Shifts in Cayley Graphs

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Outline of the talk

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- Conclusion

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- $\alpha \in \mathrm{Aut}(X)$ is called a *shift* if $v \sim \alpha(v), \forall v \in V(X)$.
- A graph that does not admit a shift is called *shiftless*.

Cayley Graph definitions

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- Define $\sigma_g : G \to G, \sigma_g(v) = gv$. Then $\sigma_g \in \mathrm{Aut}(X)$.
- If $\mathrm{Cay}(G, S)$ is shiftless, then S is also called *shiftless*.

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- In particular, if G is abelian, then X has a shift unless it's edgeless.
- The converse is false : a Cayley Graph admitting a shift is not necessarily isomorphic to a Cayley Graph on an abelian group.

Non-abelian Cayley Graph

- Let $G =< a, b; a^6 = b^2 = 1, bab = a^{-1} >$ be the dihedral group of order 12, and $S =< b, ba^2, ba^5, a^3 >$.

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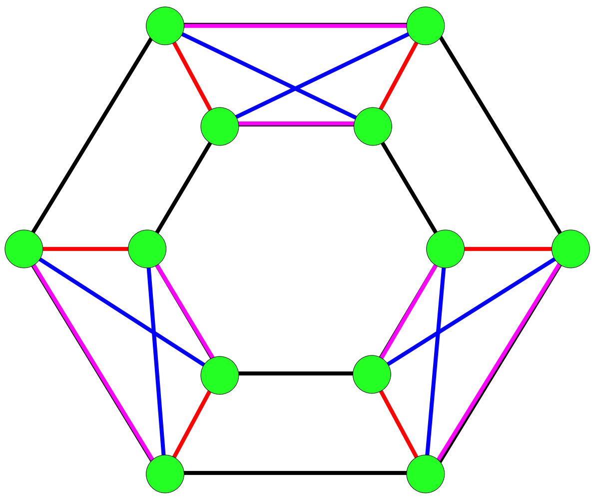
- NOTE : $a^3 \in Z(G)$ so σ_{a^3} is a shift of $\mathrm{Cay}(G, S)$.



$S=\{a^3, b, ba^2, ba^5\}$

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- Move our point of view up to the group level :
- CONJECTURE : Let G be a group with no shiftless subset. Then all Cayley Graphs on G are isomorphic to Cayley Graphs on abelian groups.

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- FACT : The groups for which all Cayley Graphs are isomorphic to Cayley Graphs on abelian groups are the abelian groups, the generalized dicyclic groups, dihedral groups of order 6, 8, 10.

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- We can rephrase conjecture :
- CONJECTURE : Let G be a group with no shiftless subset. Then G is abelian, generalized dicyclic or dihedral of order 6, 8 or 10.

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- We can prove that a graph with a shift has a certain structure.
- We figure out what this structure implies about the connection set of a Cayley Graph with a shift.
- We see what this implies about a group with no shiftless subset.

Shifts and 4-cycles

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- $\alpha(u) = v$.

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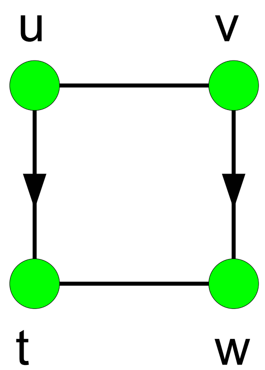
- Let $t = \alpha(u), w = \alpha(v)$, so $t \neq w$. But since u and v are adjacent, so are t and v and we get a 4-cycle $uvwt$ containing uv :



Proof, case 2

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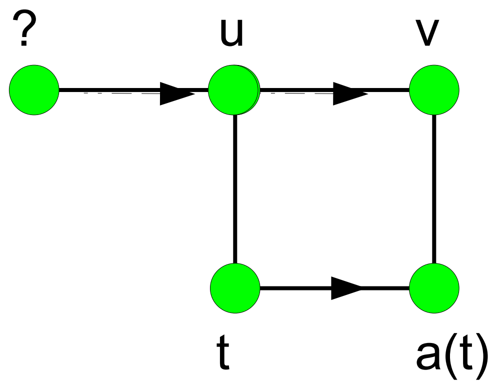
- Then u, v both have degree at least 3, so there exists a vertex t adjacent to u such that $\alpha(t) \neq u$. $ut\alpha(t)v$ is a 4-cycle containing uv :



Cayley graphs and 4-cycles

- COROLLARY : Let G be a group and S be an identity-free inverse-closed subset of G with $|S| \geq 3$ that is not shiftless. Then, $\forall a \in S$, $\exists b, c, d \in S$ such that $ba = dc \neq 1$ and $a \neq c$.

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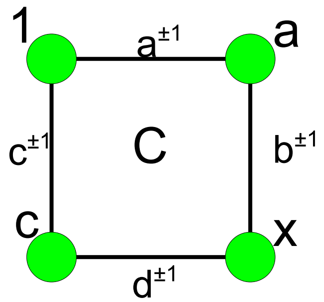
- PROOF : Take $a \in S$. The edge $1a$ is part of a 4-cycle C in $\mathrm{Cay}(G, S)$. Call c the other vertex adjacent to 1 in C (so $a \neq c$). Call x the last vertex in C. x is adjacent to a and c so there must exist $b, d \in S$ such that $ba = dc = x \neq 1$.



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- $ba^{\pm 1} = a^{\pm 1} b \Rightarrow bab = a^{\pm 1}$.
- $ba = a^{-2} \Rightarrow b = a^{-3} \Rightarrow bab = a$.
- $ba^{-1} = a^2 \Rightarrow b = a^3 \Rightarrow bab = a$.

Questions?

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Workshops, Ottawa, May 12-16

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