On geodesics in random regular graphs

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Outline

- Probabilistic preliminaries
- Random regular graphs
- Geodesics in random regular graphs

Probabilistic preliminaries

- Probability Space:

Ω = (S, Prob)

S finite, Prob: S $\to$ [0,1], $\Sigma_{s \in S}$ Prob(s) = 1

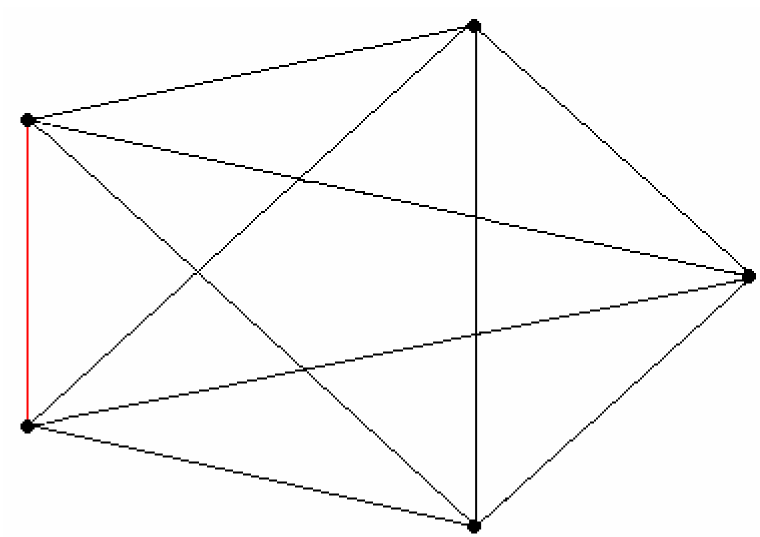
For instance, $\mathcal{G}_{n,p}$:

S = {labelled graphs on n vertices}

Prob(G) = $p^{|E(G)|} (1-p)^{n(n-1)/2 - |E(G)|}$, $0 \leq p \leq 1$

The probability space $\mathcal{G}_{n,p}$

- Consider the probability space $\mathcal{G}_{5,1/2}$:



Probabilistic preliminaries

- Event in Ω = (S, Prob):

  Subset E of S

  E = {labelled 3-regular graphs on n vertices}

- Random variable over Ω:

  A function $X : S \rightarrow \mathbf{R}$

  $$X(G) = \begin{cases} 1, & \text{if } G \text{ is 3-regular} \\ 0, & \text{otherwise} \end{cases}$$

Probabilistic preliminaries

- An event A in $\mathcal{G}_{n,p}$ holds asymptotically almost surely (a.a.s.) if

$$\lim_{n\to\infty} \text{Prob}(A) = 1$$

For example, if $A = \{K_n\}$ with $p<1$,

$$\text{Prob}(A) = p^{n(n-1)/2}$$

Probabilistic preliminaries

- Expected value of X:

$$E(X) = \sum_{s \in S} X(s)\,\mathrm{Prob}(s)$$

- Variance of X:

$$\mathrm{Var}(X) = E(|X - E(X)|^2)$$

Basic inequalities

- Markov's inequality:

If X is a non-negative random variable and t>0, then

$$\text{Prob}(X \geq t) \leq E(x)/t$$

Proof:

$$E(X) = \sum_{s \in S} X(s)\,\text{Prob}(s)$$

$$= \sum_{X(s) \geq t} X(s)\,\text{Prob}(s) + \sum_{X(s)<t} X(s)\,\text{Prob}(s)$$

$$\geq t\,\text{Prob}(X \geq t)$$

- Chebyshev's inequality:

If X is a random variable and t>0, then

$$\text{Prob}(|X - E(X)| \geq t) \leq \text{Var}(X)/t^2$$

Proof:

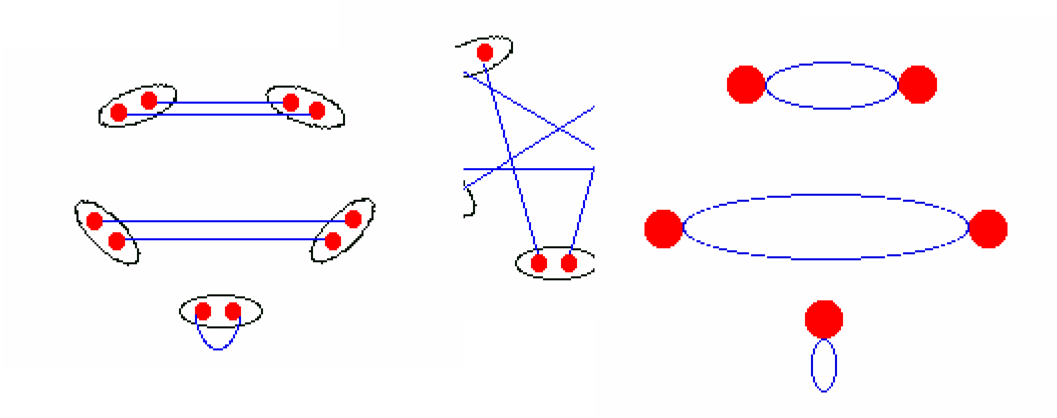
$$\text{Prob}(|X - E(X)| \geq t) = \text{Prob}(|X - E(X)|^2 \geq t^2)$$

$$\text{Prob}(X \geq t) \leq E(x)/t \quad \text{(Markov)}$$

$$\leq E(|X - E(X)|^2)/t^2 = \text{Var}(X)/t^2$$

Random regular graphs

- For regular graphs, edges do not appear independently.
- Consider $\mathcal{G}_{5,2}$



