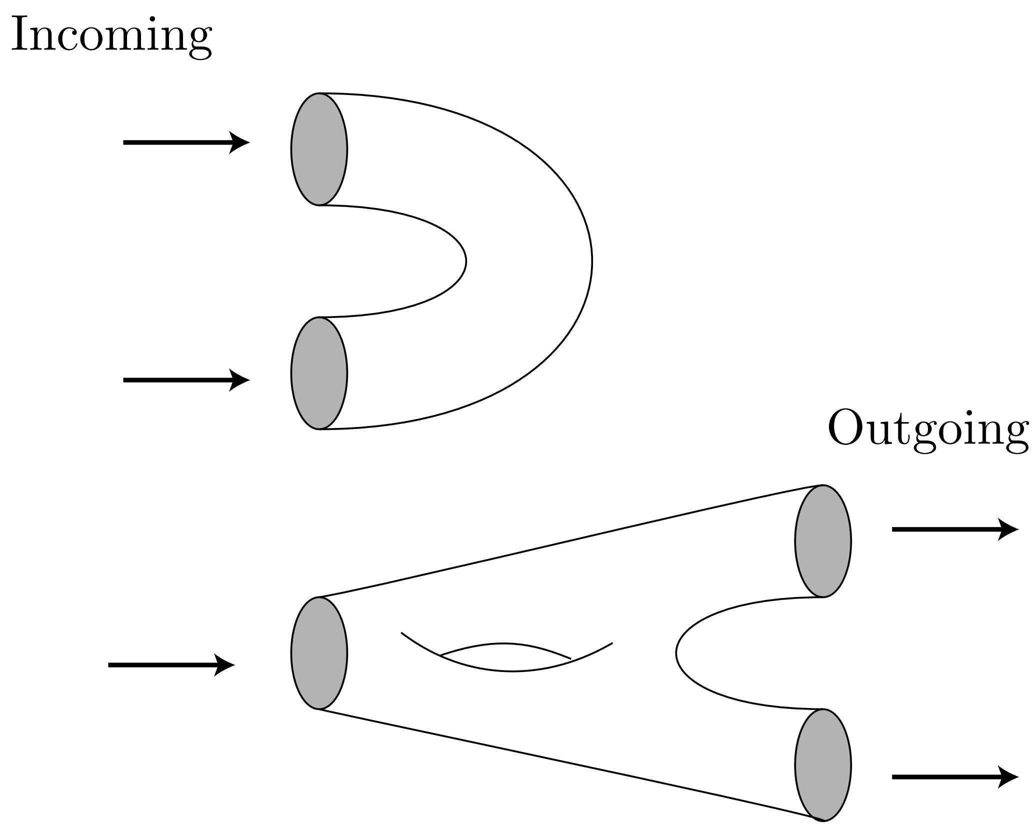
Topological conformal field theories from Calabi-Yau categories

Segal's definition of conformal field theory.

Category $\mathcal{M}$, objects are finite sets, morphisms are Riemann surfaces with parameterised boundary.



$$\Sigma \in \mathcal{M}(3, 2)$$

Boundary of Σ is split into incoming and outgoing.

Composition in $\mathcal{M}$: gluing of surfaces.

Disjoint union makes $\mathcal{M}$ into a symmetric monoidal category.

A CFT is a tensor functor from $\mathcal{M}$ to vector spaces.

If Φ is a CFT, for a finite set I, $\Phi(I)$ is a vector space.

There are maps

$$\Phi(I) \otimes \Phi(J) \to \Phi(I \amalg J)$$

Usually these are assumed to be isomorphisms; this case is called split.

Each Riemann surface $\Sigma \in \mathcal{M}(I, J)$ give a map

$$\Phi(\Sigma) : \Phi(I) \to \Phi(J)$$

Disjoint union corresponds to tensor product. The diagram

$$\begin{array}{ccc} \Phi(I_1) \otimes \Phi(I_2) & \longrightarrow & \Phi(I_1 \amalg I_2) \\ \downarrow{\scriptstyle \Phi(\alpha_1)\otimes\Phi(\alpha_2)} & & \downarrow{\scriptstyle \Phi(\alpha_1 \amalg \alpha_2)} \\ \Phi(J_1) \otimes \Phi(J_2) & \longrightarrow & \Phi(J_1 \amalg J_2) \end{array}$$

commutes.

