Graph Isomorphism Completeness for Subclasses of Perfect Graphs

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(joint work with D. Loker)

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Graph Isomorphism

Definition

We define two graphs G_1 and G_2 to be **isomorphic** if there is a bijection $\varphi : V_1 \to V_2$ such that $(u, v) \in E_1$ if and only if $(\varphi(u), \varphi(v)) \in E_2$.

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Some Basic Facts:

- GI is in the class NP
- No known polynomial time algorithm for GI
- Strong evidence that the problem is not NP-complete

The Computational Complexity Class *GI*

Definition

A problem is said to be ***GI*-complete** if it is provably as hard as GI.

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***GI*-Complete**
bipartite graphs
split graphs
chordal graphs

The Computational Complexity Class *GI*

Definition

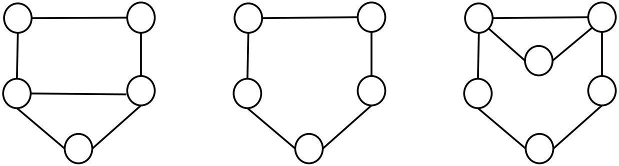
A problem is said to be ***GI*-complete** if it is provably as hard as GI.

***GI*-Complete**	**Polynomial Time Solvable**
bipartite graphs	planar graphs
split graphs	interval graphs
chordal graphs	convex graphs

Perfect Graphs

Definition

A graph G is defined to be **perfect** if for every induced subgraph of G, the **size of the largest clique** equals the **chromatic number**.



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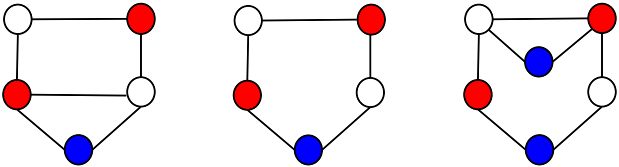

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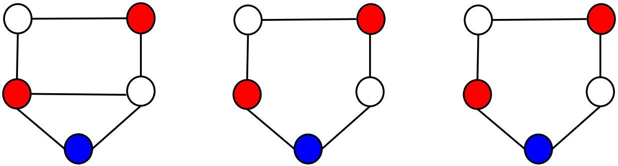

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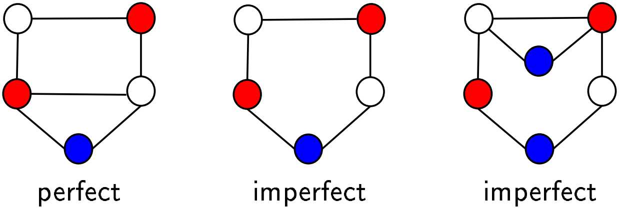

Preliminaries

A **chord** is an edge which joins two vertices of a path or cycle but is not itself not in the path or cycle.

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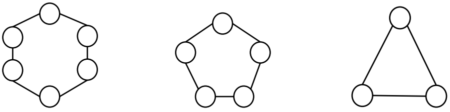
A **hole** is a chordless cycle of length greater than or equal to 4. An **antihole** is a hole in the complement graph.



C_6 C_5 $C_3 = K_3$

Berge Graphs

Berge Graphs are graphs that contain no odd hole and no odd antihole.

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In 1961, Claude Berge made two Conjectures:

Weak Perfect Graph Conjecture

The complement of every perfect graph is perfect.

The Strong Perfect Graph Conjecture (SPGC)

A graph is perfect if and only if it is Berge.

Berge and Perfect Graphs

Berge Graphs are graphs that contain no odd hole and no odd antihole.

In 1961, Claude Berge made two Conjectures:

Lovász's Theorem (1972)

The complement of every perfect graph is perfect.

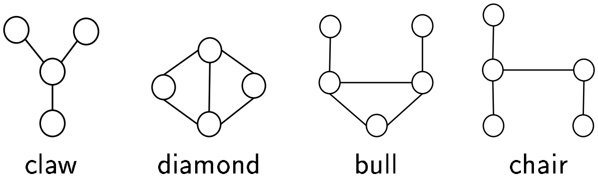
The Strong Perfect Graph Theorem (2002 to 2005)

A graph is perfect if and only if it is Berge.

