COMBINATORIAL DECOMPOSABLE STRUCTURES WITH RESTRICTED PATTERN

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INTRODUCTION

Cycle index for permutations

It is well known that a permutation can be decomposed into combination of cycles. We are interested in the number of permutations with cycles of given length.

Let $\mathbf{a} = a_1, a_2, a_3, \ldots$ be a given sequence of nonnegative integers for which $n = a_1 + 2a_2 + 3a_3 + \cdots$ is finite. How many permutations of n letters have exactly a_1 cycles of length 1, exactly a_2 cycles of length 2, and so on? In this case, $\mathbf{a}$ is called the *pattern*.

Let $C(\mathbf{a})$ denote the number of permutations with pattern $\mathbf{a}$, and write

$$F_n(x) = \sum_{a_1+2a_2+\cdots=n} C(\mathbf{a}) x_1^{a_1} x_2^{a_2} \ldots .$$

Then, $F_n(x)$ is called the **cycle index of the permutations**.

We are interested in

A. Other Decomposable Object

We are interested in generalizations of the cycle index to other decomposable structures: polynomials over finite fields, 2-regular graphs, random mappings, etc.

B. Restricted Patterns

A natural generalization of the cycle index is when we have partial information on the pattern. For example, how many permutations on n letters are there with 5 cyles of length 2 and 3 cycles of length 7?

In general, given a restricted pattern for a decomposable combinatorial object, how many objects are there with that pattern?

Furthermore, we can also study objects with restricted pattern and rth smallest or largest components. The smallest components case has been studied by Panario and Richmond; the largest case was studied by Gourdon.

COMBINATORIAL OBJECTS

Definitions and Examples

A combinatorial *object* decomposes into *components*; each component has a *size*. A permutation decomposes into cycles; the size of the cycles is its length. A polynomial over a finite field decomposes into irreducible factors; the size of an irreducible factor is its degree. A 2-regular graph decomposes into connected components; the size of a connected component is the number of vertices, and so on.

Let $C(z) = \sum_{i \geq 1} c_i z^i$ denote the generating function of components, that is, c_n, the coefficient of z^n is the number of distinct components of size n. Let $U(z)$ denote the generating function of unlabelled objects, and $L(z)$ denote the generating function of labelled ones.

RESTRICTED PATTERNS

Labelled Case

Theorem 1. Let α_i denote the number of components whose size is i and let $S = \{\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_j}\}$ be a given restricted pattern. The exponential generating function for labelled objects of size n having restricted pattern S is

$$L_s(z) = e^{C(z) - \sum_{l=1}^{j} C_{i_l} \frac{z^{i_l}}{i_l!}} \prod_{l=1}^{j} \frac{C_{i_l}^{\alpha_{i_l}} z^{i_l \alpha_{i_l}}}{(i_l!)^{\alpha_{i_l}} \alpha_{i_l}!}.$$

Example

For the labelled case, the simplest example comes from permutations viewed as a product of cycles. The generating functions of permutations is well known to be

$$C(z) = \ln \frac{1}{1-z} = z + \frac{z^2}{2!} + 2!\frac{z^3}{3!} + 3!\frac{z^4}{4!} + \cdots$$

Let us consider the pattern $\alpha_2 = 2$ and $\alpha_4 = 1$. So, from theorem 1, the generating function is

$$\left(1 + \left(z + 2\frac{z^3}{3!} + \cdots\right) + \frac{\left(z + 2\frac{z^3}{3!} + \cdots\right)^2}{2!} + \cdots\right)$$

$$\cdot \frac{1^2 z^4}{2^2 2!} \cdot \frac{6z^4}{4!}$$

If we ask for the number of different decomposition of permutations of size 11 with 2 cycle of length 2 and 1 cycle of length 4, we have

$$\left[\frac{z^{11}}{11!}\right] L(z) = 11!\frac{1}{64} = 623700$$

Unlabelled Case

Theorem 2. Let $S = \{\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_j}\}$ be given. The generating function for unlabelled objects of size n having restricted pattern S is

$$U_s(z) = \left(\prod_{l=1}^{j} (1 - z^{i_l})^{C_{i_l}} \binom{C_{i_l} + \alpha_{i_l} - 1}{\alpha_{i_l}} z^{i_l \alpha_{i_l}} \right) \exp\left(C(z) + \frac{C(z^2)}{2} + \cdots \right).$$

Example

The typical unlabelled structure is polynomials over finite fields. Consider the binary field $\mathbb{F}_2$ and assume $n = 6$ and $\alpha_2 = 1$, that is, we look for the number of polynomials of degrees 6 over $\mathbb{F}_2$ with exactly one irreducible factor of degree 2. The generating functions of irreducible polynomials over $\mathbb{F}_2$ is well known to be

$$C(z) = 2z + z^2 + 2z^3 + 3z^4 + \cdots$$

From Theorem 2, we have

$$U(z) = (z^2 + 2z^3 + 3z^4 + 6z^5 + 12z^6 + \cdots)$$

Hence, $[z^6]U(z) = 12$. Namely there are 12 polynomials over $\mathbb{F}_2$ of degree 6 with exactly one irreducible factor of degree 2.

