Computing Hodge Integrals with one λ-class

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What is Hodge integral?

Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford moduli stack of stable curves of genus g with n marked points. Consider the universal curve $\pi: \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, and let ω_π be the dualizing sheaf, $s_i: \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ denote the section of π which corresponds to the i-th marked point. There are two natural classes on $\overline{\mathcal{M}}_{g,n}$:

- $\psi_i = c_1(s_i^* \omega_\pi) \qquad s_i^*\omega_\pi \big|_{[(C,x_1,\cdots,x_n)]} = T_{x_i}^* C$
- $\lambda_j = c_j(\mathbb{E})$

$$\mathbb{E} = \pi_* \omega_\pi \quad \text{Hodge bundle of rank } g$$

$$\mathbb{E}\big|_{[(C,x_1,\cdots,x_n)]} = H^0(C, \omega_C)$$

Hodge integral is an integral of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1} \cdots \psi_n^{j_n} \lambda_1^{k_1} \cdots \lambda_g^{k_g}$$

Hodge integrals naturally arise in the calculations of Gromov-Witten invariants by localization techniques.

Goal My goal of this talk is to

- Obtain Recursion Relation between Hodge integrals
- Give an algorithm to compute all Hodge integrals with up-to one λ-class, λ_j

Moduli Space of Relative Morphisms

Double Hurwitz numbers

Localization Formula

Recursion Formula

$$[\lambda^{\ell(\mu)-\chi}] \sum_{|\nu|=|\mu|} \Phi^{\bullet}_{\mu,\nu}(-\lambda)\, z_\nu\, D^{\bullet}_{\nu,e}(\lambda) = 0$$

$\int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_j$ = Polynomial in terms of lower dimensional Hodge integrals w/one λ-class

Examples ...

Notations

$$
D_{g,\nu,e} = \begin{cases}
\dfrac{1}{\ell(e)!\,|\mathrm{Aut}\,\nu|}\left[\displaystyle\prod_{i=1}^{\ell(\nu)} \frac{\nu_i^{\nu_i}}{\nu_i!}\right]\displaystyle\int_{\overline{\mathcal{M}}_{g,\ell(\nu)+\ell(e)}} \frac{\Lambda_g^\vee(1)\prod(1-\psi_j)^{e_j}}{\prod(1-\nu_i\psi_i)} & \text{S otherwise} \\[2ex]
\nu_1^{\nu_1-2}/\nu_1! \quad \begin{matrix} g=0=\ell(e) \\ \ell(\nu)=1 \end{matrix} \ \mathrm{I} \qquad \dfrac{1}{|\mathrm{Aut}\,\nu|}\dfrac{\nu_1^{\nu_1}\nu_2^{\nu_2}}{\nu_1!\,\nu_2!}\dfrac{1}{\nu_1+\nu_2} & \begin{matrix} g=0 \\ \ell(\nu)=2 \\ \ell(e)=0 \end{matrix}\ \mathrm{II}_1 \\[2ex]
\dfrac{\nu_1^{\nu_1}}{\nu_1!}\displaystyle\sum_{k=0}^{e_1}\frac{1}{\nu_1^{1+k}}\binom{e_1}{k} & g=0,\ \ell(\nu)=1,\ \ell(e)=1 \quad \mathrm{II}_2
\end{cases}
$$

$$D(\lambda,p,q) = \sum_{|\nu|\geq 1}\sum_{g\geq 0}\lambda^{2g-2+\ell(\nu)} p_\nu q_e D_{g,\nu,e}$$

$$D^\bullet(\lambda,p,q) := \exp\left(D(\lambda,p,q)\right) = \sum_{|\nu|\geq 1} p_\nu q_e D^\bullet_{\nu,e}(\lambda)$$

$H^\bullet_\chi(\mu,\nu)$ = Double Hurwitz number with Euler characteristic χ, i.e. weighted counting of covers $C \longrightarrow \mathbb{P}^1$ with prescribed ramification type at two points are given by μ,ν and C is of Euler characteristic χ and possibly disconnected.

$$\Phi^\bullet_{\mu,\nu}(\lambda) = \sum_\chi H^\bullet_\chi(\mu,\nu)\frac{\lambda^{-\chi+\ell(\nu)+\ell(\mu)}}{(-\chi+\ell(\nu)+\ell(\mu))!}$$

e.g. $\nu=(2,1,1) \Rightarrow p_\nu = p_1p_1p_2 = p_1^2p_2$

$e=(3,1,0) \Rightarrow q_e = q_0q_1q_3.$

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \lambda_2 u^{g-2} + \dots + (-1)^g\lambda_g$$

Recursion Formula

For any partition μ and e such that $|e| < |\mu| + \ell(\mu) - \chi$, we have a convolution formula

$$[\lambda^{\ell(\mu)-\chi}] \sum_{|\nu|=|\mu|} \Phi^{\bullet}_{\mu,\nu}(-\lambda)\, z_\nu\, D^{\bullet}_{\nu,e}(\lambda) = 0$$

where the sum is taken over all partitions ν of size $|\mu|$ and $[\lambda^a]$ means taking the coefficient of λ^a in the following formula.

Idea of proof

Apply virtual localization formula to $\omega = \prod_{j=1}^{n} \psi_j^{k_j} ev_j^* H$ with prescribed values on H, $H(0)=0$, $H(\infty)=u$, to give

$$0 = \int_{[\overline{\mathcal{M}}^{\bullet}_{\chi_n}(\mathbb{P}^1,\mu)]^{vir}} \prod \psi_j^{k_j} ev_j^* H \quad \text{since } \deg\omega < \dim[\overline{\mathcal{M}}^{\bullet}_{\chi_n}(\mathbb{P}^1,\mu)]^{vir}$$

$$= \sum_{\Gamma_0} \frac{1}{|A_{\Gamma_0}|} \int_{[\overline{\mathcal{M}}_{\Gamma_0}]^{vir}} \frac{\prod (u-\psi_j)^{k_j} ev_j^* H_T}{e_T(N_{\Gamma_0}^{vir})} + \sum_{\Gamma} \frac{1}{|A_\Gamma|} \int_{[\overline{\mathcal{M}}_{\Gamma}]^{vir}} \frac{\prod (u-\psi_j)^{k_j} ev_j^* H_T}{e_T(N_{\Gamma}^{vir})}$$

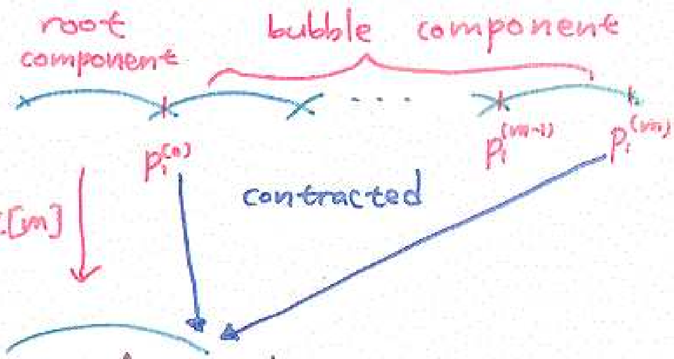
Can be decomposed into contributions of Hodge integrals and Double Hurwitz numbers.

$$H^{\bullet}_{\chi}(\mu,\nu) = \frac{1}{|\mathrm{Aut}\,\mu|\,|\mathrm{Aut}\,\nu|} \int_{[\overline{\mathcal{M}}^{\bullet}_{\chi}(\mu,\nu)]^{vir}} Br^*(H^{-\chi+\ell(\mu)+\ell(\nu)})$$

Moduli Space of relative morphisms

For any non-negative integer m, let $\mathbb{P}^1[m]$ be a chain of $m+1$ copies of $\mathbb{P}^1$ such that $\mathbb{P}^1_{(l)}$ is glued to $\mathbb{P}^1_{(l+1)}$ at $p_1^{(l)}$ for $0 \le l \le m-1$. Points $p_1^{(l)} \ne p_1^{(l+1)}$ are fixed in $\mathbb{P}^1_{(l)}$. Denote by $\pi[m]$ the contraction map and $\mathbb{P}^1(m) = \mathbb{P}^1_{(1)} \cup \cdots \cup \mathbb{P}^1_{(m)}$ the union of bubble components.

$$\mathbb{P}^1[m] = \mathbb{P}^1_{(0)} \cup \mathbb{P}^1_{(1)} \cup \cdots \cup \mathbb{P}^1_{(m)}$$



For a fixed partition μ, let $\overline{\mathcal{M}}^{\circ}_{\chi,n}(\mathbb{P}^1,\mu)$ be the moduli space of relative morphisms

$$f : (C, x_1, \cdots, x_{\ell(\mu)}, z_1, \cdots, z_n) \longrightarrow (\mathbb{P}^1[m], p_1^{(m)})$$

such that

- $(C, x_1, \cdots, x_{\ell(\mu)}, z_1, \cdots, z_n)$ is a possibly-disconnected prestable curve of Euler characteristic χ with $\ell(\mu)+n$ marked points.
- $f^{-1}(p_1^{(m)}) = \sum_{i=1}^{\ell(\mu)} \mu_i x_i$ as Cartier divisors and $\deg(\pi[m] \circ f) = |\mu|$.
- The preimage of each node in $\mathbb{P}^1[m]$ consists of nodes of C with same contact order.