Φ respects composition.

$$\Phi(\Sigma_1 \circ \Sigma_2) = \Phi(\Sigma_1) \circ \Phi(\Sigma_2)$$

if $\Sigma_2 \in \mathcal{M}(I, J)$, $\Sigma_1 \in \mathcal{M}(J, K)$.

For example : in a topological field theory, operations $\Phi(\Sigma)$ are independent of the conformal structure on Σ.

A split topological field theory is the same as a Frobenius algebra.

In TFT, we are taking locally constant functions on moduli space, which is a sheaf with higher cohomology.

Topological conformal field theory is the derived analogue of this.

In TCFT, $\Phi(I)$ are complexes. Each surface $\Sigma \in \mathcal{M}(I, J)$ gives a map of complexes

$$\Phi(\Sigma) : \Phi(I) \to \Phi(J)$$

If Σ_1, Σ_2 are connected by a path α in moduli space, then operations $\Phi(\Sigma_1), \Phi(\Sigma_2)$ are connected by a chain homotopy, $\Phi(\alpha)$.

$\Phi(\alpha)$ is a chain homotopy means

$$[\mathrm{d}, \Phi(\alpha)] = \Phi(\mathrm{d}\alpha) = \Phi(\Sigma_1) - \Phi(\Sigma_0)$$

Similarly, any n parameter family of surfaces α, i.e. singular n simplex, gives an operation $\Phi(\alpha)$ with

$$\Phi(\mathrm{d}\alpha) = [\mathrm{d}, \Phi(\alpha)]$$

Define a category $\mathscr{C}$, with the same objects as $\mathcal{M}$, but whose morphisms are the singular chain complexes

$$\mathscr{C}(I,J) = C_*(\mathcal{M}(I,J))$$

$\mathscr{C}$ is a differential graded symmetric monoidal category, monoidal structure given by disjoint union.

Then a TCFT is a tensor functor from $\mathscr{C}$ to complexes, compatible with the differential.

Each $I \in \operatorname{Ob}\mathscr{C}$, a finite set, have a complex $\Phi(I)$.

Maps $\Phi(I) \otimes \Phi(J) \to \Phi(I \amalg J)$; extra technical condition, these are quasi-isomorphisms.

Each $\alpha \in \mathscr{C}(I,J)$ gives map $\Phi(I) \to \Phi(J)$. The maps are compatible with composition, differential, and monoidal structure.

Technical points:

- Category of TCFTs is independent of chain model for moduli space.
- We should take chains with coefficients in the flat line bundle

  $$(\det(H^*(\Sigma))[\chi(\Sigma)])^{\otimes d}$$

  on moduli space. Σ is the universal surface. d is the dimension of the theory.

Dimension will correspond to complex dimension of Calabi-Yau in A or B model.

$d = 3$ is critical dimension :

$$(\det(H^*(\Sigma))[\chi(\Sigma)])^{\otimes d}$$

is the dualizing complex on $\mathcal{M}_g$.

Let $\mathscr{O}_\Lambda = C_*(\mathcal{M}_\Lambda^{open})$.

An open TCFT is a tensor functor from $\mathscr{O}_\Lambda$ to complexes. For open TCFT of dimension d we use chains with coefficients in the local system $\det(H^*(\Sigma))[\chi(\Sigma)]^{\otimes d}$.

Each pair λ_1, λ_2 of D-branes defines an object of $\mathscr{O}_\Lambda$. If Φ is an open TCFT, $\Phi(\lambda_1, \lambda_2)$ is a complex; morphisms between D-branes.

If O is a set, and $s, t : O \to \Lambda$ are maps, then $\Phi(O)$ is a complex. There is a map $\Phi(O_1) \otimes \Phi(O_2) \to \Phi(O_1 \amalg O_2)$; we assume this is a quasi-isomorphism.

If $\alpha \in \mathscr{O}_\Lambda(O_1, O_2)$ is a chain in moduli space of Riemann surfaces, with O_1 incoming, O_2 outgoing, open boundaries, D-brane labels $s, t : O_i \to \Lambda$, we have a map

$$\Phi(\alpha) : \Phi(O_1) \to \Phi(O_2)$$

with various compatibility conditions.