EXTENSION OF RESTRICTED PATTERNS

For a more general case, we want to know the number of objects with a restricted pattern and such that the size of rth smallest component is bigger than or equal to m. For here, we only discuss the case of the smallest component.

We consider the generating function of components in the part $r_m(z)$.
In the unlabelled case, $r_m(z) = \sum_{k \geq m} C_k z^k$.
In the labelled case, $r_m(z) = \sum_{k \geq m} C_k \frac{z^k}{k!}$.

For later discussion, let S_n be a random variable indicating the size of the smallest component of an object of size n.

Moreover, we take $I_l = I(\alpha_{i_l} = 0) = 1$ if $\alpha_{i_l} = 0$ and 0 otherwise.

Unlabelled Case

Theorem 3. Given a restricted pattern $S = \{\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_j}\}$ and $\alpha_{i_1} \leq m$. Let $U_m(z)$ be the ordinary generating function for unlabelled objects of size n having pattern S and $S_n \geq m$, $1 \leq m \leq n$.

We have

$$U_m(z) \;=\; \left(\prod_{l=1}^{j}(1-z^{i_l})^{C_{i_l}}\binom{C_{i_l}+\alpha_{i_l}-1}{\alpha_{i_l}}z^{i_l\alpha_{i_l}}\right) \left(\exp\left(r_m(z)+\frac{r_m(z^2)}{2}+\cdots\right)-\prod_{l=1}^{j}I_l\right)$$

Labelled Case

Theorem 4. Given a restricted pattern $S = \{\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_j}\}$ and $\alpha_{i_1} \leq m$. Let $L_m(z)$ be the exponential generating function for labelled objects of size n having $S_n \geq m$, $1 \leq m \leq n$. Then we have

$$L_m(z) = \left(\exp\left(r_m(z) - \sum_{l=1}^{j} C_{i_l}\frac{z^{i_l}}{i_l!}\right) - \prod_{l=1}^{j} I_l\right) \prod_{l=1}^{j}\frac{\left(C_{i_l}\frac{z^{i_l}}{i_l!}\right)^{\alpha_{i_l}}}{(\alpha_{i_l})!}$$

Proof of Theorem 1 (sketch).

It is well known the generating function of a labelled object without parttern is

$$L(z) = \exp(C(z)).$$

Given a pattern fixing the number of components of some sizes, we first remove those sizes from the original generating function $C(z)$. It is clear that the generating function of these pieces is $\sum_{l=1}^{j} C_{i_l} \frac{z^{i_l}}{i_l!}$.

Then we need to consider the possible different combination of these components which are given by the pattern, that is, $\prod_{l=1}^{j} \frac{C_{i_l}^{\alpha_{i_l}} z^{i_l \alpha_{i_l}}}{(i_l!)^{\alpha_{i_l}} \alpha_{i_l}!}$.

Then we combine them to get the generating function for the objects with the given pattern. □

ASYMPTOTICS

The Exp-Log Class

We focus on the exp-log class introduced by Flajolet and Soria.

We say that $C(z)$ is of logarithmic type with multiplicity constant $a > 0$ if

$$C(z) = a \log\left(\frac{1}{1 - z/\rho}\right) + R(z)$$

where $R(z)$ is analytic in certain region.We say that $e^{C(z)}$ is in the $\exp - \log$ class if $C(z)$ is of logarithmic type.

Labelled examples in exp-log class are permutations, 2-regular graphs, permutations without cycles of certain length, children's yard, etc.

Unlabelled examples are polynomials over finite fields, random mapping patterns, squarefree polynomials, arithmetical groups, etc.

Let's conside a decomposable structure in the exp-log class with restricted pattern. Assume we have a pattern $S = \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_j}$, where α_{i_j} denotes the number of components of size i_j.
Let $P(S_n)$ be the probability that a decomposable object of size n has pattern S.

Theorem 5.
With a finite pattern S, we have

$$P(S_n) = m^{a-1} R \exp\left(-\sum_{l=1}^{q} C_{i_l} \frac{\rho^{i_l}}{i_l!}\right)$$

where $m = \frac{n+1-\sum_{l=1}^{j} i_l \alpha_{i_l}}{n}$, $R = \prod_{l=1}^{j} \frac{C_{i_l}^{\alpha_{i_l}} \rho^{i_l \alpha_{i_l}}}{(i_l!)^{\alpha_{i_l}} \alpha_{i_l}!}$ and ρ is the singularity of $C(z)$.

Remark. We can expand the pattern to some infinite case, but not all.

