Constant scalar curvature Kähler metrics and stability of algebraic varieties

Joint work with **JULIUS ROSS**

1. Geometric Invariant Theory

2. Symplectic reduction

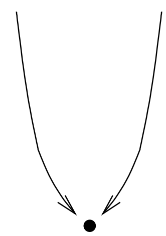
3. Balanced varieties, cscK metrics

4. Bundle analogue

5. Stability of algebraic varieties

Geometric Invariant Theory

$$\begin{array}{ccc} G & \curvearrowright & X \\ \cap & & \cap \\ SL(n+1,\mathbb{C}) & & \mathbb{P}^n \end{array} \qquad X/G\ ?$$



G-action not proper.

Quotient not Hausdorff (not separated).

GIT chooses certain "unstable" orbits to remove to give a projective quotient. Also identifies some "semistable" orbits to compactify.

$$(X, L = \mathcal{O}(1)) \longleftrightarrow \bigoplus_r H^0(X, \mathcal{O}(r)),$$

$$X/G \longleftrightarrow \bigoplus_r H^0(X, \mathcal{O}(r))^G.$$

$$(f_1 = 0 = \ldots = f_k) \subset \mathbb{P}^n \quad \longleftrightarrow \quad \frac{\mathbb{C}[x_0, \ldots, x_n]}{(f_1, \ldots, f_k)}.$$

G acts on $\mathbb{C}^{n+1}$ so on $\mathcal{O}(-1) \to \mathbb{P}^n$ so on $\mathcal{O}(r) \to X$.

$H^0(X, \mathcal{O}(r)) = \{$degree r homogeneous polynomials on $\widetilde{X} \subset \mathbb{C}^{n+1}\}$.

$x \in X$ *semistable* iff $\exists f \in H^0(X, \mathcal{O}(r))^G$ such that $f(x) \neq 0$.

So the Kodaira "embedding" of X/G,

$$X \dashrightarrow \mathbb{P}((H^0(X, \mathcal{O}(r))^G)^*)$$

is well defined at x.

x is *stable* iff $\bigoplus_r H^0(X, \mathcal{O}(r))^G$ separates orbits at x and the stabiliser of x is finite.

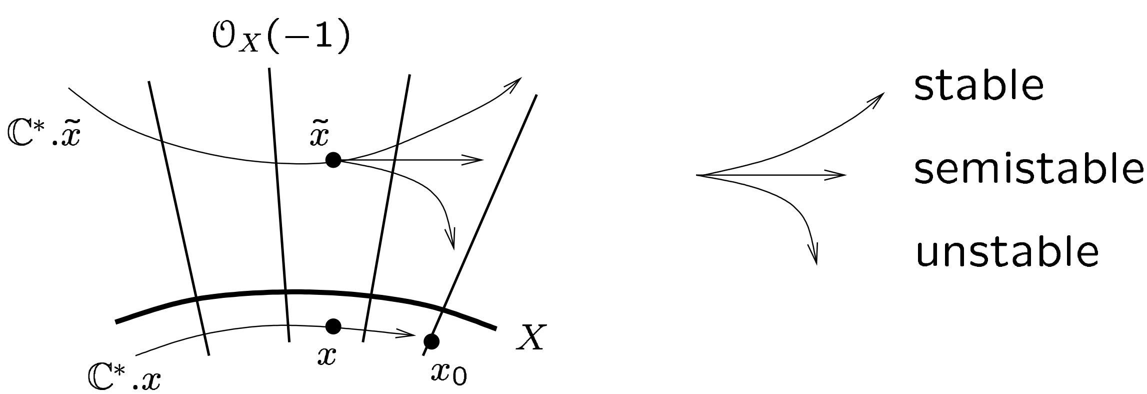
Theorem 1 [Mumford]
x is stable $\iff$ $G.\tilde{x}$ is closed in $\mathbb{C}^{n+1}$ and $\dim G.\tilde{x} = \dim G$.
($G.\tilde{x}$ just closed = polystable.)
x is semistable $\iff$ $0 \notin \overline{G.\tilde{x}}$.