Main result: allows algebraic construction of open and closed TCFTs.

Theorem 1. *1. There is a homotopy equivalence of categories between open TCFTs of dimension d, and extended Calabi-Yau A_∞ categories of dimension d.*

2. For each open TCFT, there is a homotopy universal open-closed TCFT.

3. The homology of the closed states of this is the Hochschild homology of the A-infinity category.

Calabi-Yau category is a linear category with an invariant pairing between $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(B, A)$. Categorical generalisation of Frobenius algebra. A_∞: only associative up to homotopy.

Part 1 means there are functors

$$F : \text{Open TCFTs} \rightleftarrows \text{ CY categories } : G$$

$F \circ G$, $G \circ F$ quasi-isomorphic to identity functors.

Example : X a projective Calabi-Yau, then $\mathcal{D}^b(X)$, derived category of coherent sheaves, is a Calabi-Yau category, with

$$\mathrm{Hom}(A, B) = \oplus_i \mathrm{Ext}^i(A, B)$$

Serre pairing

$$\mathrm{Ext}^i(A, B) \cong \mathrm{Ext}^{d-i}(B, A)^\vee$$

Then we should have

$$HH_k(\mathcal{D}^b(X)) \stackrel{?}{=} \oplus_{j-i=k} H^i(X, \Omega^j)$$

TCFT structure induces operations

$$H_*(\mathcal{M}(I, J)) \to \mathrm{Hom}(HH_*(\mathcal{D}^b(X))^{\otimes I}, HH_*(\mathcal{D}^b(X))^{\otimes J})$$

B model mirror to operations

$$H_*(\mathcal{M}(I, J)) \to \mathrm{Hom}(H_*(X^\vee)^{\otimes I}, H_*(X^\vee)^{\otimes J})$$

arising geometrically (Gromov-Witten type invariants).

Functors

$$j : \mathscr{O}_\Lambda \to \mathscr{OC}_\Lambda \leftarrow \mathscr{C} : i$$

If $\Phi : \mathscr{O}_\Lambda \to$ Complexes is an open TCFT, there is a universal extension $j_* \Phi : \mathscr{OC}_\Lambda \to$ Complexes.

$\mathbb{L} j_* \Phi$: homotopy universal extension (replace Φ by resolution, apply j_*).

Any functor $\Psi : \mathscr{OC}_\Lambda \to$ Complexes pulls back to $i^* \Psi : \mathscr{C} \to$ Complexes.

Set

$$\mathscr{H}(\Phi) = i^* \mathbb{L} j_* \Phi$$

$\mathscr{H}$ is a functor from open to closed TCFTs. Main result says that

$$H_*(\mathscr{H}(\Phi)(I)) = HH_*(\Phi)^{\otimes I}$$

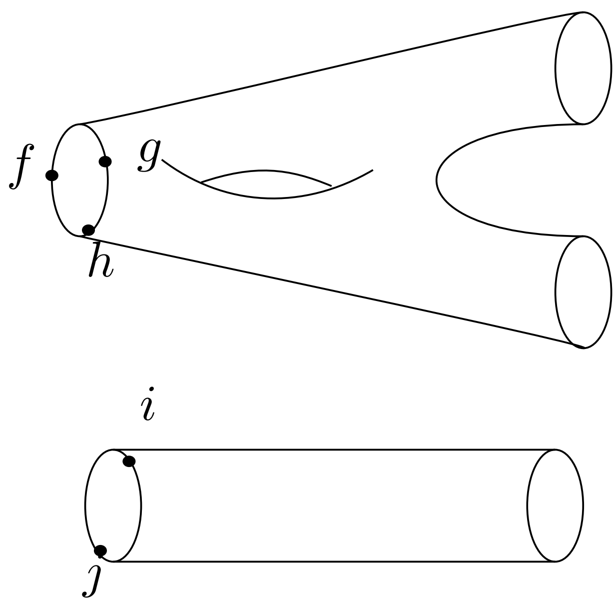
A category is like an algebra, a functor to complexes is like a left module. Then

$$j_* \Phi = \mathscr{OC}_\Lambda \otimes_{\mathscr{O}_\Lambda} \Phi$$

$$\mathbb{L} j_* \Phi = \mathscr{OC}_\Lambda \otimes^{\mathbb{L}}_{\mathscr{O}_\Lambda} \Phi$$