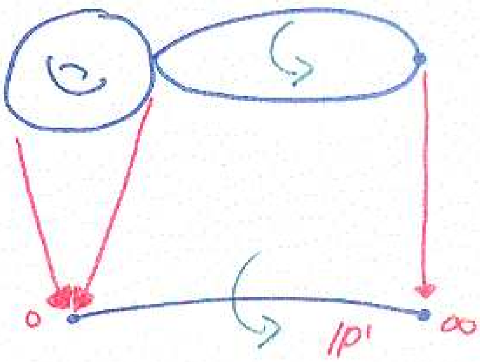
- The automorphism group of f is finite.
  Automorphism of f consists of an automorphism of the domain curve and automorphism of the pointed curve $(\mathbb{P}^1(m), p_1^{(0)}, p_1^{(m)})$

$\overline{\mathcal{M}}^{\bullet}_{\chi,n}(\mathbb{P}^1,\mu)$ is a separated, proper Deligne-Mumford stack with a perfect obstruction theory of virtual dimension $r=-\chi+|\mu|+\ell(\mu)+n$, and hence has a virtual fundamental class of degree r.

We can introduce $\mathbb{C}^*$-action on $\overline{\mathcal{M}}^{\bullet}_{\chi,n}(\mathbb{P}^1,\mu)$ from the natural $\mathbb{C}^*$-action on $\mathbb{P}^1$;

$$t\cdot[z:w]=[tz:w]$$

by moving the target $\mathbb{P}^1$.



Equipped with this $\mathbb{C}^*$-action, we can apply localization formula to any equivariant class ω;

$$\int_{[\overline{\mathcal{M}}^{\bullet}_{\chi,n}(\mathbb{P}^1,\mu)]^{vir}}\omega=\sum_{F_\Gamma}\int_{[F_\Gamma]^{vir}}\frac{i^*_{F_\Gamma}(\omega)}{e_T(N^{vir}_{F_\Gamma})}$$

Virtual Localization Formula.

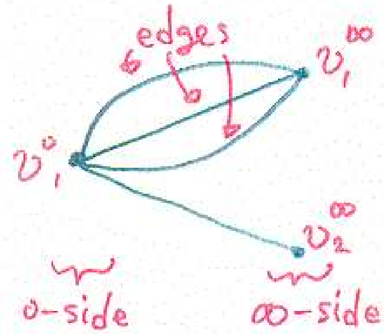
where F_Γ's are fixed components of the $\mathbb{C}^*$-action on $\overline{\mathcal{M}}^{\bullet}_{\chi,n}(\mathbb{P}^1,\mu)$ and $i_{F_\Gamma}:F_\Gamma\hookrightarrow\overline{\mathcal{M}}^{\bullet}_{\chi,n}(\mathbb{P}^1,\mu)$ is the inclusion map. Also $N^{vir}_{F_\Gamma}$ is the virtual normal bundle of F_Γ in $\overline{\mathcal{M}}^{\bullet}_{\chi,n}(\mathbb{P}^1,\mu)$.

Fixed locus and Graph notation

Each fixed locus can be associated to a graph as follows: Let $f : (C, x_1, \cdots, x_{\ell(\mu)}, z_1, \cdots, z_n) \longrightarrow (\mathbb{P}^1[m], p_i^{(m)})$ be a representative of the fixed locus F_Γ and consider a graph Γ which is constructed as follows:

Vertex v $\longleftrightarrow$ Connected component of $\tilde{f}^{-1}(\{0,\infty\})$

Edge e $\longleftrightarrow$ Rational irreducible component of C



Each vertex v is equipped with ;

$g(v)$: genus of the component contracted to v

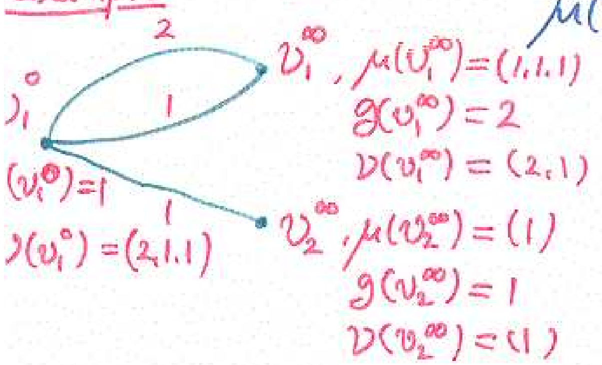
$\nu(v)$: partition determined by edges attached to v

$j(v)$: number of marked points $\{z_1, \cdots, z_n\}$

In case $m>0$ and v is on ∞-side ;

$\mu(v)$: partition determined by the ramification of $f : C_v \to \mathbb{P}^1(m)$ over $p_i^{(m)}$

Example



Then define

$$\begin{cases} \overline{\mathcal{M}}_{\Gamma_0} = \prod_{1\le i\le k} \overline{\mathcal{M}}_{g_i,\, \ell(\mu(v_i^0))+j(v_i^0)}, & m=0 \\ \overline{\mathcal{M}}_{\Gamma} = \overline{\mathcal{M}}_{\Gamma}^{(0)} \times \overline{\mathcal{M}}_{\Gamma}^{(1)}, & m>0 \end{cases}$$

With this setting, $\overline{\mathcal{M}}_\Gamma / A_\Gamma \cong F_\Gamma$ where A_Γ is the automorphism group of any morphism associated to the graph Γ, which satisfies

$$|A_{\Gamma_0}| = [\prod_{i=1}^{\ell(\mu)} \mu_i] \times \prod_{k=1}^{m} m_k! \, |j(v_k)! \, \mathrm{Aut}\, \mu(v_k)|^{m_k}$$

$$|A_\Gamma| = |\mathrm{Aut}\, \Gamma_0| \cdot |\mathrm{Aut}\, \mu| \left(\prod_{k=1}^{m} n_k!\right) \left(\prod_{v \in V(\Gamma)^{(\infty)}} j(v)! \, |\mathrm{Aut}\, \mu(v)| \, |\mathrm{Aut}\, \nu(v)|\right)$$

With this graph notation, we have

$$\int_{[\overline{M}^{\bullet}_{X,n}(\mathbb{P}^1,\mu)]^{vir}} \omega = \sum_{F_\Gamma} \int_{[F_\Gamma]^{vir}} \frac{i_{F_\Gamma}^*(\omega)}{e_T(N_{F_\Gamma}^{vir})}$$

$$= \sum_{\Gamma} \frac{1}{|A_\Gamma|} \int_{[\overline{M}_\Gamma]^{vir}} \frac{i_\Gamma^*(\omega)}{e_T(N_\Gamma^{vir})}$$

Hence, if we know how to find all graphs corresponding to fixed locus and how to compute the Euler class of equivariant Normal bundle, we can integrate any equivariant class over the moduli space of relative morphisms.

Finding all graphs are straightforward in this case:

For each partition ν, there can be $1, 2, \cdots$ $\cdots \ell(\nu)$ vertices on 0-side and $1, 2, \cdots, \min\{\ell(\mu), \ell(\nu)\}$ vertices on ∞-side. So it amounts to finding all groupings of ν and μ as well as $z_1, \cdots, z_n$. Genus on each vertex can be assigned independently with one condition that $\sum g(v_i^0) + \sum g(v_j^\infty)$

$$= m + \overline{m} - \ell(\nu) - \frac{\chi}{2}.$$

Computation of $e_T(\mathcal{N}_{\Gamma}^{vir})$

For a fixed component associated to a graph Γ, the equivariant Euler class of the virtual normal bundle $\mathcal{N}_{\Gamma}^{vir}$ can be computed as:

$$e_T(\mathcal{N}_{\Gamma}^{vir}) = e_T(\hat{T}^1) / e_T(\hat{T}^2)$$

where T^1, T^2 are the tangent space and the obstruction space of $\bar{\mathcal{M}}^{\bullet}_{\chi,n}(\mathbb{P}^1,\mu)$, and $\hat{T}^1, \hat{T}^2$ means the moving parts of T^1, T^2, respectively.