Classes of Perfect Graphs

The SPGC was first verified for several graph classes, including:

- claw-free graphs (Parthasarathy and Ravindra, 1976)
- K_4-free graphs (Tucker, 1977)
- bull-free graphs (Chvátal and Sbihi, 1987)
- chair-free graphs (Sassano, 1997)



How the Strong Perfect Graph Theorem was Proved

Theorem (Chudnovsky, Robertson, Seymour and Thomas [C+06])

Every Berge graph G satisfies one of the following:

- G or $\overline{G}$ is bipartite,
- G or $\overline{G}$ is the line graph of a bipartite graph,
- G is a double split graph

or

- G or $\overline{G}$ admits a 2-join,
- G or $\overline{G}$ admits a homogeneous pair,
- G admits a balanced skew partition.

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GI-Completeness for Perfect Graphs

Lemma 1

Given the graph classes α and β such that $\beta \subseteq \alpha$, if GI for β is *GI*-complete then GI for α is *GI*-complete.

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Corollary 1

GI for the class of perfect graphs is *GI*-complete.

Proof:

- GI for chordal graphs has been shown to be *GI*-complete [BL79]
- Chordal graphs is a subclass of perfect graphs

Two General Results

Lemma 2

The GI problem for any restricted proper graph class is in *GI*.

Lemma 3

Given two graphs G_1 and G_2 and their respective complements $\bar{G}_1$ and $\bar{G}_2$, $G_1 \sim G_2$ if and only if $\bar{G}_1 \sim \bar{G}_2$.

Five Basic Graph Classes

We show the *GI*-completeness for the first five basic graph classes:

- Bipartite graphs and their complements
- Line graphs of bipartite graphs and their complements
- Double Split Graphs

These results are trivial from the result that bipartite graphs are *GI*-complete.

We consider an reduction from **split graphs** to double split graphs to show the *GI*-hardness.

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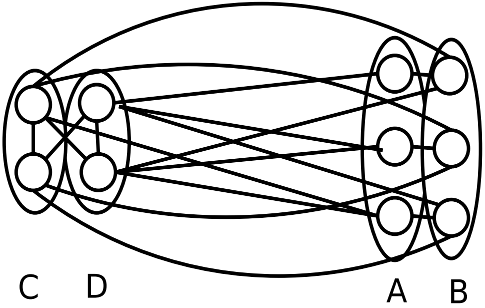
Definition of a Double Split Graph

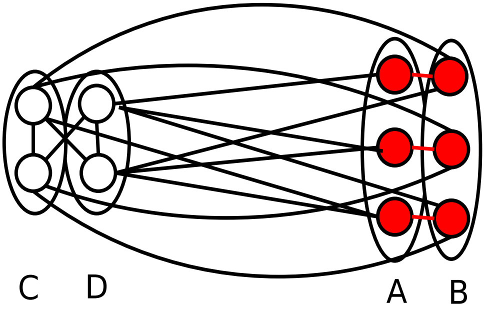
Definition

A graph is a **double split graph** if V can be partitioned into four sets A, B, C, D as follows:



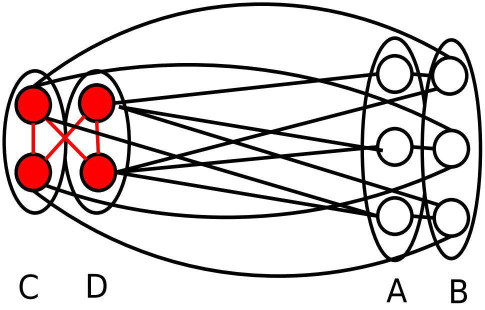
Definition of a Double Split Graph

- a_i is adjacent to b_i for $1 \leq i \leq m$



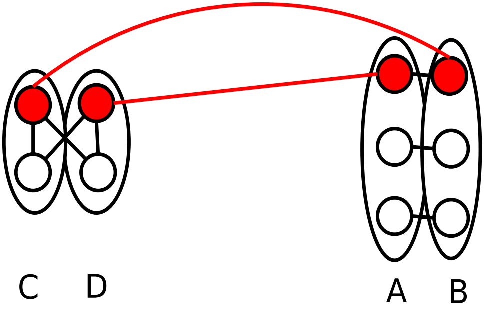
Definition of a Double Split Graph

- c_j is nonadjacent to d_j for $1 \leq j \leq n$



Definition of a Double Split Graph

- There exists a P_4 joining the pairs $\{a_i, b_i\}$ and $\{c_j, d_j\}$ (and the edges of the P_4 have no common end).



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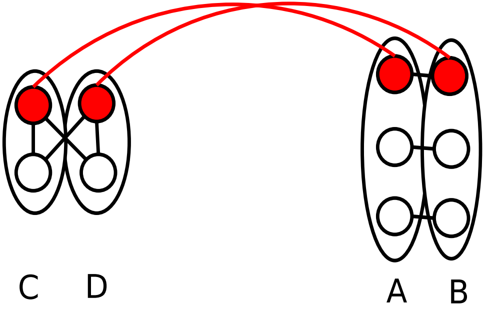

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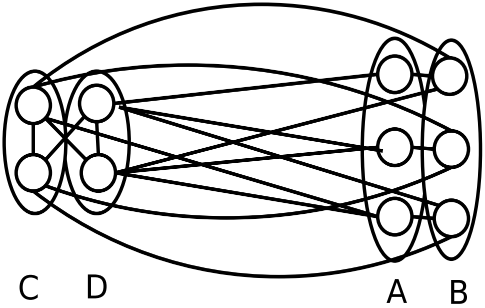


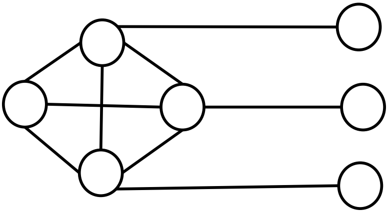
Definition of a Split Graph

Definition

A **split graph** is a graph whose vertex set can be partitioned into a clique and stable set.

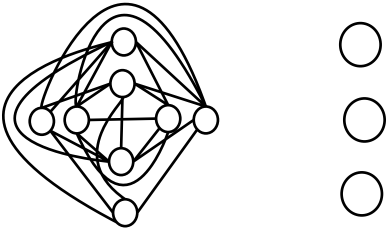
Reduction: Split Graphs $\rightarrow$ Double Split Graphs

We let $G = (Q \cup S, E)$ be our split graph.



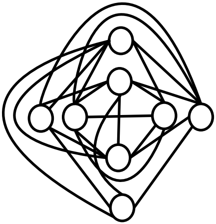
Reduction: Split Graphs → Double Split Graphs

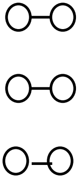
- Replace every $q_i \in Q$ by two non-adjacent vertices c_i, d_i.



Reduction: Split Graphs → Double Split Graphs

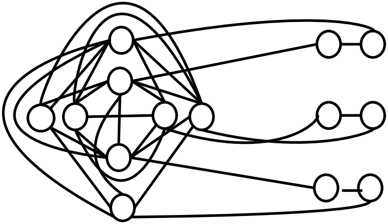
- Replace every $s_j \in S$ by two adjacent vertices a_j, b_j.





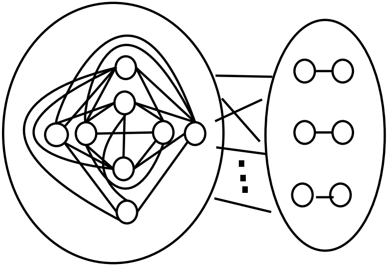
Reduction: Split Graphs → Double Split Graphs

- For every edge $\{q_i, s_j\} \in E$ we have $\{d_i, b_j\} \in \mathcal{E}$, $\{a_i, c_j\} \in \mathcal{E}$.