Think of $\mathscr{OC}_\Lambda$ as a $\mathscr{C} - \mathscr{O}_\Lambda$ bimodule. So that if Φ is an $\mathscr{O}$ module – i.e. open TCFT – then $\mathscr{OC}_\Lambda \otimes_{\mathscr{O}_\Lambda} \Phi$ is a $\mathscr{C}$ module.

Picture of tensor product:



Free boundaries labelled by D-branes, $f, \ldots, j$ morphisms between them.

Geometric examples of TCFT (largely conjectural):

If X is compact, symplectic, Fukaya category is open (tree level) TCFT.

D branes : graded Lagrangians in $L \subset X$;

$$\mathrm{Hom}_i(L_1, L_2) = CF^{-i}(L_1, L_2)$$

the Floer complex.

Homology $H_{*+d}(X)$ of X is a closed homological CFT of dimension d. Operations

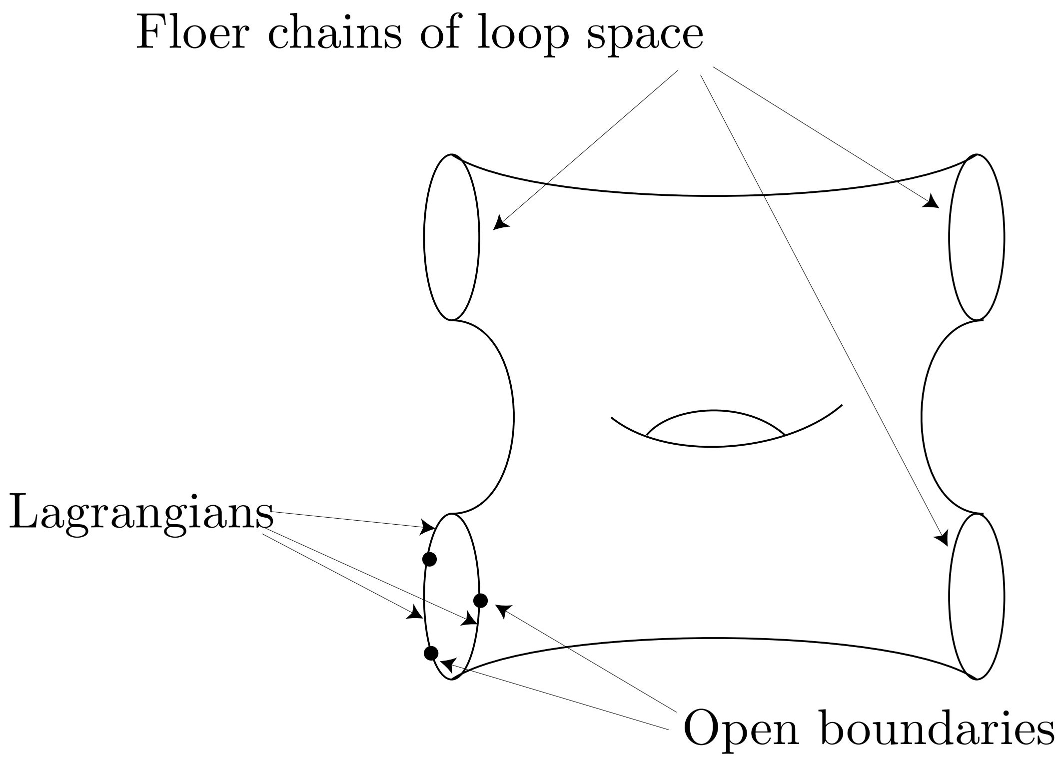
$$H_*(\mathcal{M}(I, J)) \to \mathrm{Hom}(H_{*+d}(X)^{\otimes I}, H_{*+d}(X)^{\otimes J})$$

from GW theory.