Theorem 2 [Hilbert-Mumford criterion]
The same result is true iff it is true for all one parameter subgroups (1-PS) $\mathbb{C}^* \subset SL(n+1, \mathbb{C})$. So everything reduces to the $\mathbb{C}^*$-action on the line over the limit point $x_0 = \lim_{\lambda \to 0} \lambda.x$.
x_0 fixed point of $\mathbb{C}^*$-action, so get action on $\mathcal{O}_{x_0}(-1)$.

Weight $\rho \in \mathbb{Z}$ of action, $\lambda \mapsto \lambda^\rho$,

- $\rho < 0$ *stable*
- $\rho = 0$ *semistable*
- $\rho > 0$ *unstable*

So "just" compute this weight for all $\mathbb{C}^* \subset SL(n+1, \mathbb{C})$; x is stable $\iff$ weight always < 0.



Fundamental example – points in $\mathbb{P}^1$

n points in $\mathbb{P}^1 \;\leftrightarrow\;$ 0-dim algebraic subvariety!

(Points with multiplicities $\leftrightarrow$ length-n 0-dim subscheme)

$$
\begin{aligned}
& SL(2,\mathbb{C}) \curvearrowright \mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2) \\
\Rightarrow\ & SL(2,\mathbb{C}) \curvearrowright S^n(\mathbb{C}^2)^* \\
& \qquad = \{\deg n \text{ polys on } \mathbb{C}^2\} = H^0(\mathcal{O}_{\mathbb{P}^1}(n)).
\end{aligned}
$$

But $\{n \text{ points}\} = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(n)))$ as roots of the degree n polynomial.

Theorem 3 *n points in $\mathbb{P}^1$.*
Semistable $\iff$ each multiplicity $\leq n/2$.
Stable $\iff$ each multiplicity $< n/2$.

Proof. Diagonalise a given $\mathbb{C}^* \subset SL(2,\mathbb{C})$:

$\begin{pmatrix} \lambda^k & 0 \\ 0 & \lambda^{-k} \end{pmatrix}$ w.r.t. $[x:y]$ coords on $\mathbb{P}^1$. $(k \geq 0.)$

Polynomial $f = \sum_{i=0}^n a_i x^i y^{n-i}$.
$\lambda.f$ tends to ∞ iff there are more ys than xs in a nonzero summand.

I.e. stable unless $a_i = 0$ for $i \leq n/2$.

Alternatively, use Hilbert-Mumford criterion. After rescaling, $\lambda.f \to f_0 = a_j x^j y^{n-j}$, where j is smallest such that $a_j \neq 0$.
$(f = a_j x^j y^{n-j}(1 + \frac{a_{j+1}}{a_j} x y^{-1} + \dots.)$

Weight on $\mathbb{C}.f_0$ is $k(j-(n-j)) = k(2j-n)$.
So stable $\iff$ $k(2j-n) < 0$ $\iff$ $j < n/2$ $\iff$ f vanishes to order $< n/2$ at $x = 0$ $\forall \mathbb{C}^* \subset SL(2,\mathbb{C})$. $\square$

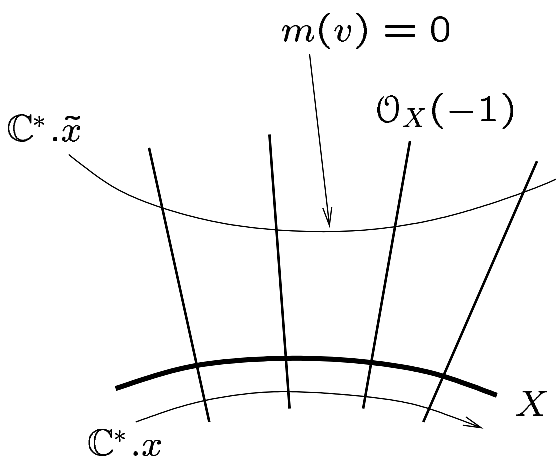
Symplectic reduction

$G \subset SL(N+1, \mathbb{C})$ has compact subgroup $K = G \cap SU(N+1)$. $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$.
K acts on $\mathbb{P}^N$, preserves J *and* g, and so ω too.

