The Crossing Number of K_{11} is 100

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Definition of Drawing

Definition

Vertices $\longrightarrow$ Points
Edges $\longrightarrow$ Simple Curves

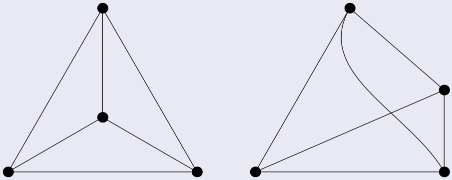

Figure: Drawings of K_4

Definition of Crossing Number

Definition

$cr(G)$: minimum number of edge crossings over all drawings of G.

Example: $cr(K_4) = 0$, $cr(K_5) = 1$.

Crossing Number of Complete Graphs

Conjecture (Guy's Conjecture)

$$cr(K_n) \stackrel{?}{=} Z(n),$$

where

$$Z(n) := \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$

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- True for $n \leqslant 10$;
- For $n = 4, 5, 6, 7, 8,$ there are $1, 1, 5, 3$ *optimal drawings*, respectively.

Cylindrical Drawings of K_n

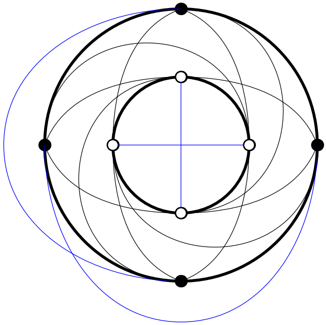
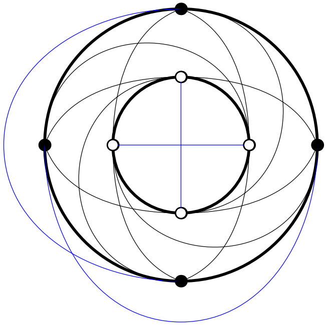


Figure: A cylindrical drawing of K_8 [Richter and Thomassen]

Cylindrical Drawings of K_n

Cylindrical drawings $\Longrightarrow cr(K_n) \leqslant Z(n)$.



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Good Drawing

Definition (Good Drawing)

A drawing without forbidden crossings.

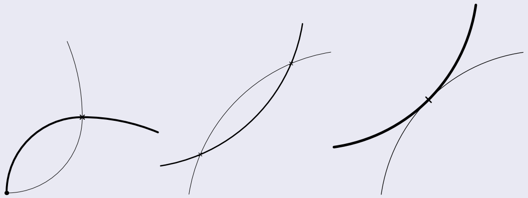

Figure: Forbidden crossings

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Any optimal drawing is good.

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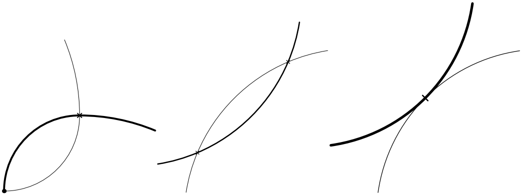

Figure: Forbidden crossings

Good Drawing

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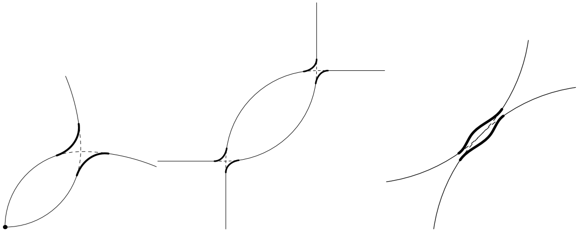

Figure: Eliminating forbidden crossings

Parity Property

Theorem (Kleitman)

For an **odd** *n, in any good drawing of K_n,*

$$\#crossings \equiv Z(n) \pmod 2.$$

Counting Properties

Lemma (Lower Bound)

For $n > 4$,

$$cr(K_n) \geqslant \left\lceil \frac{n}{n-4} \cdot cr(K_{n-1}) \right\rceil .$$

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Corollary

For an **odd** n,

$$cr(K_n) = Z(n) \implies cr(K_{n+1}) = Z(n+1).$$

Counting Properties

Lemma (Sub-drawing)