Conjecture 1. *This lifts to chain level, so* $C_{*+d}(X)$ *is a closed TCFT of dimension* d*.*

Conjecture 2. *There is open-closed TCFT structure on* $(\mathrm{Fuk}(X), C_{*+d}(X))$*, generalising the open structure on* $\mathrm{Fuk}(X)$ *and closed structure on* $C_{*+d}(X)$*.*

This structure should come from open-closed GW theory.



Open boundaries are shrunk to points. Free boundaries map to Lagrangians, and closed boundaries to critical points of loop space Floer functional, which are periodic orbits of a generic time-dependent Hamiltonian.

Compactify by bubbling. Note $CF_*(X) \simeq C_*(X)$, so gives TCFT structure on either Floer or ordinary chains.

If X is compact, symplectic, $\mathrm{Fuk}(X)$ is a unital Calabi-Yau A_∞ category, and so an open TCFT.

Universal closed TCFT $\mathscr{H}(\mathrm{Fuk}(X))$,

$$H_*(\mathscr{H}(\mathrm{Fuk}(X))) = HH_*(\mathrm{Fuk}(X))$$

If $(\mathrm{Fuk}(X), C_{*+d}(X))$ is an open-closed TCFT, then there is a map

$$\mathscr{H}(\mathrm{Fuk}(X)) \to C_{*+d}(X)$$

of closed TCFTs : intertwines operations from $\mathscr{C} = C_*(\mathcal{M})$.

This induces a map

$$HH_*(\mathrm{Fuk}(X)) \to H_{*+d}(X)$$

For all $\alpha \in H_*(\mathcal{M}(I,J))$ diagram

$$\begin{tikzcd} HH_*(\mathrm{Fuk}(X))^{\otimes I} \arrow[r] \arrow[d, "\alpha"] & H_{*+d}(X)^{\otimes I} \arrow[d, "\alpha"] \\ HH_*(\mathrm{Fuk}(X))^{\otimes J} \arrow[r] & H_{*+d}(X)^{\otimes J} \end{tikzcd}$$

commutes.

In particular : dualise

$$H^*(X) \to HH^*(\mathrm{Fuk}(X))$$

takes quantum product to Hochschild cup product.

Conjecture 3. *The map* $HH_*(\mathrm{Fuk}(X)) \to H_{*+d}(X)$ *is an isomorphism, for reasonable* X*.*

This seems to be an integral part of the homological mirror symmetry picture.

Equivalently, the formal neighbourhood of $\mathrm{Fuk}(X)$ in moduli of A_∞ categories is the same as the formal neighbourhood of symplectic class in $H^*(X)$.

If this was true, and $X^\vee$ is mirror to X, we would see that $\oplus H^i(X^\vee, \Omega^j)$ is isomorphic to $H_*(X)$ in a way compatible with operations from $H_*(\mathcal{M})$.

Proof of main theorem : existence of universal open-closed TCFT. Let Φ be an open TCFT. Then define open-closed TCFT by

$$\mathscr{OC}_\Lambda \otimes_{\mathscr{O}_\Lambda} \Phi$$

For $b \in \operatorname{Ob} \mathscr{OC}_\Lambda$, $(\mathscr{OC}_\Lambda \otimes_{\mathscr{O}_\Lambda} \Phi)(b)$ is spanned by

$$\mathscr{OC}_\Lambda(a, b) \otimes \Phi(a)$$

where $a \in \operatorname{Ob} \mathscr{O}_\Lambda$. Quotient by the relation that the diagram

$$\begin{tikzcd}
\mathscr{OC}_\Lambda(a,b) \otimes \mathscr{O}_\Lambda(a',a) \otimes \Phi(a') \arrow[r] \arrow[d] & \mathscr{OC}_\Lambda(a,b) \otimes \Phi(a) \arrow[d] \\
\mathscr{OC}_\Lambda(a',b) \otimes \Phi(a') \arrow[r] & (\mathscr{OC}_\Lambda \otimes_{\mathscr{O}_\Lambda} \Phi)(b)
\end{tikzcd}$$

commutes.

Homotopy universal TCFT : left derived version of this. Replace Φ by a flat resolution.