Informations of T^1, T^2 are obtained through the following two exact sequences:

$$0 \longrightarrow \mathrm{Ext}^0(\Omega_C(D), \mathcal{O}_C) \longrightarrow H^0(\mathbf{D}^\bullet) \longrightarrow T^1$$
$$\longrightarrow \mathrm{Ext}^1(\Omega_C(D), \mathcal{O}_C) \longrightarrow H^1(\mathbf{D}^\bullet) \longrightarrow T^2 \longrightarrow 0$$

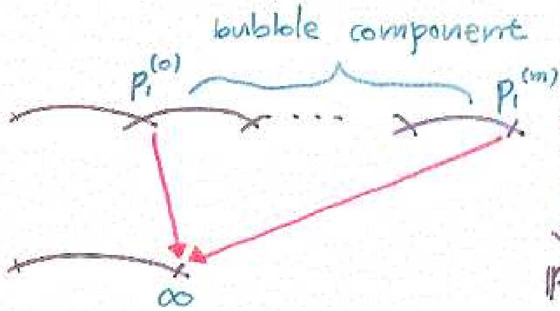
, and

$$0 \longrightarrow H^0(C, f^*(\omega_{\mathbb{P}^1[m]}(\log p_1^{(m)}))^\vee) \longrightarrow H^0(\mathbf{D}^\bullet)$$
$$\longrightarrow \bigoplus_{l=0}^{m-1} H^0_{et}(\mathbf{R}^\bullet_l) \longrightarrow H^1(C, f^*(\omega_{\mathbb{P}^1[m]}(\log p_1^{(m)}))^\vee)$$
$$\longrightarrow H^1(\mathbf{D}^\bullet) \longrightarrow \bigoplus_{l=0}^{m-1} H^1_{et}(\mathbf{R}^\bullet_l) \longrightarrow 0$$

[J. Li] "Relative Gromov-Witten invariants and a degeneration formula of Gromov-Witten invariants".

Here, $\omega_{\mathbb{P}^1[m]}$ is the dualizing sheaf of $\mathbb{P}^1[m]$ and $D = x_1 + \cdots + x_{\ell(\mu)}$ is the branch divisor.

By contraction of bubble components $\mathbb{P}^1(m)$ of $\mathbb{P}^1[m]$ to the point at infinity $\pi[m] : \mathbb{P}^1[m] \longrightarrow \mathbb{P}^1$



$$\begin{array}{c} \mathbb{P}^1[m] \\ \downarrow \pi[m] \\ \mathbb{P}^1 \end{array}$$

, we have

$$f^*(\omega_{\mathbb{P}^1[m]}(\log p_1^{(m)}))^\vee \cong \tilde{f}^* \mathcal{O}_{\mathbb{P}^1}(1)$$

where $\tilde{f} = \pi[m] \circ f$.

Then the $\mathbb{C}^*$-actions on $\overline{\mathcal{M}}^\bullet_{\chi,n}(\mathbb{P}^1, \mu)$ and $\mathbb{P}^1$ induces $\mathbb{C}^*$-actions on the following groups;

$$\mathrm{Ext}^0(\Omega_C(D), \mathcal{O}_C), \quad H^0(C, \tilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1)), \quad \bigoplus_{\ell=0}^{m-1} H^0_{et}(\boldsymbol{R}^\bullet_\ell)$$

$$\mathrm{Ext}^1(\Omega_C(D), \mathcal{O}_C), \quad H^1(C, \tilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1)), \quad \bigoplus_{\ell=0}^{m-1} H^1_{et}(\boldsymbol{R}^\bullet_\ell)$$

The moving parts of each of these groups form vector bundles over $\overline{\mathcal{M}}_\Gamma$, and in this case we have

$$\frac{1}{e_T(N^{vir}_\Gamma)} = \frac{e_T(\hat{\boldsymbol{T}}^2)}{e_T(\hat{\boldsymbol{T}}^1)}$$

$$= \frac{e_T(\widehat{\mathrm{Ext}^0(\Omega_C(D), \mathcal{O}_C)})\, e_T(\widehat{H^1(C, \tilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1))})\, e_T(\widehat{\bigoplus_{\ell=0}^{m-1} H^1_{et}(\boldsymbol{R}^\bullet_\ell)})}{e_T(\widehat{H^0(C, \tilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1))})\, e_T(\widehat{\mathrm{Ext}^1(\Omega_C(D), \mathcal{O}_C)})}$$

Now their contributions can be expressed in terms of representations of $\mathbb{C}^*$-actions.

We can compute their contributions in case of Γ_0 to be;

$$e_T\big(\widehat{\bigoplus_{l=0}^{m-1} H^1_{et}(\mathbf{R}_l^\bullet)_v}\big) = 1$$

$$e_T\big(\mathrm{Ext}^0(\widehat{\Omega_C(D), \mathcal{O}_C})_v\big) = \begin{cases} \frac{u}{\mu_{v,1}}, & \text{if } v \in \mathrm{I} \\ 1, & \text{if } v \in \mathrm{II} \text{ or } \mathrm{S} \end{cases}$$

$$e_T\big(\mathrm{Ext}^1(\widehat{\Omega_C(D), \mathcal{O}_C})_v\big) = \begin{cases} 1, & \text{if } v \in \mathrm{I} \\ \frac{u}{\mu_{v,1}} + \frac{u}{\mu_{v,2}}, & \text{if } v \in \mathrm{II}_1 \\ \frac{u}{\mu_{v,1}}, & \text{if } v \in \mathrm{II}_2 \\ \prod_{i=1}^{l(\mu(v))}\left(\frac{u}{\mu_{v,i}} - \psi_{v,i}\right), & \text{if } v \in \mathrm{S} \end{cases}$$

For the contributions from the rest, consider the normalization sequence when $v \in \mathrm{S}$;

$$0 \to \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1) \to \big(\hat{f}\mid_{C_v}\big)^*\mathcal{O}_{\mathbb{P}^1}(1) \oplus \bigoplus_{i=1}^{l(\mu(v))} \big(\hat{f}\mid_{C_{e_{v,i}}}\big)^*\mathcal{O}_{\mathbb{P}^1}(1) \to \bigoplus_{i=1}^{l(\mu(v))} \mathcal{O}_{\mathbb{P}^1}(1)_{p_0} \to 0$$

The corresponding long exact sequence becomes

$$\begin{aligned} 0 \longrightarrow & H^0\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \longrightarrow H^0\big(C_v, (\hat{f}\mid_{C_v})^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \oplus \bigoplus_{i=1}^{l(\mu(v))} H^0\big(C_{e_{v,i}}, (\hat{f}\mid_{C_{e_{v,i}}})^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \\ \longrightarrow & \bigoplus_{i=1}^{l(\mu(v))} \mathcal{O}_{\mathbb{P}^1}(1)_{p_0} \longrightarrow H^1\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \\ \longrightarrow & H^1\big(C_v, (\hat{f}\mid_{C_v})^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \oplus \bigoplus_{i=1}^{l(\mu(v))} H^1\big(C_{e_{v,i}}, (\hat{f}\mid_{C_{e_{v,i}}})^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \longrightarrow 0 \end{aligned}$$

and the representations of $\mathbb{C}^*$ are given by

$$\begin{aligned} 0 \to & H^0\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \to H^0\big(C_v, \mathcal{O}_{C_v}\big) \otimes (1) \oplus \bigoplus_{i=1}^{l(\mu(v))} \Big(\bigoplus_{a=1}^{\mu_{v,i}} \big(\frac{a}{\mu_{v,i}}\big)\Big) \to \\ & \bigoplus_{i=1}^{l(\mu(v))} (1) \to H^1\big(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1)\big) \to H^1\big(C_v, \mathcal{O}_{C_v}\big) \otimes (1) \to 0 \end{aligned}$$

Hence their ratio can computed as;

$$\frac{e_T\big(\widehat{H^1(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1))}\big)}{e_T\big(\widehat{H^0(C, \hat{f}^*\mathcal{O}_{\mathbb{P}^1}(1))}\big)} = \prod_v \Big[\Lambda^\vee_{g(v)}(u) u^{l(\mu(v))-1} \prod_{i=1}^{l(\mu(v))} \big(\frac{\mu_{v,i}^{\mu_{v,i}}}{\mu_{v,i}!} u^{-\mu_{v,i}}\big)\Big]$$

where $\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g$ is the dual Hodge bundle.