Any good drawing D_n of K_n contains a good drawing D_{n-1} of K_{n-1} such that

$$cr(D_{n-1}) \leqslant cr(D_n) - \lceil 4cr(D_n)/n \rceil .$$

Containing Argument 1

Theorem (Containing-1)

For $n \leqslant 8$, any optimal drawing of K_n contains an optimal drawing of K_{n-1}.

n	4	5	6	7	8	9
$cr(K_n)$	0	1	3	9	18	36
$cr(D_{n-1}) \leqslant$	0	0	1	3	9	$20 > 18$

Containing Argument 2

Theorem (Containing-2)

Any optimal drawing of K_9 contains a good drawing of K_8 with at most 20 crossings, which contains an optimal drawing of K_7.

18	19	20
$\lceil 18 \cdot 4/8 \rceil = 9$ $18 - 9 = 9$	$\lceil 19 \cdot 4/8 \rceil = 10$ $19 - 10 = 9$	$\lceil 20 \cdot 4/8 \rceil = 10$ $20 - 10 = 10$ $\rightarrow 9$

Containing Argument 3

Theorem (Containing-3)

Any good drawing of K_{11} with less than $Z(11) = 100$ crossings, if there is any, contains a good drawing of K_{10} with at most 62 crossings, which contains an optimal drawing of K_9.

96	98
$\lceil 96 \cdot 4/11 \rceil = 35$ $96 - 35 = 61$	$\lceil 98 \cdot 4/11 \rceil = 36$ $98 - 36 = 62$

60	61	62
$\lceil 60 \cdot 4/10 \rceil = 24$ $60 - 24 = 36$	$\lceil 61 \cdot 4/10 \rceil = 25$ $61 - 25 = 36$	$\lceil 62 \cdot 4/11 \rceil = 25$ $62 - 25 = 37$ $\rightarrow 36$

Basic Idea - Generating Drawings by Computer

$\mathcal{D}_n^c$: set of all good drawings of K_n with c crossings.

$$
\begin{aligned}
\mathcal{D}_4^0 &\longrightarrow \mathcal{D}_5^1 \longrightarrow \mathcal{D}_6^3 \longrightarrow \mathcal{D}_7^9 \\
&\longrightarrow \mathcal{D}_8^{18} \cup \mathcal{D}_8^{19} \cup \mathcal{D}_8^{20} \\
&\longrightarrow \mathcal{D}_9^{36} \\
&\longrightarrow \mathcal{D}_{10}^{60} \cup \mathcal{D}_{10}^{61} \cup \mathcal{D}_{10}^{62} \\
&\longrightarrow \mathcal{D}_{11}^{\leqslant 98}
\end{aligned}
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Expected: $\mathcal{D}_{11}^{\leqslant 98} = \emptyset$.

Problem 1: Generating New Drawings

?

D_n: A good drawing of K_n

$\downarrow$

D_{n+1}: A good drawing of K_{n+1},

- which contains D_n, and
- $cr(D_{n+1}) \leqslant cr(K_{n+1}) + \delta$, $\delta \in \{0, 1, 2\}$.

Generating New Drawings

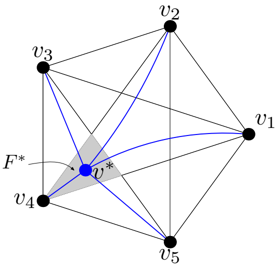
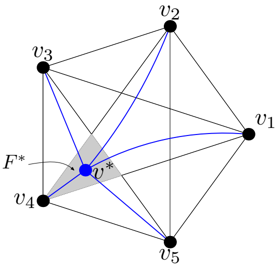


Figure: $D_5 \longrightarrow D_6$

Generating New Drawings



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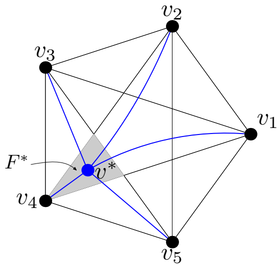

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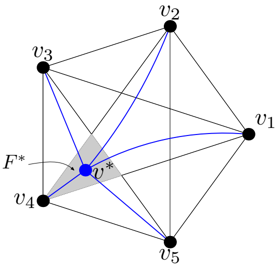
<u>New edges:</u>

- walks in the dual graph, W_i, $i = 1, 2, \cdots, n$;
- $len(W_i) \leqslant d(F^*, v_i) + \delta$.