So $\forall v \in \mathfrak{k} = LK$ the infinitesimal action X_v is *Hamiltonian*, $X_v \lrcorner\, \omega = dm_v$.

Gives **moment map** $\quad m : X \to \mathfrak{k}^*$.
(Collection of r hamiltonians m_v, $r = \dim K$.)

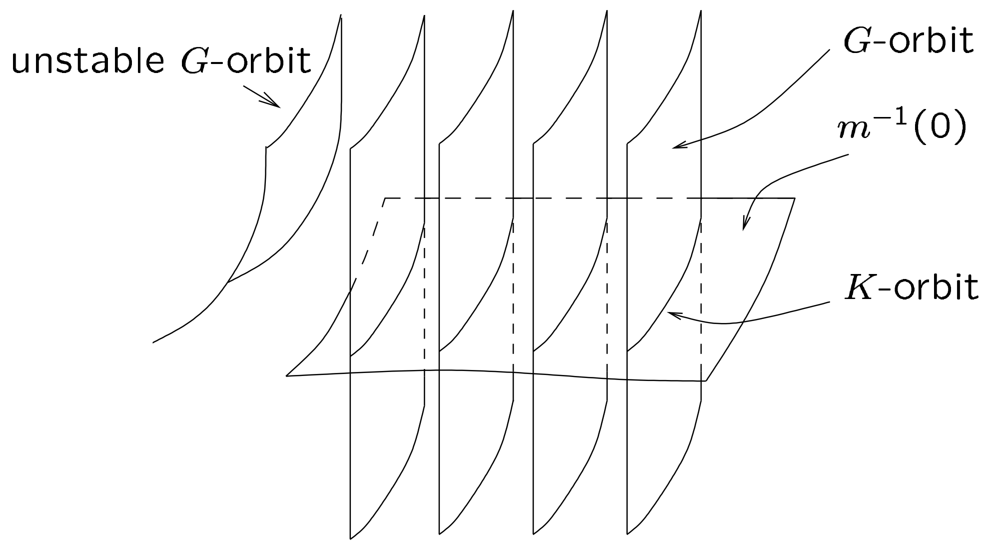
m_v = derivative down $(0, \infty) \subset \mathbb{C}^*$ orbit of $\log ||\lambda \tilde{x}||_{\lambda \in (0,\infty)}$, i.e. down $JX_v = X_{iv}$.



(Poly)Stable $\iff ||\lambda \tilde{x}||$ achieves min on all $\mathbb{C}^*$-orbits $\iff m(v) = 0$ somewhere on orbit $\forall v$.

Theorem 4 [Kempf-Ness]

$$\frac{X}{G} \cong \frac{m^{-1}(0)}{K}.$$



$m^{-1}(0)$ provides slice to $i\mathfrak{k} \subset \mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ part of orbit; K-equivariant.

E.g. $U(1) \subset \mathbb{C}^* \curvearrowright \mathbb{C}^n$, moment map $= |\underline{z}|^2 - a^2$.

$$\frac{\mathbb{C}^n \backslash \{0\}}{\mathbb{C}^*} \cong \frac{S^{2n-1} = \{\underline{z} \,:\, |\underline{z}|^2 = a^2\}}{U(1)} \cong \mathbb{P}^{n-1}.$$

E.g. n points in $\mathbb{P}^1$ again.

$$SL(2,\mathbb{C}) \supset SU(2) \curvearrowright \mathbb{P}^1 \xrightarrow{m} \mathfrak{su}(2)^*$$

is the inclusion $S^2 \subset \mathbb{R}^3$.

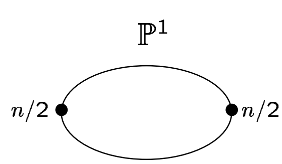
Adding gives, for n points, $m = \sum_{i=1}^n m_i$:

$$S^n\mathbb{P}^1 \longrightarrow \mathbb{R}^3,$$

the sum of n points in $\mathbb{R}^3$ ("centre of mass").

So $m^{-1}(0) = \{$**Balanced configurations**$\}$
(Centre of mass $0 \in \mathbb{R}^3$).

Stable $\iff$ $\exists$ $SL(2,\mathbb{C})$ transformation of $\mathbb{P}^1$
such that points are balanced
$\iff$ mass at each point $< n/2$.