Combining all contributions gives

$$\frac{1}{e_T(N_{\Gamma_0}^{\mathrm{vir}})} = \left[\prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} u^{-\mu_i}\right] \times \left[\prod_{v\in \mathrm{I}} \frac{u}{\mu_{v,1}}\right]$$

$$\times \left[\prod_{v\in \mathrm{II}_1} \frac{u}{\frac{u}{\mu_{v,1}} + \frac{u}{\mu_{v,2}}}\right] \times \left[\prod_{v\in \mathrm{II}_2} \frac{u}{u/\mu_{v,1}}\right]$$

$$\times \left[\prod_{v\in S} \frac{\Lambda^\vee_{g(v)}(u)}{u} \left(\prod_{i=1}^{\ell(\mu(v))} \frac{u}{\frac{u}{\mu_{v,i}} - \psi_{v,i}}\right)\right]$$

Similar computations for the case of Γ, $m>0$ gives

$$\frac{1}{e_T(N_{\Gamma}^{\mathrm{vir}})} = \left[\prod_{i=1}^{\ell(\nu)} \frac{\nu_i^{\nu_i}}{\nu_i!} u^{-\nu_i}\right] \times \left[\prod_{v\in \mathrm{I}} \frac{u}{\nu_{v,1}}\right]$$

0-side of Γ

$$\times \left[\prod_{v\in \mathrm{II}_1} \frac{u}{\frac{u}{\nu_{v,1}} + \frac{u}{\nu_{v,2}}}\right] \times \left[\prod_{v\in \mathrm{II}_2} \frac{u}{u/\nu_{v,1}}\right]$$

$$\times \left[\prod_{v\in S} \frac{\Lambda^\vee_{g(v)}(u)}{u} \left(\prod_{i=1}^{\ell(\nu(v))} \frac{u}{\frac{u}{\nu_{v,i}} - \psi_{v,i}}\right)\right]$$

∞-side of Γ

$$\times \left[\frac{-\prod_{i=1}^{\ell(\nu)} \nu_i}{u + \psi^t}\right]$$

0.1. **Relating Double Hurwitz Numbers with Hodge Integrals.** We can extend the notion of $\mathbb{P}^1[m]$ to have bubble components on both directions. Denote by

$$\mathbb{P}^1[m_0, m_\infty] = \mathbb{P}^1_{(-m_0)} \cup \cdots \cup \mathbb{P}^1_{(-1)} \cup \mathbb{P}^1_{(0)} \cup \mathbb{P}^1_{(1)} \cup \cdots \cup \mathbb{P}^1_{(m_\infty)}$$

As before, we call $\mathbb{P}^1_{(0)}$ the *root component* , $\mathbb{P}^1_0(m_0) = \mathbb{P}^1_{(-m_0)} \cup \cdots \cup \mathbb{P}^1_{(-1)}$ the *bubble component at* 0, and $\mathbb{P}^1_\infty(m_\infty) = \mathbb{P}^1_{(1)} \cup \cdots \cup \mathbb{P}^1_{(m_\infty)}$ the *bubble component at* ∞. Define $\overline{\mathcal{M}}^\bullet_{\chi,n}(\mu^0, \mu^\infty)$ as the moduli space of morphisms $f : C \longrightarrow \mathbb{P}^1[m_0, m_\infty]$ such that

(1) $(C; x_1, \cdots, x_{l(\mu^\infty)}, y_1, \cdots, y_{l(\mu^0)})$ is a possibly-disconnected prestable curve with $l(\mu^0)+l(\mu^\infty)$ unordered marked points.
(2) $\chi = \sum_i (2 - 2g_i)$ where g_i is the genus of each connected component of C.
(3) As Cartier divisors, $f^{-1}(p_0^{(-m_0)}) = \sum_{j=1}^{l(\mu^0)} \mu^0_j y_j$, $\quad f^{-1}(p_1^{(m_\infty)}) = \sum_{i=1}^{l(\mu^\infty)} \mu^\infty_i x_i$
(4) The automorphism group of f is finite. Here, an automorphism of f consists of an automorphism of the domain curve C and automorphisms of the pointed curves $(\mathbb{P}^1_0(m_0), p_0^{(-m_0)}, p_1^{(-1)})$ and $(\mathbb{P}^1_\infty(m_\infty), p_0^{(1)}, p_1^{(m_\infty)})$, which is an element of $(\mathbb{C}^*)^{m_0}$ and $(\mathbb{C}^*)^{m_\infty}$, respectively.

We can extend the standard action $t \cdot [z, w] = [tz, w]$ on $\mathbb{P}^1$ to $\mathbb{P}^1[m_0, m_\infty]$ by trivial action on the bubble components at 0 and ∞. Then this action induces $\mathbb{C}^*$-action on $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^0, \mu^\infty)$. Let $\pi[m_0, m_\infty] : \mathbb{P}^1[m_0, m_\infty] \longrightarrow \mathbb{P}^1$ be the projection which contracts both bubble components and $f_r = \pi[m_0, m_\infty] \circ f$. Denote by ν the partition of $d = |\mu^0| = |\mu^\infty|$ by the degrees of f_r on each rational irreducible components. For any morphism $f : (C, x_i, y_j) \longrightarrow \mathbb{P}^1[m_0, m_\infty]$ which represents a fixed point of $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^0, \mu^\infty)$ under this action, one of the following four cases must hold:

- $\underline{m_0 = m_\infty = 0}$: We have $f = f_r$, $\mu^0 = \mu^\infty = \nu$
- $\underline{m_0 = 0, m_\infty > 0}$: Let $\chi_\infty = \sum(2-2g_i^\infty)$ where g_i^∞ is the genus of each connected component of $f^{-1}(\mathbb{P}^1_\infty(m_\infty))$. In this case, we have $\mu^0 = \nu, \chi_\infty = \chi, \chi_0 = 2l(\nu)$
- $\underline{m_0 > 0, m_\infty = 0}$: Let $\chi_0 = \sum(2 - 2g_j^0)$ where g_j^0 is the genus of each connected component of $f^{-1}(\mathbb{P}^1_0(m_0))$. In this case, we have $\mu^\infty = \nu, \chi_0 = \chi, \chi_\infty = 2l(\nu)$.
- $\underline{m_0 > 0, m_\infty > 0}$: We have $\chi = \chi_0 + \chi_\infty - 2l(\nu)$.

Hence, we can see that χ_0, χ_∞, ν determines each connected component of $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^0, \mu^\infty)^{\mathbb{C}^*}$.

Consider the branch morphism

$$\mathrm{Br} : \overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1, \mu^0, \mu^\infty) \longrightarrow \mathrm{Sym}^{-\chi+l(\mu^0)+l(\mu^\infty)}\mathbb{P}^1 \cong \mathbb{P}^{-\chi+l(\mu^0)+l(\mu^\infty)}$$

The Double Hurwitz numbers for possibly disconnected covers of $\mathbb{P}^1$ can be defined by

$$H^\bullet_\chi(\mu^0, \mu^\infty) = \frac{1}{|\operatorname{Aut}(\mu^0)||\operatorname{Aut}(\mu^\infty)|} \int_{[\overline{\mathcal{M}}^\bullet_\chi(\mu^0,\mu^\infty)]^{vir}} \mathrm{Br}^*\left(H^{-\chi+l(\mu^0)+l(\mu^\infty)}\right)$$

under the assumption $-\chi + l(\mu^0) + l(\mu^\infty) > 0$ where $H \in H^2(\mathbb{P}^{-\chi+l(\mu^0)+l(\mu^\infty)})$ is the hyperplane class. We want to compute this integration by virtual localization. The connected components of

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$\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^0,\mu^\infty)^{\mathbb{C}^*}$ can be described as follows:

- $\underline{m_0=0, m_\infty>0}$: $\mathcal{F}(\nu;2l(\nu),\chi)\cong\Big(\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\nu,\mu^\infty)//\mathbb{C}^*\Big)/\coprod\mathbb{Z}_{\mu_i^\infty}$
- $\underline{m_0>0, m_\infty=0}$: $\mathcal{F}(\nu;\chi,2l(\nu))\cong\Big(\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^0,\nu)//\mathbb{C}^*\Big)/\coprod\mathbb{Z}_{\mu_j^0}$
- $\underline{m_0>0, m_\infty>0}$:

$$\mathcal{F}(\nu;\chi_0,\chi_\infty)\cong\Big(\big(\overline{\mathcal{M}}^\bullet_{\chi_0}(\mathbb{P}^1,\mu^0,\nu)//\mathbb{C}^*\big)\times\big(\overline{\mathcal{M}}^\bullet_{\chi_\infty}(\mathbb{P}^1,\nu,\mu^\infty)//\mathbb{C}^*\big)\Big)/(\mathrm{Aut}(\nu)\prod_{i=1}^{l(\nu)}\mathbb{Z}_{\nu_i})$$