Random regular graphs

- We consider the probability space of pairings $\mathcal{P}_{n,d}$
- All simple graphs correspond to the same number of pairings
- If A is an event in $\mathcal{G}_{n,d}$ and A' is the set of pairings corresponding to the graphs in A, then

$$\text{Prob}_{\mathcal{G}_{n,d}}(A) = \text{Prob}_{\mathcal{P}_{n,d}}(A')/\ \text{Prob (Simple)}$$

Prob (Simple) > k(d) > 0, so

a.a.s in $\mathcal{P}_{n,d} \Rightarrow$ a.a.s in $\mathcal{G}_{n,d}$

Geodesics

- A geodesic in a graph is a shortest path between two of its vertices.

Theorem 1: If u, v are vertices in $G \in \mathcal{G}_{n,d}$, where d is a constant, and ω is a function of n satisfying $\lim_{n\to\infty}\omega = \infty$, then, a.a.s.,

$$\log_{d-1} n - \omega < \mathrm{dist}(u,v) < \log_{d-1} n + \omega$$

Proof of Theorem 1

- Lower bound: $\log_{d-1} n - \omega < \text{dist}(u,v)$ a.a.s.

To prove: $\text{Prob}(\text{dist}(u,v) \leq \log_{d-1} n - \omega) \rightarrow 0$

X_l : random variable counting the number of uv-paths of length at most l in G.

By Markov's inequality with t=1, $\text{Prob}(X_l \geq 1) \leq E(X_l)$

So, $\text{Prob}(X_l=0) = \text{Prob}(X_l<1) \geq 1 - E(X_l)$

Proof of Theorem 1

To prove: $E(X_l) \to 0$ when $l = \log_{d-1} n - \omega$

$\mathbf{P}_k$: family of uv-paths of length k in K_n

For $P \in \mathbf{P}_k$,

$$X_P^k = \begin{cases} 1, & \text{if } E(P) \subseteq E(G) \\ 0, & \text{otherwise} \end{cases}$$

So,

$$X_l = \Sigma_{k=1..l} \Sigma_{P \in \mathbf{P}_k} X_P^k$$

$$E(X_l) = \Sigma_{k=1..l} \Sigma_{P \in \mathbf{P}_k} E(X_P^k)$$

Proof of Theorem 1

To prove: $E(X_l) \to 0$ when $l = \log_{d-1} n - \omega$

$$E(X_l) = \Sigma_{k=1..l} \Sigma_{P \in \mathbf{P}_k} E(X_P^k)$$

$$E(X_P^k) = \text{Prob}(X_P^k = 1) \sim d/n \left((d-1)/n \right)^{k-1}$$

$$|\mathbf{P}_k| = C_{n-2,k-1}\,(k-1)!$$

Thus,

$$E(X_l) = \Sigma_{k=1..l} |\mathbf{P}_k|\, E(X_P^k) \sim \Sigma_{k=1}^{l}\, d/n\, (d-1)^{k-1}$$

$$= O((d-1)^l/n) = O((d-1)^{-\omega}) \to 0$$

Proof of Theorem 1

Lower bound established!

Proof of Theorem 1

- Upper bound: dist(u,v) < $\log_{d-1} n + \omega$ a.a.s.

To prove: Prob (dist(u,v) $\geq \log_{d-1} n + \omega$) $\rightarrow 0$

X_l : random variable counting the number of uv-paths of length at most l in G.

By Chebyshev's inequality with $t=E(X_l)$,

$$\text{Prob}(X_l=0) \leq \text{Prob}(|X_l - E(X_l)| \geq E(X_l)) \leq \text{Var}(X_l)/E(X_l)^2$$

A similar result

Theorem 2:

Let u, v be vertices in $G \in \mathcal{G}_{n,d}$, where d is a constant. Consider a function ω of n satisfying $\lim_{n\to\infty}\omega = \infty$.

Then a.a.s., if P, Q are two geodesics between u and v with midpoints p, q, respectively,

$$\log_{d-1}n - \omega < \mathrm{dist}(u,v) < \log_{d-1}n + \omega$$

Probability of a single geodesic

Theorem 3:

The probability that two vertices u, v in G ∈ $\mathcal{G}_{n,d}$, where d is a constant, are joined by exactly one geodesic is asymptotic to

$$\sum_{k=-\infty}^{\infty} d\,(d-1)^{(2k-2\gamma-2)}\,\mathbf{e}^{\left(-\frac{d\,(d-1)^{(2k-2\gamma-1)}}{d-2}\right)}\left(1+(d-1)\,\mathbf{e}^{\left(-d\,(d-1)^{(2k-2\gamma-1)}\right)}\right)$$

Conclusion

- Can we use basic probabilistic tools to derive interesting combinatorial results?