Reduction: Split Graphs → Double Split Graphs

- For every edge $\{q_i, s_j\} \notin E$ we have $\{d_i, a_j\} \in \mathcal{E}$, $\{c_i, b_j\} \in \mathcal{E}$.



GI-Completeness for Double Split Graphs

Theorem

GI for double split graphs is *GI*-complete.

Proof:

- GI for double split graphs is in *GI* (lemma 2).
- GI for split graphs is *GI*-complete.
- From our reduction it follows that GI for double split graphs is *GI*-hard.

Graph Classes Defined by Graph Decompositions

We show GI for the following graph classes is *GI*-complete:

- Graphs admitting a balanced skew partition
- Graphs admitting a 2-join
- Graphs whose complement admits a 2-join

The result follows from a reduction from split graphs.
We consider an reduction from bipartite graphs to graphs admitting a 2-join to show the *GI*-hardness.
Follows from the previous result and lemma 3.

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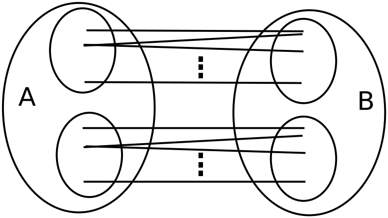
Definition of a 2-join Graph

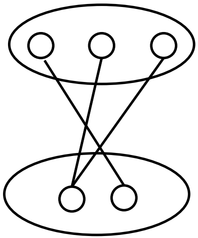
Definition

G admits a **2-join** if $V(G)$ can be partitioned A and B, where A_1, A_2 are disjoint subsets of A and B_1, B_2 are disjoint subsets of B, such that:



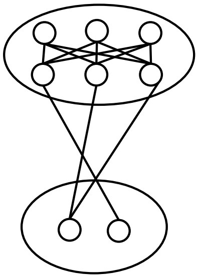
Reduction: Bipartite Graphs $\rightarrow$ Graphs Admitting a 2-join

We let $G = (A \cup B, E)$ be our bipartite graph.



Reduction: Bipartite Graphs → Graphs Admitting a 2-join

- The vertex set $\mathcal{V}$ of the reduced graph $\mathcal{G} = (\mathcal{V} = A \cup A' \cup B \cup B', \mathcal{E})$.



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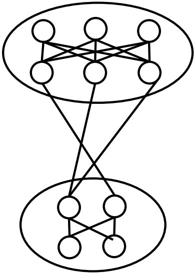

GI-Completeness for Graphs Admitting a 2-join

Theorem

GI for graphs admitting a 2-join is *GI*-complete.

Proof:

- GI for the class of graphs admitting a 2-join is in *GI* (lemma 2).
- GI for bipartite graphs is *GI*-complete
- From our reduction it follows that GI for graphs admitting a 2-join is *GI*-hard.

Extensions to Other Graph Classes

GI for chordal bipartite graphs and strongly chordal graphs is *GI*-complete [UTN05].

chordal bipartite $\subseteq$ strongly chordal $\subseteq$ SQP $\subseteq$ QP.

- GI for **strict quasi-parity (SQP)** and **quasi-parity (QP)** is *GI*-complete.

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- GI for **strict quasi-parity (SQP)** and **quasi-parity (QP)** is *GI*-complete.
- **perfectly contractile graphs** is a *GI*-complete graph class since strongly chordal graphs is a subclass of perfectly contractile graphs.

Future Work

GI-completeness for the following two classes is open:

- **clique separable**
- **trapezoid graphs**

Future Work

- **clique separable**

clique separable $\subseteq$ perfectly contractile $\subseteq$ SQP

Future Work

- **trapezoid graphs**

Conjecture

There exists a polynomial-time GI algorithm for trapezoid graphs.

Final Notes

- All graph classes that have shown to be *GI*-complete are subclasses of perfect graphs.

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- All graph classes that have shown to be *GI*-complete are subclasses of perfect graphs.
- Are there other complexity problems that could be formulated into a GI problem and shown to be *GI*-complete?

References

[BL79] K.S. Booth and G.S. Leuker, “A linear time algorithm for deciding interval graph isomorphism”, *Journal of the ACM*, 26(2): 183-195, 1979.

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