Let $\mathcal{N}^{vir}_{\nu;\chi_0,\chi_\infty}$ be the pull-back of the virtual normal bundle of $\mathcal{F}(\nu,\chi_0,\chi_\infty)$ in $\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^0,\mu^\infty)$. By computations similar to those $e_T(\mathcal{N}^{vir}_\Gamma)$, we obtain

$$\frac{1}{e_T(\mathcal{N}^{vir}_{\nu;2l(\nu),\chi})}=\frac{-a_\nu}{u+\psi^0},\qquad \frac{1}{e_T(\mathcal{N}^{vir}_{\nu;\chi,2l(\nu)})}=\frac{a_\nu}{u-\psi^\infty},\qquad \frac{1}{e_T(\mathcal{N}^{vir}_{\nu;\chi_0,\chi_\infty})}=\frac{-A_\nu}{u+\psi^0}\times\frac{a_\nu}{u-\psi^\infty}$$

where ψ^0 and ψ^∞ are the first Chern classes of the cotangent line bundle $T^*_{p_0^{(0)}}\mathbb{P}^1[m_0,m_\infty]$ and $T^*_{p_1^{(0)}}\mathbb{P}^1[m_0,m_\infty]$, respectively. Let $r=-\chi+l(\mu^0)+l(\mu^\infty)$ and observe that

$$\mathrm{Br}(\mathcal{F}(\nu;\chi_0,\chi_\infty))=(-\chi_\infty+l(\nu)+l(\mu^\infty))p_1+(-\chi_0+l(\mu^0)+l(\nu))p_0\quad\in\ \mathbb{P}^r$$

$$i^*_{\nu,\chi_0,\chi_\infty}\mathrm{Br}^*\Big(\prod_{k=1}^r(H-w_k)\Big)=\Big(\prod_{k=1}^r(-\chi_0+l(\mu^0)+l(\nu)-w_k)\Big)u^r$$

By taking special values for w:

$$w=(0,1,\cdots,-\chi+l(\mu^0)+l(\mu^\infty)-1)$$

, and applying virtual functorial localization formula gives the following relation

$$\frac{H^\bullet_\chi(\mu^0,\mu^\infty)}{(-\chi+l(\mu^0)+l(\mu^\infty))!}=\frac{1}{|\mathrm{Aut}(\mu^0)||\mathrm{Aut}(\mu^\infty)|}\int_{[\overline{\mathcal{M}}^\bullet_\chi(\mathbb{P}^1,\mu^0,\mu^\infty)//\mathbb{C}^*]^{vir}}(\psi^0)^{-\chi+l(\mu^0)+l(\mu^\infty)-1}$$

After counting multiplicities, it is related to the integral over $[\overline{\mathcal{M}}^{(1)}_\Gamma]^{vir}$ as follows:

$$\Big(\prod n_k!\Big)\Big(\prod_{V(\Gamma)^{(1)}}|\mathrm{Aut}\ \mu(v)|\ |\mathrm{Aut}\ \nu(v)|\Big)\int_{[\overline{\mathcal{M}}^\bullet_{\chi_\infty}(\mathbb{P}^1,\mu,\nu)//\mathbb{C}^*]^{vir}}(\psi^0)^{-\chi_\infty+l(\mu)+l(\nu)-1}$$
$$=\int_{[\overline{\mathcal{M}}^{(1)}_\Gamma]^{vir}}(\psi^t)^{-\chi_\infty+l(\mu)+l(\nu)-1}$$

$$0=\sum_{\Gamma_0\in\overline{G}^0_{\chi,n}(\mathbb{P}^1,\mu)}\frac{1}{|A_{\Gamma_0}|}\int_{\overline{\mathcal{M}}_{\Gamma_0}}\frac{\prod_{j=1}^n(1-\psi_j)^{k_j}}{e_T(\mathcal{N}^{vir}_{\Gamma_0})}+\sum_{\Gamma\in\overline{G}^\infty_{\chi,n}(\mathbb{P}^1,\mu)}\frac{1}{|A_\Gamma|}\int_{\overline{\mathcal{M}}_\Gamma}\frac{\prod_{j=1}^n(1-\psi_j)^{k_j}}{e_T(\mathcal{N}^{vir}_\Gamma)}$$

$$\begin{aligned}&=\sum_{\Gamma_0\in\overline{G}^0_{\chi,n}(\mathbb{P}^1,\mu)}\frac{1}{|V(\Gamma_0)|!}\binom{|V(\Gamma_0)|}{m_1,\cdots,m_l}\prod_k\mathcal{D}_{g(v_k),\mu(v_k),e(v_k)}\\&\quad+\sum_{\Gamma\in\overline{G}^\infty_{\chi,n}(\mathbb{P}^1,\mu)}\Big[\frac{1}{|V(\Gamma)^{(0)}|!}\binom{|V(\Gamma)^{(0)}|}{m_1,\cdots,m_l}\prod_k\mathcal{D}_{g(v_k),\nu(v_k),e(v_k)}\\&\quad\times\frac{(-1)^{-\chi_\infty+l(\mu)+l(\nu)}\prod_{i=1}^{l(\nu)}\nu_i}{|\mathrm{Aut}\,\mu|\left(\prod n_k!\right)\left(\prod_{V(\Gamma)^{(1)}}|\mathrm{Aut}\,\mu(v)|\,|\mathrm{Aut}\,\nu(v)|\right)}\int_{[\overline{\mathcal{M}}^{(1)}_\Gamma]^{vir}}(\psi^0)^{-\chi_\infty+l(\mu)+l(\nu)-1}\Big]\end{aligned}$$

term in the expansion $\mathcal{D}^\bullet_{\chi,\mu,e}=\exp\left(\sum\mathcal{D}_{g,\mu,e}\right)$

$$\begin{aligned}&=\sum_{\Gamma_0\in\overline{G}^0_{\chi,n}(\mathbb{P}^1,\mu)}\frac{1}{|V(\Gamma_0)|!}\binom{|V(\Gamma_0)|}{m_1,\cdots,m_l}\prod_k\mathcal{D}_{g(v_k),\mu(v_k),e(v_k)}\\&\quad+\sum_{\Gamma\in\overline{G}^\infty_{\chi,n}(\mathbb{P}^1,\mu)}\Big[\frac{1}{|V(\Gamma)^{(0)}|!}\binom{|V(\Gamma)^{(0)}|}{m_1,\cdots,m_l}\prod_k\mathcal{D}_{g(v_k),\nu(v_k),e(v_k)}\\&\quad\times\frac{(-1)^{-\chi_\infty+l(\mu)+l(\nu)}z_\nu}{|\mathrm{Aut}\,\mu|\,|\mathrm{Aut}\,\nu|}\int_{[\overline{\mathcal{M}}^\bullet_{\chi_\infty}(\mathbb{P}^1,\mu,\nu)//\mathbb{C}^*]^{vir}}(\psi^0)^{-\chi_\infty+l(\mu)+l(\nu)-1}\Big]\end{aligned}$$

related to double Hurwitz number $H^\bullet_{\chi_\infty}(\mu,\nu)$

$$=\mathcal{D}^\bullet_{\chi,\mu,e}+\sum_\nu\sum_{-\chi_\infty+l(\mu)+l(\nu)\neq0}\frac{(-1)^{-\chi_\infty+l(\mu)+l(\nu)}H^\bullet_{\chi_\infty}(\mu,\nu)}{(-\chi_\infty+l(\mu)+l(\nu))!}z_\nu\mathcal{D}^\bullet_{\chi_0,\nu,e}$$

Initial value of Hurwitz number $H^\bullet_{\mu,\nu}(g=0)=\frac{1}{z_\mu}\delta_{\mu,\nu}$

$$=\sum_\nu\sum_{\chi_0,\chi_\infty}\frac{(-1)^{-\chi_\infty+l(\mu)+l(\nu)}H^\bullet_{\chi_\infty}(\mu,\nu)}{(-\chi_\infty+l(\mu)+l(\nu))!}z_\nu\mathcal{D}^\bullet_{\chi_0,\nu,e}$$

$-\chi_\infty+l(\mu)+l(\nu)=0.$

$$=[\lambda^{l(\mu)-\chi}]\sum_{|\nu|=|\mu|}\Phi^\bullet_{\mu,\nu}(-\lambda)z_\nu\mathcal{D}^\bullet_{\nu,e}(\lambda)$$

$\leftarrow$ holds for all μ, e and χ such that $|e|<-\chi+|\mu|+l(\mu)$

Computing All Hodge Integrals with one λ-class

As an application of the recursion formula

$$[\lambda^{\ell(\mu)-\chi}] \sum_{|\nu|=|\mu|} \Phi^{\bullet}_{\mu,\nu}(-\lambda)\, z_\nu\, D^{\bullet}_{\nu,e}(\lambda) = 0$$

to the cases $\mu=(d)$, $d\in\mathbb{N}$, we can prove the following result.

Any given Hodge integral with one λ-class

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_j$$

where $k_1,\cdots,k_n \geq 0$, $0\leq j\leq g$, $\sum k_i + j = 3g-3+n$.
is expressed as a polynomial in terms of lower-dimensional Hodge integrals with one λ-class.

Therefore it gives an algorithm to compute all Hodge integrals with one λ-class.

Since $\lambda_0=1$, it also computes all Hodge integrals involving ψ-classes only.