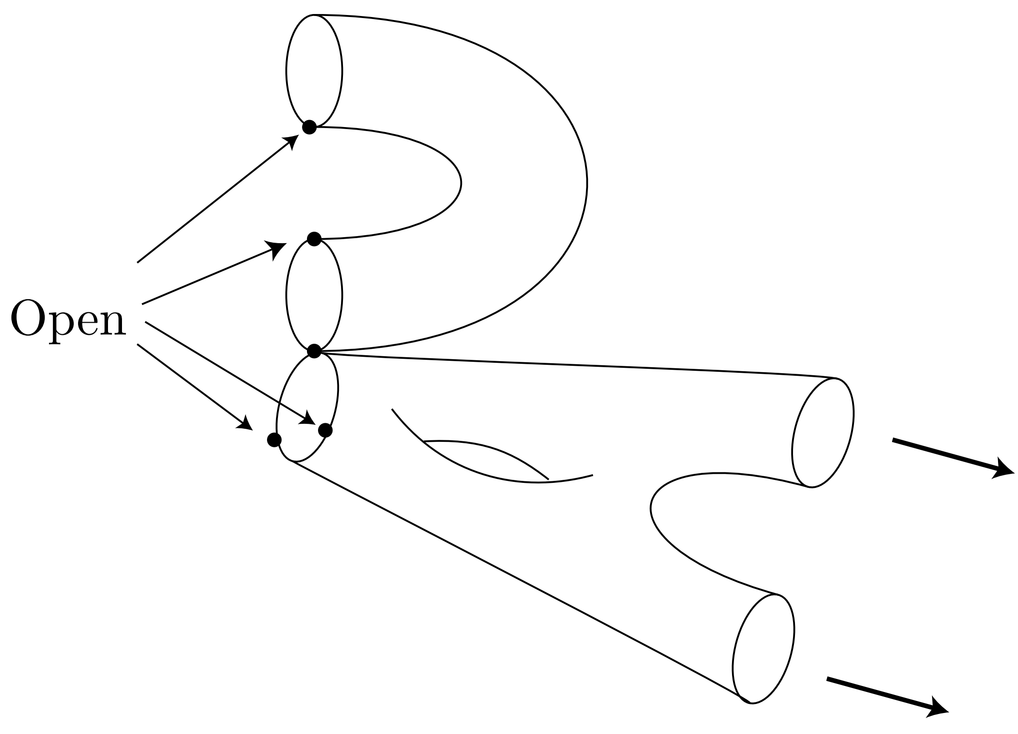
To compute the homotopy universal TCFT, we only need to know about $\mathscr{OC}_\Lambda(a, b)$ where only b has a closed part – i.e. the only closed boundaries on the surfaces are outgoing, and use only open gluing.

Give combinatorial model for this.

Use moduli of Riemann surfaces where open boundaries are contracted to points.

Open gluing forces us to allow nodes on the boundary. This is homotopy equivalent to usual moduli spaces.

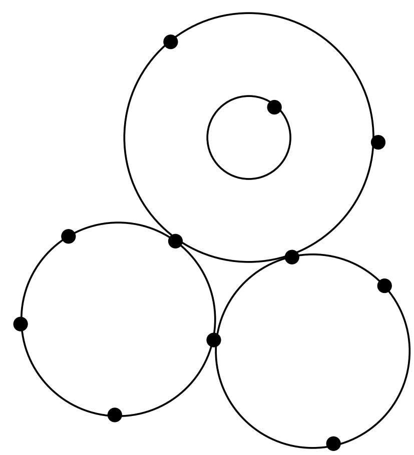
For example :



Theorem 2. *Locus where irreducible components of surface are either discs, or annuli, one of whose boundary components is closed and outgoing, is homotopy equivalent to the whole moduli space.*

(This implies a stronger version of ribbon graph decomposition).

A surface in this moduli space :

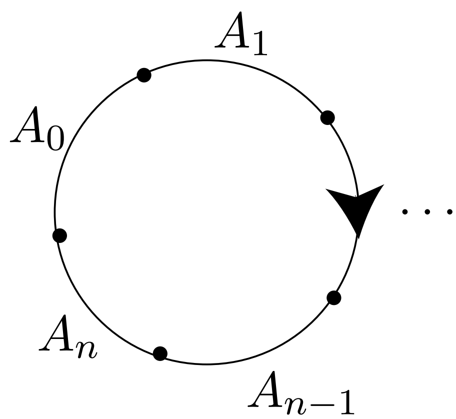


Open marked points, and closed outgoing boundary, interior of annulus : marked point there is start of parameterisation.

