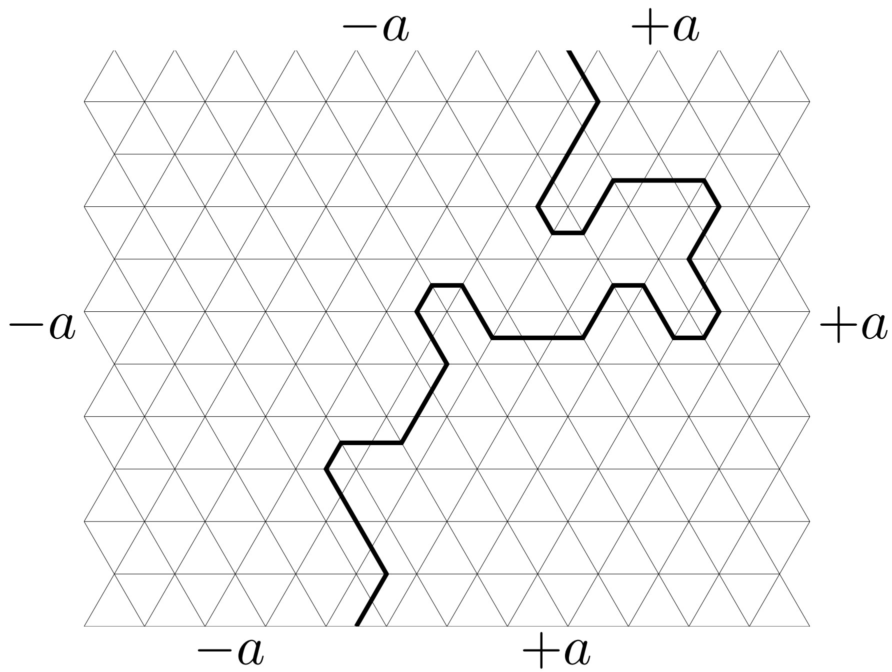
The discrete **Gaussian free field** is random a real valued function h on the vertices of the grid, such that $(h(v) : v \in V)$ is a multi-dimensional Gaussian. The probability density of h is proportional to

$$\exp\Big(-\sum_{[u,v]} \frac{(h(v)-h(u))^2}{2}\Big)\,.$$

The boundary values of h are fixed.

Rick Kenyon has shown that the Gaussian free field is the scaling limit of the domino tiling (dimer tiling) height function.

DGFF interface



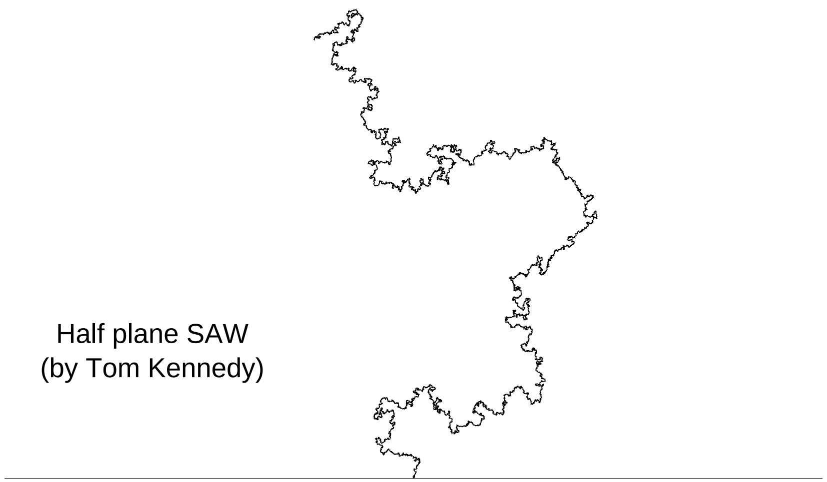
Gaussian free field interface is SLE(4)

Theorem (Schramm-Sheffield [in preparation]). The interface of the [discrete] Gaussian free field [scaling limit] is SLE(4).

The Gaussian free field is central to the (heuristic) determination of exponents for several distinct statistical physics models.

Self-avoiding walk

The half-plane SAW scaling limit is SLE(8/3) (Conj. LSW). Supported experimentally by Tom Kennedy.



Existence of the SLE path

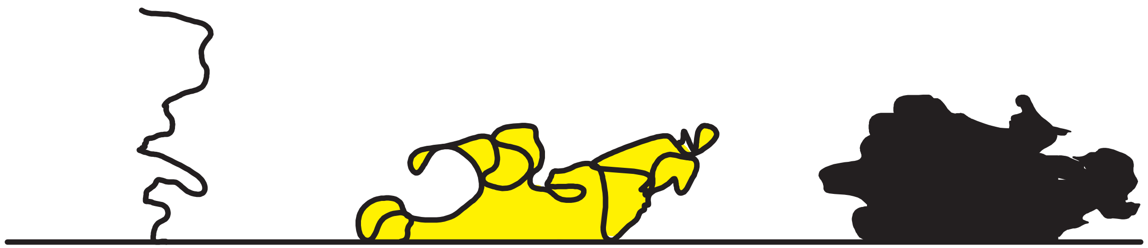
Theorem. $g_t^{-1}(w(t))$ is well-defined and is a.s. a continuous path.

(Rohde-Schramm (2001) for $\kappa \neq 8$, Lawler-Schramm-Werner (2002) for $\kappa = 8$.)

The SLE trace is the path $t \mapsto g_t^{-1}\big(w(t)\big)$.

Phases of SLE

Theorem Rohde-Schramm (2001). The SLE(κ) trace is a simple path iff $\kappa \leqslant 4$. It is space filling iff $\kappa \geqslant 8$.



$\kappa \in [0,4]$ $\kappa \in (4,8)$ $\kappa \in [8,\infty)$

In the phase $\kappa \in (4,8)$, the SLE path makes loops "swallowing" parts of the domain. However, it never crosses itself.

Hausdorff Dimension

Beffara's Theorem. The Hausdorff dimension of the SLE(κ) path is $1 + \kappa/8$ when $\kappa \in [0, 8]$.

SLE gives (proved)

- critical site percolation on the triangular grid (6)
- LERW (2)
- UST Peano curve (8)
- GFF, HE (4)

SLE gives (conj)

- other critical percolation (6)
- Ising (3,6)
- FK cluster boundaries ($q = 2 + 2\cos(8\pi/\kappa)$, $\kappa \in [4, 8]$)
- O(n) models ($n = -2\cos(4\pi/\kappa)$, Kager-Nienhuis)
- SAW (8/3)
- Double domino (4)

SLE does not give

- DLA (not conformally invariant)
- MST paths (Weiland-Wilson)
- Dimension > 2

Conjectures and problems

Smirnov's Theorem for critical bond percolation on $\mathbb{Z}^2$.

The half-plane SAW scaling limit is SLE(8/3) (Conj. LSW). Supported experimentally by Tom Kennedy.

Several other processes have SLE as scaling limit (critical Ising, FK-interfaces, double-domino,...).

Sample path properties?

Modulus of continuity?

Intersection behaviour?

Convergence speed?