This follows from the following observations :

- By choosing $\chi = 2-2g$ and $e = (k_1, \cdots, k_{n-1})$, for any $d \in \mathbb{N}$ such that $d > |e| + \chi + 1$ the recursion relation gives a linear relation between Hodge integrals with one λ-class which includes the given type of Hodge integral and those integrals have dimension $3g-3+n$.

- Since there are infinitely many relations between finitely many Hodge integrals, it can be proved that there are enough linearly independent relations to compute those Hodge integrals

- Repeat this process on each dimension to compute all Hodge integrals with one λ-class.

Double Hurwitz numbers for $\mu = (d)$

In case μ has only one part, i.e. $\mu = (d)$ for some $d \in \mathbb{N}$. the Double Hurwitz numbers $H^{\bullet}_{(d),\nu}(g)$ have closed formula

$$H^{g}_{(d),\nu} = \frac{r!}{|\mathrm{Aut}\,\nu|} d^{r-1} [t^{2g}] \prod_{k\geq 1} \left(\frac{\sinh(kt/2)}{kt/2} \right)^{c_k}$$

[I.P. Goulden, D.M. Jackson, R. Vakil "Towards the Geometry of Double Hurwitz numbers"

where $\begin{cases} r = 2g - 1 + \ell(\nu) & \text{and} \\ c_1 = \#\text{ of } 1\text{'s in } \nu - 1 & \\ c_k = \#\text{ of } k\text{'s in } \nu & \text{for } k > 1 \end{cases}$

and in this case, the recursion formula becomes

$$\sum_{|\nu|=d} \sum_{\chi_0, \chi_\infty} (-1)^{-\chi_\infty + \ell(\nu)+1} \frac{Z_\nu}{|\mathrm{Aut}\,\nu|} d^{-\chi_\infty + \ell(\nu)} [t^{2g}] \prod_{k\geq 1} \left(\frac{\sinh(kt/2)}{kt/2} \right)^{c_k} D^{\bullet}_{\chi_0,\nu,e} = 0$$

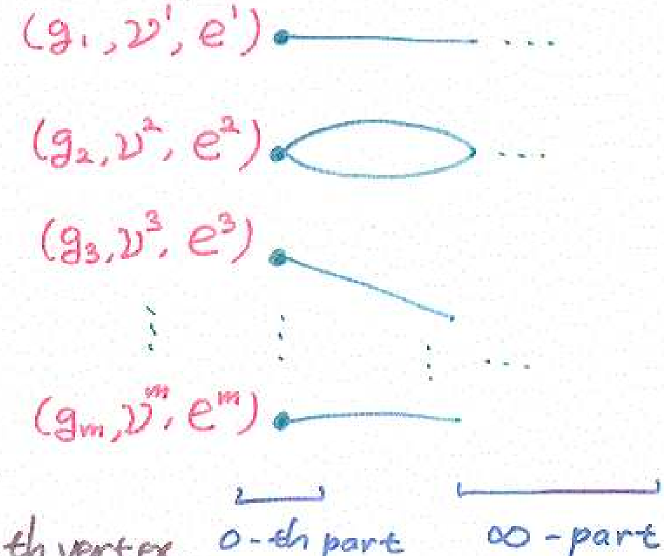
Now $D^{\bullet}_{\chi_0,\nu,e}$ will give certain Hodge integrals. Since

$$D^{\bullet}_{\chi_0,\nu,e} = \sum_{\text{splittings}} \frac{1}{m!} \binom{m}{m_1, \cdots, m_\ell} \prod_{k=1}^{m} C_k \int_{\overline{\mathcal{M}}_{g_k, \ell(\nu^k)+\ell(e^k)}} \frac{\Lambda_{g_k}(1)}{\prod_i (1-\psi_i)} \prod_j \psi_j^{e_j^k}$$

, it will be enough to show that highest dimension that can occur in $D^{\bullet}_{\chi_0,\nu,e}$ is $3g-3+n$ and their coefficients form invertible matrix.

• Upper bound of Hodge integral dimensions in the recursion formula.

For any given fixed graph, assume there are m vertices in the 0-th part. Then we have splittings of the total genus g, the ramification type ν, and the prescribed ψ-class exponent type e.



Denote, g_i, ν^i, e^i to be the genus, ramification type, and the ψ-class exponent type at i-th vertex.

The vertex v_i will contribute to the recursion relation on Hodge integral of dimension

$$(*) \cdots \dim \overline{\mathcal{M}}_{g_i, \ell(\nu^i)+\ell(e^i)} = 3g_i - 3 + \ell(\nu^i) + \ell(e^i)$$

$\{g_i\}$, $\{\nu^i\}$, $\{e^i\}$ satisfy following conditions

$$\sum g_i = g, \quad \bigcup \nu^i = \nu, \quad \bigcup e^i = e$$

$$\chi_0 = \sum (2 - 2g_i), \quad m \le \ell(\nu)$$

$$\chi_\infty \le 2 \cdot \min\{1, \ell(\nu)\} = 2, \quad \chi = \chi_0 + \chi_\infty - 2\ell(\nu)$$

From these conditions, we can find upper bound of (*)

$$3g_i - 3 + \ell(\nu^i) + \ell(e^i)$$

$$\leq 3g_i - 3 + \left[\ell(\nu) - (m-1)\right] + \left[\ell(e)\right]$$

$$= 3\cdot\underbrace{\left[m - \frac{\chi_0}{2} - \sum_{j\neq i} g_j\right]}_{\text{from } \chi_0 = \sum(2-2g_i) = 2m - 2g_i - 2\sum_{j\neq i} g_j} + \ell(\nu) + \ell(e) - m + 1 - 3$$

$$\leq -\frac{3\chi_0}{2} + 2m - 2 + \ell(\nu) + \ell(e) \quad \text{since } \sum_{j\neq i} g_j \geq 0$$

$$= -\frac{3}{2}\underbrace{\left[2\ell(\nu) + \chi - \chi_\infty\right]}_{\text{from } \chi = \chi_0 + \chi_\infty - 2\ell(\nu)} + 2m - 2 + \ell(\nu) + \ell(e)$$

$$= -\frac{3}{2}\chi + \ell(e) + 2\underbrace{(m - \ell(\nu))}_{\leq 0} + \frac{3}{2}\underbrace{(\chi_\infty - 2)}_{\leq 0 \text{ since } \chi_\infty \leq 2} + 1$$

$$\leq -\frac{3}{2}\chi + \ell(e) + 1 = \underbrace{3g - 3 + \ell(e) + 1}_{= \dim \overline{\mathcal{M}}_{g,\, \ell(\nu) + \ell(e)}}$$

So it attains maximum dimension if and only if

$\ell(\nu^i) = \ell(\nu) - (m-1) \Leftrightarrow v_i$ contains maximum # of parts of ν

$\ell(e^i) = \ell(e) \Leftrightarrow v_i$ contains all extra marked points

$\sum_{j\neq i} g_j = 0,\ \chi_\infty = 2 \Leftrightarrow$ All genus are concentrated on v_i

$m = \ell(\nu) \Leftrightarrow$ The graph has maximum # of vertices

Hence the Recursion formula gives linear relation

$$\sum_{k+\sum a_j = 3g-3+n} C_d(k,(a_j)) \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \lambda_k$$

$=$ polynomial in terms of lower-dim'l Hodge integrals.

where $C_d(k,(a_j))$ are defined as

$$C_d(k,(a_j)) = \sum_{|\nu|=d} \left[\left(\prod_{i=1}^{\ell(\nu)} \frac{\nu_i^{\nu_i - 1}}{\nu_i !} \right) \frac{(-1)^{\ell(\nu)-1} d^{\ell(\nu)-2}}{(-1+\gamma)!} \sum_{i=1}^{\ell(\nu)} \frac{\nu_i^2}{|\mathrm{Aut}\,\hat{\nu}_i|} \left(\nu_i^{a_n} \prod_{j=1}^{n-1} \binom{k_j}{a_j} \right) \right]$$

- Since it gives linear relation for all $d > |e| + \chi - 1$, there are infinitely many linear relations between finitely many Hodge integrals
- Moreover, $C_d(k,(a_j))$'s form a Vandermonde-type matrix.

We can always find values of $d > |e| + \chi - 1$ which will solve the linear system, hence it solves for all Hodge integrals with one λ-class.

Examples

1) Computing $\int_{\overline{\mathcal{M}}_{1,1}} \psi^1$ and $\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1$

Let $\chi = 2-2g = 0$ and $e = \emptyset$.

Need two relations. From the condition

$|e| < -\chi + d + 1$, we can use $d = 1, 2$.