Proof of theorem : push open boundary components inwards, eventually becomes singular; normalise, repeat.

Pass to chain level : using open gluing, these moduli spaces are freely generated by discs and annuli.

Chain of a disc with $n+1$ moving points, n incoming one outgoing, free boundaries labelled by D-branes $A_0, \ldots, A_n$,

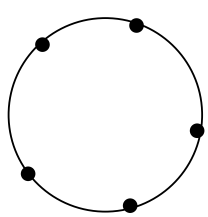
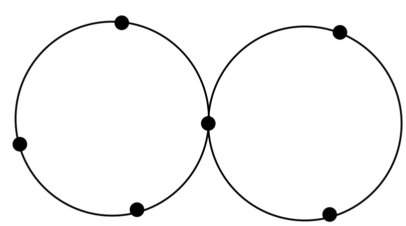


Gives map

$$\mathrm{Hom}(A_0, A_1) \otimes \ldots \mathrm{Hom}(A_{n-1}, A_n) \to \mathrm{Hom}(A_0, A_n)$$

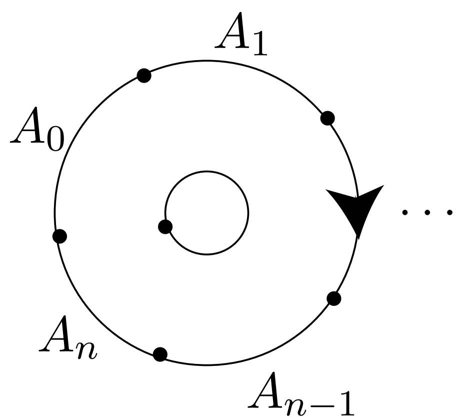
which is the A_∞ operation m_n.

The formula

d  $= \sum \pm$ 

corresponds to A_∞ axiom for operations m_n.

Annuli give Hochschild complex. Annulus with $n+1$ open marked points, free boundaries labelled by D-branes $A_0, \ldots, A_n$,

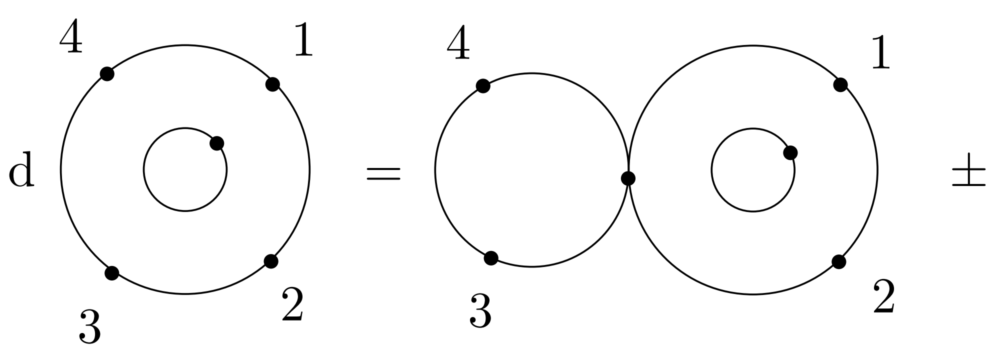


gives map

$\mathrm{Hom}(A_0, A_1) \otimes \mathrm{Hom}(A_1, A_2) \otimes \ldots \mathrm{Hom}(A_n, A_0) \to \mathscr{C}_D(1)$

If $f_i : A_i \to A_{i+1 \bmod n+1}$, get Hochschild chains

$$f_0 \otimes \ldots \otimes f_{n+1} \in \mathscr{C}_D(1)$$



This corresponds to

$$\mathrm{d}(f_1 \otimes f_2 \otimes f_3 \otimes f_4) = f_1 \otimes f_2 \otimes m_2(f_3 \otimes f_4) \pm \ldots$$