CONCLUSION

- **What has been done?**

  1. Generating functions for labelled and unlabelled cases of decomposable structures with restricted patterns.

  2. Generating functions for a further developed case, which requires rth smallest (or largest) component bigger than or equal to m.

- **Future work**

  1. Expand asymptotics of exp-log class from finite pattern case to the infinite pattern case.

  2. Obtain asympotics of exp-log class with restricted pattern such that rth smallest (or largest) component bigger (or smaller) than or equal to m.

- **Problems might be involved**

  Approaching by typical singularity analysis, infinite pattern, especially the infinite pattern with infinite components of infinite sizes might bring up infinite amount of difficulties for asymptotics.

Workshops, Ottawa, May 12-16

http://www.fields.utoronto.ca/programs/scientific/05-06/discrete_math/
http://www.fields.utoronto.ca/programs/scientific/05-06/covering_arrays/

- Ottawa-Carleton DISCRETE MATH DAY
- May 12-13 (Friday-Saturday)
- Plenary Speakers:
  Bill Cook, Anthony Evans, Jonathan Jedwab, Pierre Leroux, Kieka Mynhardt

- Workshop on COVERING ARRAYS
- May 14-16 (Sunday-Tuesday)
- Plenary Speakers:
  Rick Brewster, Charlie Colbourn, Peter Gibbons, Alan Hartman, Brett Stevens, Doug Stinson.

DEADLINE APRIL 26

- Student financial support to travel to Ottawa
- Submission of abstracts for contributed talks

EXTENSION OF RESTRICTED PATTERNS

For a more general case, we want to know the number of objects with a restricted pattern and such that the size of rth smallest component is bigger than or equal to m.

We consider the generating function of components in two parts $r_m(z)$ and $s_m(z)$. In the unlabelled case, $r_m(z) = \sum_{k \geq m} C_k z^k$, and $s_m(z) = \sum_{k<m} C_k z^k$. In the labelled case, $r_m(z) = \sum_{k \geq m} C_k \frac{z^k}{k!}$, and $s_m(z) = \sum_{k<m} C_k \frac{z^k}{k!}$.

For later discussion, let $S_n^{[r]}$ be a random variable indicating the size of the rth smallest component of an object of size n.

Moreover, we take $I(i_l < m) = 1$ if $i_l < m$ and 0 otherwise, and $I(\alpha_{i_l} = 0) = 1$ if $\alpha_{i_l} = 0$ and 0 otherwise.

Unlabelled Case

Theorem 3. Given a restricted given pattern S, let $U_m^{[r]}(z)$ be the ordinary generating function for unlabelled objects of size n having pattern S and $S_n^{[r]} \geq m$, $1 \leq m \leq n$.

Let

$$I_l = I(i_l < m) + I(\alpha_{i_l} = 0) - I(i_l < m)I(\alpha_{i_l} = 0).$$

Let $f(z,u)$ be the function

$$= \left(\prod_{l=1}^{j}(1 - u^{I_{i_l<m}} z^{i_l})^{C_{i_l}} \binom{C_{i_l}+\alpha_{i_l}-1}{\alpha_{i_l}} z^{i_l \alpha_{i_l}} u^{\alpha_{i_l} I_{i_l<m}}\right) \left(\exp\left(r_m(z) + \frac{r_m(z^2)}{2} + \cdots\right) - \prod_{l=1}^{j} I_l\right) \exp\left(u s_m(z) + u^2 \frac{s_m(z^2)}{2} + \cdots\right).$$

Let $f^{[j]}(z,u)$ be the coefficient of u^j in the function $f(z,u)$. Then we have

$$U_m^{[r]}(z) = f^{[0]}(z,u) + f^{[1]}(z,u) + \cdots + f^{[r-1]}(z,u).$$

Labelled Case

Theorem 4. Let S be a given restricted pattern and let $L_m^{[r]}(z)$ be the exponential generating function for labelled objects of size n having $S_n^{[r]} \geq m$, $1 \leq m \leq n$. Let $f(z,u)$ be the function

$$f(z,u) = \exp\left(us_m(z) - \sum_{l=1}^{j} uI(i_l < m)C_{i_l}\frac{z^{i_l}}{i_l!}\right) \left(\exp\left(r_m(z) - \sum_{l=1}^{j} I(i_l < m)C_{i_l}\frac{z^{i_l}}{i_l!}\right) - \prod_{l=1}^{j} I_l\right) \prod_{l=1}^{j} \frac{\left(u^{I_{i_l<m}}C_{i_l}\frac{z^{i_l}}{i_l!}\right)^{\alpha_{i_l}}}{(\alpha_{i_l})!}$$

Then we have

$$L_m^{[r]}(z) = f^{[0]}(z,u) + f^{[1]}(z,u) + \cdots + f^{[r-1]}(z,u)$$