$\boxed{d=1}$ There's only one partition of size 1

$\nu = (1)$ and there are two pairs of χ_0, χ_∞

$(\chi_0, \chi_\infty) = (0,2), (2,0)$

- $\nu = (1)$ & $(\chi_0, \chi_\infty) = (0,2)$

$$\sim \quad \int_{\overline{\mathcal{M}}_{1,1}} \frac{1-\lambda_1}{1-\psi} = -\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 + \int_{\overline{\mathcal{M}}_{1,1}} \psi^1$$

- $\nu = (1)$ & $(\chi_0, \chi_\infty) = (2,0)$

This case, we have $H^{\bullet}_{(1),(1)}(g=1) = 0$.
, hence it doesn't contribute.

$$\Rightarrow \quad -\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 + \int_{\overline{\mathcal{M}}_{1,1}} \psi^1 = 0$$

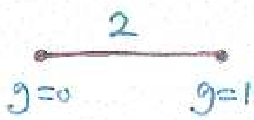
$\boxed{d=2}$ There are 2 partitions of size 2. $\underline{\nu=(2),(1,1)}$

- Case $\nu=(2)$. We have $(\chi_0,\chi_\infty)=(0,2),(2,0)$

$(\chi_0,\chi_\infty)=(0,2)$

$$\sim \quad \underbrace{\tfrac{1}{2}}_{H^\circ_{(2),(2)}(g=0)}\times \underbrace{2}_{Z_\nu}\times \underbrace{2\int_{\overline{\mathcal{M}}_{1,1}}\frac{1-\lambda_1}{1-2\psi}}_{D^\circ_\nu(g=1)\text{ with }\nu=(2)} = -2\int_{\overline{\mathcal{M}}_{1,1}}\lambda_1 + 4\int_{\overline{\mathcal{M}}_{1,1}}\psi'$$

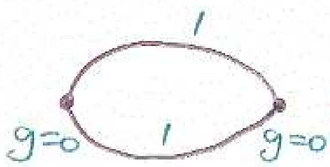
$(\chi_0,\chi_\infty)=(2,0)$



$$\sim \quad \underbrace{\tfrac{1}{4}}_{H^\circ_{(2),(2)}(g=1)}\times \underbrace{2}_{Z_\nu}\times \underbrace{2\int_{\overline{\mathcal{M}}_{0,1}}\frac{1}{1-2\psi}}_{D^\circ_\nu(g=0)\text{ with }\nu=(2)} = \frac{1}{4}$$

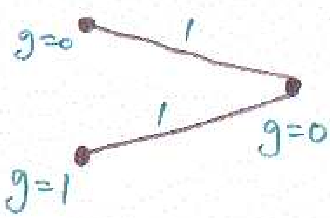
- Case $\nu=(1,1)$ We have $(\chi_0,\chi_\infty)=(2,2),(4,0)$

$(\chi_0,\chi_\infty)=(2,2)$ $\quad H^\circ_{(2),(1,1)}(g=0)=\frac{1}{2}$



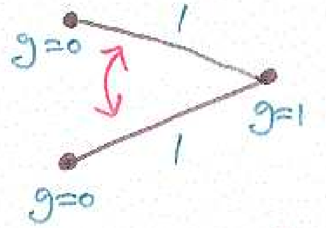
$$\sim \quad -\frac{1}{2}\times 2\times\frac{1}{2}\int_{\overline{\mathcal{M}}_{0,2}}\frac{1}{(1-\psi)(1-\psi)} = -\frac{1}{4}$$

$-$ sign comes from $(-1)^r=(-1)^{-\chi_\infty+\ell(\nu)+1}$



$$\sim \quad -\frac{1}{2}\times 2\times\int_{\overline{\mathcal{M}}_{0,1}}\frac{1}{1-\psi}\int_{\overline{\mathcal{M}}_{1,1}}\frac{1-\lambda_1}{1-\psi} = \int_{\overline{\mathcal{M}}_{1,1}}\lambda_1 - \int_{\overline{\mathcal{M}}_{1,1}}\psi$$

$(\chi_0, \chi_\infty) = (4, 0)$ $\qquad H^{\circ}_{(2),(1,1)}(g=1) = \frac{1}{12} \times 3!$



$$\sim \quad \frac{-1}{12} \times 2 \times \frac{1}{2}\left[\int_{\overline{\mathcal{M}}_{0,1}} \frac{1}{1-\psi}\right]^2 = -\frac{1}{12}$$

Comes from symmetry of the graph.

Summing up these contributions gives a relation

$$-\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 + 3\int_{\overline{\mathcal{M}}_{1,1}} \psi' = \frac{1}{12}$$

Hence we have two linear relations

$$\begin{cases} -\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 + \int_{\overline{\mathcal{M}}_{1,1}} \psi' = 0 & \text{from } d=1,\ \chi=0,\ e=\emptyset \\ -\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 + 3\int_{\overline{\mathcal{M}}_{1,1}} \psi' = \frac{1}{12} & \text{from } d=2,\ \chi=0,\ e=\emptyset \end{cases}$$

Solving these relations gives the values

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi' = \frac{1}{24} \quad , \quad \int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 = \frac{1}{24}$$

which recovers the exceptional case contribution and the λ_g-formula.

Dimension 2 : There are 5 Hodge integrals with one λ-class of dimension 2;

$$\int_{\overline{\mathcal{M}}_{0,5}} \psi^2, \qquad \int_{\overline{\mathcal{M}}_{0,5}} \psi^1\psi^1, \qquad \int_{\overline{\mathcal{M}}_{1,2}} \psi^1\lambda_1, \qquad \int_{\overline{\mathcal{M}}_{1,2}} \psi^2, \qquad \int_{\overline{\mathcal{M}}_{1,2}} \psi^1\psi^1$$

And these integrals can be computed as follows:

$d = 3, g = 0, e = (0, 0, 0, 0)$;

$$25\int_{\overline{\mathcal{M}}_{0,5}} \psi^2 - 25 = 0 \implies \int_{\overline{\mathcal{M}}_{0,5}} \psi^2 = 1$$

$d = 4, g = 0, e = (1, 0, 0, 0)$;

$$-10\int_{\overline{\mathcal{M}}_{0,5}} \psi^1\psi^1 + 20 = 0 \implies \int_{\overline{\mathcal{M}}_{0,5}} \psi^1\psi^1 = 2$$

$d = 1, g = 1, e = (0)$, and $d = 2, g = 1, e = (0)$;

$$-\int_{\overline{\mathcal{M}}_{1,2}} \psi^1\lambda_1 + \int_{\overline{\mathcal{M}}_{1,2}} \psi^2 = 0, \qquad -3\int_{\overline{\mathcal{M}}_{1,2}} \psi^1\lambda_1 + 7\int_{\overline{\mathcal{M}}_{1,2}} \psi^2 - \frac{1}{6} = 0$$

$$\implies \int_{\overline{\mathcal{M}}_{1,2}} \psi^1\lambda_1 = \int_{\overline{\mathcal{M}}_{1,2}} \psi^2 = \frac{1}{24}$$

$d = 2, g = 1, e = (1)$;

$$-3\int_{\overline{\mathcal{M}}_{1,2}} \psi^1\psi^1 + \frac{1}{8} = 0 \implies \int_{\overline{\mathcal{M}}_{1,2}} \psi^1\psi^1 = \frac{1}{24}$$

The first two values matches with the Witten's formula for the $g = 0$ case which says that

$$(1) \qquad \int_{\overline{\mathcal{M}}_{0,n}} \psi^{k_1}\cdots\psi^{k_n} = \binom{n-3}{k_1, \cdots, k_n}, \qquad \text{when } k_1 + \cdots + k_n = n - 3$$

Dimension 3 : There are 8 Hodge integrals with one λ-class of dimension 3;

$$\int_{\overline{\mathcal{M}}_{0,6}} \psi^3, \quad \int_{\overline{\mathcal{M}}_{0,6}} \psi^2\psi^1, \quad \int_{\overline{\mathcal{M}}_{0,6}} \psi^1\psi^1\psi^1, \quad \int_{\overline{\mathcal{M}}_{1,3}} \psi^2\lambda_1,$$
$$\int_{\overline{\mathcal{M}}_{1,3}} \psi^3, \quad \int_{\overline{\mathcal{M}}_{1,3}} \psi^1\psi^1\lambda_1, \quad \int_{\overline{\mathcal{M}}_{1,3}} \psi^2\psi^1, \quad \int_{\overline{\mathcal{M}}_{1,3}} \psi^1\psi^1\psi^1$$

And these integrals can be computed as follows:

$d=3, g=0, e=(0,0,0,0,0)$;

$$90\int_{\overline{\mathcal{M}}_{0,6}} \psi^3 - 90 = 0 \implies \int_{\overline{\mathcal{M}}_{0,6}} \psi^3 = 1$$

$d=4, g=0, e=(1,0,0,0,0)$;

$$-65\int_{\overline{\mathcal{M}}_{0,6}} \psi^2\psi^1 + 195 = 0 \implies \int_{\overline{\mathcal{M}}_{0,6}} \psi^2\psi^1 = 3$$