(Note that balanced 

has dim 1 stabiliser.)

Polarised algebraic varieties (X, L)

$$X \hookrightarrow \mathbb{P}(H^0(X, L^r)^*) = \mathbb{P}^N, \quad r \gg 0.$$

Defines a point in Hilb $\subset Gr \subset \mathbb{P}^M$ by the subspace

$$H^0(\mathbb{P}^N, \mathscr{I}_X(k)) \subset H^0(\mathbb{P}^N, \mathcal{O}(k)) = S^k H^0(X, L^r)$$

of deg k polys on $\mathbb{P}^N$ vanishing on X.

I.e. point of $\Lambda^{\dim H^0_{\mathbb{P}^N}(\mathscr{I}_X(k))} S^k H^0(X, L^r), \quad r, k \gg 0.$

Divide by autos $SL(N+1, \mathbb{C})$ of $\mathbb{P}^N$ to get moduli of polarised varieties.

Choice of line bundle on Hilb $\Rightarrow$ notion of stability for (X, L).

Moment map for appropriate ample line bundle / symplectic structure on Hilb.

Let $m\colon \mathbb{P}^N \to \mathfrak{su}(N+1)^*$ denote the usual moment map.

Then moment map takes $X \subset \mathbb{P}^N$ to the centre of mass

$$\int_X m \ \mathrm{vol}_{FS} \ \in \ \mathfrak{su}(N+1)^*.$$

Zeros of moment map = **Balanced** varieties $X \subset \mathbb{P}^N$.

Theorem 5 [Zhang/Luo] *Balanced + finite automorphism group $\Rightarrow$ HM stable.*

As $r \to \infty$ ($\Rightarrow N \to \infty$) moment map has an expansion (Catlin, Z. Lu, W.-D. Ruan, Tian, Zelditch) involving $s =$ scalar curvature of g_{FS}. Roughly, balanced metrics "tend towards" cscK metrics with $[\omega] = [c_1(L)]$.

Theorem 6 [Donaldson] (Aut(X) discrete.)
(X, L) admits cscK metric in $[c_1(L)]$ $\Rightarrow$ (X, L^r) balanced for $r \gg 0$.
(Zhang/Luo $\Rightarrow$ HM-stable, Chen-Tian $\Rightarrow$ K-stable.)

Partial result in converse direction: If $(X, L^r) \subset \mathbb{P}^{N(r)}$ balanced for $r \gg 0$ and resulting $\omega_{FS,r}$ convergent, then limit metric has csc. Also generalisation due to Mabuchi for arbitrary X.

Donaldson also gives an infinite dimensional GIT/moment map formulation.
(Think of as lim $r \to \infty$, where balanced condition has become cscK condition.)

(Hamiltonian diffeomorphisms) $\curvearrowright (X, \omega = c_1(L))$ so $\curvearrowright$ {compatible complex structures on X}.

Moment map = scalar curvature + const.

Zeros = cscK metrics.

(When $L = K_X^{\pm 1}$, $\omega = \mp c_1(X)$, cscK=KE. Yau suggested the relationship stability $\leftrightarrow$ KE metrics. Tian proved this for surfaces and also suggested the general polarisation / cscK relationship.)

So we have the infinite dimensional analogue of the balanced condition for points in $\mathbb{P}^1$ (i.e. cscK metrics) and part of the relationship to stability (Donaldson, Luo), but not the algebro-geometric description of stability. I.e. the Hilbert-Mumford criterion, giving the analogue of the $< n/2$ points condition, is missing.

In the bundle case, *all* of these ingredients are worked out.

Moduli of bundles over (X, L)

Given $E \to X$, form $E(r) := E \otimes L^r$ for $r \gg 0$,

$$H^0(E(r)) \to E(r) \to 0 \qquad \text{on } X.$$

Gives map $X \to Gr$. Action of $SL(H^0(E(r)))$ on Maps(X, Gr). $\qquad Gr \subset \mathbb{P}^N \subset \mathfrak{su}(N+1)^*$.

So can again talk about *balanced* $X \to Gr$ and asymptotics as $r, N(r) \to \infty$.