Generating New Drawings



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New edges:

- walks in the dual graph, W_i, $i = 1, 2, \cdots, n$;
- $len(W_i) \leqslant d(F^*, v_i) + \delta$.

Claim: For $\delta \in \{0, 1, 2\}$, W_i's are *paths* in the dual graph.

Induced Planar Graph

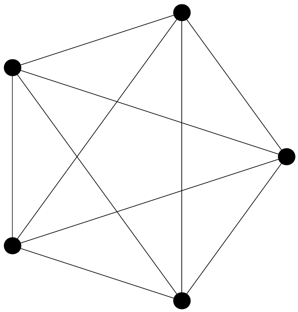

Figure: A good drawing of K_5

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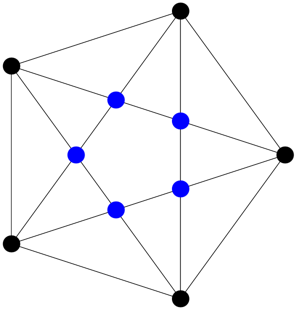

Figure: Induced planar graph

Proof of W_i's being Paths

Lemma (3-connectivity)

For any good drawing of K_n, $n \geqslant 4$, the induced planar graph is 3-connected.

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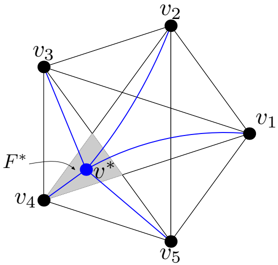

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For any good drawing of K_n, $n \geqslant 4$, the induced planar graph is 3-connected.

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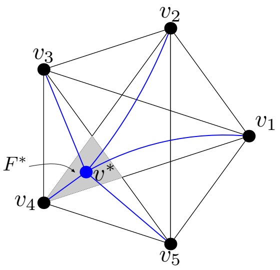

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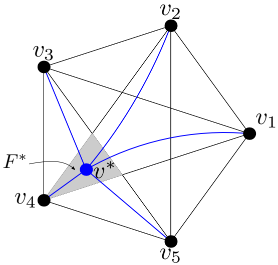
Lemma (3-connectivity)

For any good drawing of K_n, $n \geqslant 4$, the induced planar graph is 3-connected.

- $\Longrightarrow$ the dual graph is simple,
- $\Longrightarrow$ $len(W_i) \geqslant d(F^*, v_i) + 3$,
- $\Longrightarrow$ W_i's are paths.



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$\Longrightarrow$ checking *graph isomorphism*.
(**nauty** = **no** **aut**omorphisms, **y**es?)

Proof of $cr(K_{11}) = 100$

$$\begin{aligned}
\mathcal{D}_4^0 &\longrightarrow \mathcal{D}_5^1 \longrightarrow \mathcal{D}_6^3 \longrightarrow \mathcal{D}_7^9 \\
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\end{aligned}$$

Expected: $\mathcal{D}_{11}^{\leqslant 98} = \varnothing$.

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Output: $\mathcal{D}_{11}^{\leqslant 98} = \varnothing$.

Number of Optimal Drawings

n	3	4	5	6	7	8	9	10
$cr(K_n)$	0	0	1	3	9	18	36	60
#drawings	1	1	1	1	5	3		

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n	3	4	5	6	7	8	9	10
$cr(K_n)$	0	0	1	3	9	18	36	60
#drawings	1	1	1	1	5	3	**3080**	**5679**

References

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