$d=5, g=0, e=(1,1,0,0,0)$;

$$15\int_{\overline{\mathcal{M}}_{0,6}} \psi^1\psi^1\psi^1 - 90 = 0 \implies \int_{\overline{\mathcal{M}}_{0,6}} \psi^1\psi^1\psi^1 = 6$$

$d=1, g=1, e=(0,0)$, and $d=2, g=1, e=(0,0)$;

$$-\int_{\overline{\mathcal{M}}_{1,3}} \psi^2\lambda_1 + \int_{\overline{\mathcal{M}}_{1,3}} \psi^3 = 0, \qquad -7\int_{\overline{\mathcal{M}}_{1,3}} \psi^2\lambda_1 + 15\int_{\overline{\mathcal{M}}_{1,3}} \psi^3 - \frac{1}{3} = 0$$
$$\implies \int_{\overline{\mathcal{M}}_{1,3}} \psi^2\lambda_1 = \int_{\overline{\mathcal{M}}_{1,3}} \psi^3 = \frac{1}{24}$$

$d=2, g=1, e=(1,0)$, and $d=3, g=1, e=(1,0)$;

$$3\int_{\overline{\mathcal{M}}_{1,3}} \psi^1\psi^1\lambda_1 - 7\int_{\overline{\mathcal{M}}_{1,3}} \psi^2\psi^1 + \frac{1}{3} = 0,$$
$$6\int_{\overline{\mathcal{M}}_{1,3}} \psi^1\psi^1\lambda_1 - 25\int_{\overline{\mathcal{M}}_{1,3}} \psi^2\psi^1 + \frac{19}{12} = 0$$
$$\implies \int_{\overline{\mathcal{M}}_{1,3}} \psi^1\psi^1\lambda_1 = \int_{\overline{\mathcal{M}}_{1,3}} \psi^2\psi^1 = \frac{1}{12}$$

$d=3, g=1, e=(1,1)$;

$$6\int_{\overline{\mathcal{M}}_{1,3}} \psi^1\psi^1\psi^1 - \frac{1}{2} = 0 \implies \int_{\overline{\mathcal{M}}_{1,3}} \psi^1\psi^1\psi^1 = \frac{1}{12}$$

These values match with (1), λ_g-formula and previously known results.

Recursion relation for $d = 1, g = 2$, and $e = \emptyset$:

$$\int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2 - \int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1 + \int_{\overline{\mathcal{M}}_{2,1}} \psi^4 = 0$$

Recursion relation for $d = 2, g = 2$, and $e = \emptyset$:

$$\begin{aligned}7\int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2 - 15\int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1 + 31\int_{\overline{\mathcal{M}}_{2,1}} \psi^4 = &-\int_{\overline{\mathcal{M}}_{1,2}} \psi^1\lambda_1 + \int_{\overline{\mathcal{M}}_{1,2}} \psi^2 + \frac{1}{2}\int_{\overline{\mathcal{M}}_{1,2}} \psi^1\psi^1 \\ &+ \frac{1}{2}\Big[\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1\Big]^2 - \int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 \int_{\overline{\mathcal{M}}_{1,1}} \psi^1 + \frac{5}{6}\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 \\ &+ \frac{1}{2}\Big[\int_{\overline{\mathcal{M}}_{1,1}} \psi^1\Big]^2 - \frac{11}{6}\int_{\overline{\mathcal{M}}_{1,1}} \psi^1 + \frac{1}{40}\end{aligned}$$

Recursion relation for $d = 3, g = 2$, and $e = \emptyset$:

$$\begin{aligned}25\int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2 - 90\int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1 + 301\int_{\overline{\mathcal{M}}_{2,1}} \psi^4 = &-9\int_{\overline{\mathcal{M}}_{1,2}} \psi^1\lambda_1 + \frac{13}{2}\int_{\overline{\mathcal{M}}_{1,2}} \psi^1\psi^1 + 17\int_{\overline{\mathcal{M}}_{1,2}} \psi^2 \\ &+ \frac{5}{2}\Big[\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1\Big]^2 - 9\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 \int_{\overline{\mathcal{M}}_{1,1}} \psi^1 + \frac{63}{8}\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 \\ &+ \frac{13}{2}\Big[\int_{\overline{\mathcal{M}}_{1,1}} \psi^1\Big]^2 - \frac{231}{8}\int_{\overline{\mathcal{M}}_{1,1}} \psi^1 + \frac{3}{8}\end{aligned}$$

Example 2) To compute

$$\int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2 \quad , \quad \int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1 \quad , \quad \int_{\overline{\mathcal{M}}_{2,1}} \psi^4$$

We need 3 relations!. From the condition

$|e| < -\chi + d + 1$, $d = 1, 2, 3$ will work.

After substituting values for lower-dimensional Hodge integrals, the linear system

$$\int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2 - \int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1 + \int_{\overline{\mathcal{M}}_{2,1}} \psi^4 = 0$$

$$7\int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2 - 15\int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1 + 31\int_{\overline{\mathcal{M}}_{2,1}} \psi^4 = \frac{1}{240}$$

$$25\int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2 - 90\int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1 + 301\int_{\overline{\mathcal{M}}_{2,1}} \psi^4 = \frac{5}{48}$$

is obtained. Solving this system yields

$$\int_{\overline{\mathcal{M}}_{2,1}} \psi^2\lambda_2 = \frac{7}{5760} \quad , \quad \int_{\overline{\mathcal{M}}_{2,1}} \psi^3\lambda_1 = \frac{1}{480}$$

$$\int_{\overline{\mathcal{M}}_{2,1}} \psi^4 = \frac{1}{1152}$$

These values match with previously known methods as follows :

① $\int_{\overline{\mathcal{M}}_{2,1}} \psi^2 \lambda_2 = \frac{7}{5760}$

$\boxed{\lambda_g\text{-formula}}$ computes in case of top Hodge class :

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_g = \binom{2g+n-3}{k_1, \cdots, k_n} \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}$$

Here, B_i's are Bernoulli numbers.

We know that

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0,$$

and $B_4 = -\frac{1}{30}$.

So in this case,

$$\int_{\overline{\mathcal{M}}_{2,1}} \psi^2 \lambda_2 = \binom{2}{2} \frac{2^3-1}{2^3} \cdot \frac{1}{30} \cdot \frac{1}{4!} = \frac{7}{5760}$$

② $\int_{\overline{\mathcal{M}}_{2,1}} \psi^3 \lambda_1 = \frac{1}{480}$

In case of λ_{g-1} with one marked point, we have the following formula :

$$\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-1} \lambda_{g-1} = b_g \sum_{i=1}^{2g-1} \frac{1}{i} - \frac{1}{2} \sum_{g_1+g_2=g} \frac{(2g_1-1)!\,(2g_2-1)!}{(2g-1)!} b_{g_1} b_{g_2}$$

where $b_g = \begin{cases} 1 & \text{if } g=0 \\ \dfrac{2^{2g-1}-1}{2^{2g-1}} \dfrac{|B_{2g}|}{(2g)!} & \text{if } g>0 \end{cases}$

So in this case,

$$\int_{\overline{\mathcal{M}}_{2,1}} \psi^3 \lambda_1 = \frac{2^3-1}{2^3} \frac{1}{30} \cdot \frac{1}{4!} \cdot \left(1 + \frac{1}{2} + \frac{1}{3}\right)$$

$$- \frac{1}{2} \left(\frac{1}{3!} \cdot \left[\frac{1}{2}\left(\frac{1}{6}\right)\right]^2 \right) = \underline{\underline{\frac{1}{480}}}$$

③ $\int_{\overline{\mathcal{M}}_{2,1}} \psi^4 = \frac{1}{1152}$

This value can be computed from the result of C. Itzykson and J.-B. Zuber which says that

$$F_g = \sum_{\{l_k\}} \langle \tau_2^{l_2} \tau_3^{l_3} \cdots \tau_{3g-2}^{l_{3g-2}} \rangle \frac{1}{(1-I_1)^{2(g-1)+\sum l_p}} \cdot \prod_{p=2}^{3g-2} \frac{I_p^{l_p}}{l_p!}$$

holds where $\{l_k\}$ satisfies

$$\sum_{2 \leq k \leq 3g-2} (k-1)\, l_k = 3g-3$$

and $$F = \sum_{g \geq 0} F_g = \ln Z$$

Z is a τ-function for the KdV equation.

So in this case

$$F_2 = \frac{1}{5760}\left[5 \cdot \frac{I_4}{(1-I_1)^3} + 29 \frac{I_3 I_2}{(1-I_1)^4} + 28 \frac{I_2^3}{(1-I_1)^5}\right]$$

and $$\langle \tau_4 \rangle = 1 \times 5 \times \frac{1}{5760} = \underline{\underline{\frac{1}{1152}}}$$

Thank you!