Stable bundles admit balanced maps $X \to Gr$. Pulling back the canonical quotient connection on Gr gives, in the limit, a HYM connection (X.-W. Wang. Donaldson-Uhlenbeck-Yau: polystable $\Rightarrow$ HYM. This connection is then unique).

Atiyah-Bott gave an infinite dimensional GIT / moment map formulation.
$U(E) = \{$unitary gauge transformations$\}$,
$\mathcal{A} = \{$connections A with $F_A^{0,2} = 0\}$.
$U(E) \curvearrowright \mathcal{A}$.
Moment map = HYM = $\omega^{n-1} \wedge F_A^{1,1}$.

In this case HM-criterion can be manipulated (Gieseker, Maruyama, Simpson) to give an algebro-geometric understanding of stability.

Hilbert poly $h^0(E(r)) = a_0 r^n + a_1 r^{n-1} + \dots$
$a_0 = \operatorname{rk} E \int_X \omega^n/n!, \quad a_1 = \int_X c_1(E).\omega^{n-1}/(n-1)! + \varepsilon(X).$

Reduced Hilbert poly $p_E(r) = r^n + \frac{a_1}{a_0} r^{n-1} + \dots$.

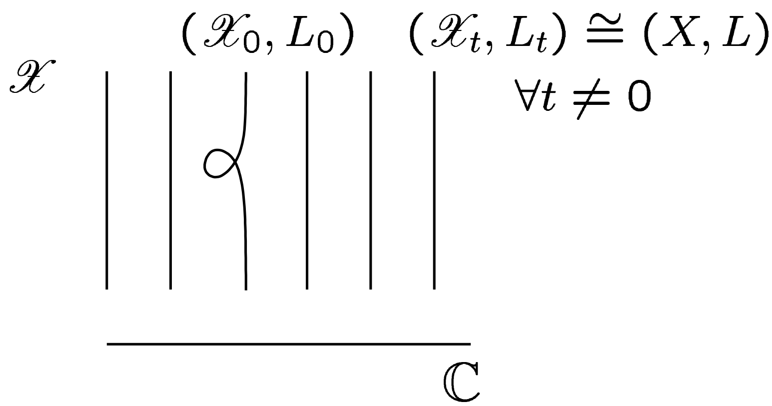
E stable $\iff \forall F \hookrightarrow E,\ p_F(r) < p_E(r) \quad r \gg 0.$

$$E \text{ slope-stable} \quad \begin{array}{ll} \iff & \dfrac{a_1(F)}{a_0(F)} < \dfrac{a_1(E)}{a_0(E)} \\ \iff & \mu(F) < \mu(E). \end{array}$$

($\mu(E) = \int_X c_1(E).\omega^{n-1}/\operatorname{rk}(E)$. Corresponds to a different line bundle on moduli space – Jun Li.)

So bundles/sheaves destabilised by subsheaves $F \subset E$. Can $\mathbb{P}(F) \subset \mathbb{P}(E)$ destabilise as varieties? Can subschemes $Z \subset (X, L)$ destabilise? (cf. length $\geq n/2$ subschemes of n points in $\mathbb{P}^1$.)

A $\mathbb{C}^* \subset SL(M+1,\mathbb{C})$ orbit of $X \in \mathrm{Hilb} \subset \mathbb{P}^M$ gives a $\mathbb{C}^*$-equivariant flat family (**test configuration**) $\mathscr{X} \to \mathbb{C}$



For the HM-criterion one calculates the weight $w_{r,k}$ of the $\mathbb{C}^*$-action on

$$\Lambda^{\max} H^0(\mathscr{X}_0, L_0^{rk})^* \otimes \Lambda^{\max} S^k H^0(\mathscr{X}_0, L_0^r).$$

$$w_{r,k} = a_{n+1}(r)k^{n+1} + a_n(r)k^n + \dots,$$

where

$$a_i(r) = a_{in}r^n + a_{i,n-1}r^{n-1} + \dots.$$

Definition 7 *The* $\mathbb{C}^* \subset SL(M+1, \mathbb{C})$ *destabilises* (X, L) *if* $w_{r,k} \succ 0$ *in the following sense:*

- *HM(*r*)-unstable:* $w_{r,k} > 0$ *for all* $k \gg 0$,
- *Asymptotically HM-unstable: for all* $r \gg 0$, $w_{r,k} > 0$ *for all* $k \gg 0$,
- *Chow(*r*)-unstable: leading* k^{n+1}*-coefficient* $a_{n+1}(r) > 0$,
- *Asymptotically Chow unstable:* $a_{n+1}(r) > 0$ *for* $r \gg 0$,
- *K-unstable: leading coefficient* $a_{n+1,n} > 0$.

These correspond to different line bundles on Hilb: the standard one, the Chow line, and the Paul-Tian line.

Slope for K-stability

$$Z \subset (X, L)$$

$$h^0(\mathcal{O}_X(r)) = a_0 r^n + a_1 r^{n-1} + \dots$$

$$h^0(\mathscr{I}_Z^{xr}(r)) = a_0(x) r^n + a_1(x) r^{n-1} + \dots$$

$a_i(x)$ polynomials in $x \in \mathbb{Q} \cap [0, \epsilon(Z))$ for $r \gg 0$.
(Seshadri constant $\epsilon(Z)$ defined so that $\mathscr{I}_Z^{xr}(r)$ generated by global sections for $x < \epsilon(Z)$ for $r \gg 0$).

$a_0(0) = a_0$, and $a_1(0) = a_1$ for X normal.

$$a_0 = \frac{\int_X \omega^n}{n!}, \; a_1 = \frac{\int_X c_1(X) \omega^{n-1}}{2(n-1)!}.$$

For any $c \leq \epsilon(Z)$, define slope of Z to be

$$\mu_c(\mathscr{I}_Z) = \frac{\int_0^c a_1(x) + \frac{a_0'(x)}{2} dx}{\int_0^c a_0(x) dx}.$$

$Z = \emptyset$ gives

$$\mu(X) = \frac{a_1}{a_0}.$$

Theorem 8

K-(semi)stable $\implies$ slope (semi)stable:
$\mu_c(\mathscr{I}_Z) \leq \mu(X)$ $\forall$ *closed subschemes* $Z \subset X$.
(*K-stability:* $\mu_c(\mathscr{I}_Z) < \mu(X)$ $\forall c \in (0, \epsilon(Z))$ and $\forall c \in (0, \epsilon(Z)]$ *if* $\epsilon(Z) \in \mathbb{Q}$ *and* $\mathscr{I}_Z^{\epsilon(Z)r}(r)$ *saturated by global sections for* $r \gg 0$.)

Corollary 9

If $\mu_c(\mathscr{I}_Z) > \mu(X)$ *then* X *admits no cscK metric in the class of* $c_1(L)$.
(Donaldson/Zhang/Luo & Chen-Tian: cscK $\implies$ *K-semistable.)*

Examples.

- $F \subset E$ destabilising subbundle $\implies$ $\mathbb{P}(F) \subset \mathbb{P}(E)$ destabilises, for suitable polarisations $\pi^* L^m \otimes \mathcal{O}_{\mathbb{P}(E)}(1)$, $m \gg 0$. And for **all** polsraisations if the base is a curve, in which case $\mathbb{P}(E)$ is cscK $\leftrightarrow$ E is HYM.

- -1-curves on del Pezzo surfaces for appropriate L. So Aut(X) reductive (or trivial) does not imply cscK (unless $L \neq K^{-1}$, by Tian).

- $\mathbb{P}^2$ blown up in one point. Aut(X) not reductive $\implies$ not stable. Destabilised by the -1-curve for all polarisations.

- Generically stable varieties can specialise to unstable ones. Move two -1-curves together on a del Pezzo to give a limit -2-curve.
  (Blow up 2 "infinitely near" points: blow up one, then another on the exceptional curve.)
  The -2-curve destabilises for suitable L.

- Calabi-Yau manifolds, and varieties with canonical singularities and $mK_X \sim 0$ are slope stable.

- Canonically polarised varieties with canonical singularities (i.e. the canonical models of Mori theory) are slope stable.

- Curves are K-stable ($\mathbb{P}^1$ is K-polystable).